Exact Controllability of Klein-Gordon Systems with a Time-varying Parameter

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Abstract: - In this paper we give a sufficient condition on the exact controllability of the Klein-Gordon system with a time-varying parameter. We show that under some regularity conditions on the time-varying parameter, the system is exact controllable. We prove its exact controllability by applying the Hilbert Uniqueness Method.

Key-Words: - Exact Control; Distributed Parameter System; Hilbert Uniqueness Method; Klein-Gordon

1 Introduction

Suppose Ω be an open and bounded domain in $\mathbb{R}^d$, where $d \geq 2$. Next, we let a distributed parameter system $\mathcal{M}$ be

$$w_{tt} - \Delta w + \alpha(t)w = 0 \text{ in } \Omega \times (0,T), \tag{1}$$

$$w(x,t) = u(x,t)\chi_{\omega\times(0,T)} \text{ on } \partial\Omega \times (0,T), \tag{2}$$

$$w(x,0) = w_0(x) \in L^2(\Omega), \quad \text{and} \tag{3}$$

$$w_t(x,0) = w_1(x) \in H^{-1}(\Omega), \tag{4}$$

where ω is a relatively open subset of the boundary $\partial\Omega$. Here, H^{-1} denotes the dual of the space H_0^1 and α is a smooth function of time t. The function u is called control, and the whole system is called Klein-Gordon system. To shorten the notaion, we denote the left hand side of (1) by $L(w)$.

Our problem is to find a sufficient condition for the existence of a control $u \in L^2(\omega \times (0,T))$ such that $w(.,T) = w_t(.,T) = 0$. A control satisfying this property is called *exact control.*

Positive analytical results on the existence of the exact control u have been stated for several mathematical models. For example, see [1], [5], [6], [7], and [8]. The control's relation with the damping of the systems is presented in [10].

The most recent study on wave-like equation is done by Avalos and Lasiecka in [2].

2 Exact Controllability

Before stating the exact controllability of the system above, we define the dual system $\mathcal{M}'$ of the above system as

$$v_{tt} - \Delta v + \alpha(t)v = 0 \text{ in } \Omega \times (0,T), \tag{5}$$

$$v(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \tag{6}$$

$$v(x,0) = v_0(x) \in H_0^1(\Omega), \quad \text{and} \tag{7}$$

$$v_t(x,0) = v_1(x) \in L^2(\Omega). \tag{8}$$

The dual system above is equipped with an output map

$$z(x,t) = -\partial_\nu v(x,t) \text{ on } \omega \times (0,T). \tag{9}$$

Here, ν denotes the unit normal vector field on the boundary $\partial\Omega$, pointing outward, and ∂_ν is the directional derivative on the direction of ν.

Let $S = H_0^1 \times L^2$ and $|\cdot|$ be the L^2-norm. We then define a norm

$$\|(v, v_t)\|_S = \sqrt{|\nabla v|^2 + |v_t|^2} \tag{10}$$

for every $(v, v_t) \in S$. With this norm, we define *energy* E of the state $(v(\cdot,t), v_t(\cdot,t))$ as

$$E(t) = \frac{1}{2}\left(\|(v, v_t)\|_S^2 + \alpha(t)|v|^2\right). \tag{11}$$

Next, let r be a point in $\mathbb{R}^d \setminus \bar{\Omega}$. Then we can define a set

$$\Gamma(r) = \{x \in \partial\Omega | (x - r) \cdot \nu(x) > 0\}. \tag{12}$$

Every element of $\Gamma(r)$ is called an *exit point* of Ω relative to r.

Corresponding to the geometry of Ω, one can find a constant $\lambda_0 > 0$, such that $|\nabla\varphi|^2 \geq \lambda_0|\varphi|^2$ for all $\varphi \in H_0^1$. This constant in fact is the first eigenvalue for the negative Laplacian $-\Delta$ with Dirichlet boundary condition.

Relating to the dual system $\mathcal{M}'$, we obtain the following a kind of observability property.

Lemma 1 *If $r = (r^1, r^2, \cdots, r^d) \in \mathbb{R}^d \setminus \bar{\Omega}$, and there is an $\epsilon \in (0,1)$, such that*

(A1) $(1-\epsilon)\lambda_0 \geq |\alpha(t)|$ *for all* $t \geq 0$*;*

(A2) $\mathrm{var}(\alpha) < \infty$*;*

(A3) $\Gamma(r) \subseteq \omega$*,*

then there exists a $T > 0$ and constants $k_T, K_T > 0$ such that

$$k_T\|(v_0, v_1)\|_S^2 \leq |z|^2_{L^2(\omega\times(0,T))} \leq K_T\|(v_0, v_1)\|_S^2 \tag{13}$$

for every $(v_0, v_1) \in S$.

Here, $\mathrm{var}(\alpha)$ denotes the total variation of α on $[0, \infty)$. It is defined as

$$\mathrm{var}(\alpha) = \sup_{p\in\mathcal{P}} \sum_{i=1}^{N_p} |\alpha(t_i) - \alpha(t_{i-1})|, \tag{14}$$

where $\mathcal{P}$ is the family of all finite partition $p = \{[t_{i-1}, t_i) | i = 1, 2, \cdots, N_p\}$ of $[0, \infty)$.

Proof. By (A1), one can prove that the energy E is always non negative. Moreover, if we differentiate it along the trajectory of $\mathcal{M}'$, we obtain

$$E'(t) = \frac{\alpha'(t)}{2}|v|^2. \tag{15}$$

Since $|v|^2 \leq \frac{2}{\epsilon\lambda_0}E(t)$, then

$$\frac{E(0)}{M(T)} \leq E(t) \leq M(T)\ E(0), \tag{16}$$

for every $t > 0$. Here, $M(T) = C\exp\left(C_0\int_0^T |\alpha'(t)|dt\right)$, $C_0 = 1/(\epsilon\lambda_0)$, and C is some positive number.

Next, we define a vector field

$$q(x) = (q^1(x), \cdots, q^d(x)) = ((x^1 - r^1), \cdots, (x^d - r^d)). \tag{17}$$

It is clear that $q \in \left(C^\infty(\mathbb{R}^d)\right)^d$.

We now consider the following equation

$$\int_{\Omega\times(0,T)} L(v)\ Q\ dx\ dt = 0, \tag{18}$$

where $Q = \sum_{i=1}^{d} q^i\ \partial_{x_i}v$. This Q is called the *multiplier*. This is the reason why this method is called multiplier technique [4]. On the left hand side, we have three terms. Integrating the first term, we obtain

$$\int_{\Omega\times(0,T)} v_{tt}\ Q\ dx\ dt = \rho(T) - \rho(0) + \frac{d}{2}\int_{\Omega\times(0,T)} |v_t|^2\ dx\ dt, \tag{19}$$

where $\rho(t) = \int_\Omega v_t(x,t) \sum_{i=1}^{d} q^i\ \partial_{x^i}v(x,t)\ dx$. Integrating the second term, we obtain

$$-\int_{\Omega\times(0,T)} \Delta v\ Q\ dx\ dt = -\int_{\partial\Omega\times(0,T)} \partial_\nu v\ Q\ ds\ dt + \int_{\Omega\times(0,T)} \sum_{i,j=1}^{d} \partial_{x^j}v\ \partial_{x^j}(q^i\ v)\ dx\ dt \tag{20}$$

The boundary condition $v|_{\partial\Omega} = 0$ implies that $\partial_{x^i}v = \nu^i(\partial_\nu v)$. Hence, the right hand side of (20) becomes

$$-\int_{\partial\Omega\times(0,T)} \sum_{i=1}^{d}\left(\frac{q^i\nu^i}{2}|\partial_\nu v|^2\right) ds\ dt - \frac{d-2}{2}\int_{\Omega\times(0,T)} |\nabla v|^2 dx\ dt. \tag{21}$$

And, by the boundary condition, the third term becomes

$$-\frac{d}{2}\int\limits_{\Omega\times(0,T)} \alpha(t)(v)^2\ dx\ dt. \qquad (22)$$

Thus, (18) becomes

$$0 = \rho(T) - \rho(0)$$

$$+\frac{d-1}{2}\int_0^T \left(|v_t|^2 - |\nabla v|^2 - \alpha(t)|v|^2\right) dt$$

$$\frac{1}{2}\int_0^T \left(|v_t|^2 + |\nabla v|^2 - \alpha(t)|v|^2\right) dt$$

$$-\frac{1}{2}\int\limits_{\partial\Omega\times(0,T)} \sum_{i=1}^{d} q^i\nu^i(\partial_\nu v)^2 ds\ dt. \qquad (23)$$

If $W = \max_i \sup_{x\in\Omega}\{|q^i(x)|\}$, then

$$|\rho(t)| \le W|v_t||\nabla v| \le W\ D\ E(t), \qquad (24)$$

where $D = 1/\epsilon$. By this estimate on ρ, (23) becomes

$$\frac{1}{2}\int\limits_{\partial\Omega\times(0,T)} \sum_{i=1}^{d}\left(\frac{q^i\nu^i}{2}|\partial_\nu v|^2\right) ds\ dt$$

$$\ge \frac{d-1}{2}\int_0^T \left(|v_t|^2 - |\nabla v|^2 - \alpha(t)|v|^2\right) dt$$

$$+\frac{1}{2}\int_0^T \left(|v_t|^2 + |\nabla v|^2 - \alpha(t)|v|^2\right) dt$$

$$-2\ W\ D\ E(t). \qquad (25)$$

If we let $\delta = \epsilon^2$, then there is a positive constant N, such that

$$\frac{1}{2}\int\limits_{\partial\Omega\times(0,T)} \sum_{i=1}^{d}\left(\frac{q^i\nu^i}{2}|\partial_\nu v|^2\right) ds\ dt$$

$$\ge -2\ W\ D\ E(T) - N\left(E(T) + E(0)\right)$$

$$+\delta\int_0^T E(t)\ dt. \qquad (26)$$

Then, by the estimate of growth and decrease of the energy in (16), the inequality (26) becomes

$$\frac{1}{2}\int\limits_{\partial\Omega\times(0,T)} \sum_{i=1}^{d}\left(\frac{q^i\nu^i}{2}|\partial_\nu v|^2\right) ds\ dt$$

$$\ge \left(-2\left(W\ D\ + N\right) M(T) + \delta\frac{T}{M(T)}\right) E(0). \quad (27)$$

By (A3), the left hand side of (27) is dominated by the integral of the same integrand but over $\omega\times(0,T)$. In other words, we obtain

$$\frac{1}{2}\int\limits_{\omega\times(0,T)} \sum_{i=1}^{d}\left(\frac{q^i\nu^i}{2}|\partial_\nu v|^2\right) ds\ dt$$

$$\ge \left(-2\left(W\ D\ + N\right) M(T) + \delta\frac{T}{M(T)}\right) E(0). \quad (28)$$

Thus, if $F = 2/(\sup_{x\in\omega}\{q(x)\cdot\nu(x)\})$, we obtain

$$|z|^2 \ge F\Bigg(- 2\left(W\ D\ + N\right) M(T)$$

$$+\delta\frac{T}{M(T)}\Bigg) E(0). \qquad (29)$$

By (A2), we can find T sufficiently large such that

$$\left(-2\left(W\ D\ + N\right) M(T) + \delta\frac{T}{M(T)}\right) > 0, \qquad (30)$$

because $\frac{T}{M(T)}$ grows relatively faster than $M(T)$. In addition, because $E(0)$ can be estimated by $\|(v_0, v_1)\|_S$, then there exists a $k_T > 0$, such that

$$|z|^2 \ge k_T\|(v_0, v_1)\|_S. \qquad (31)$$

Similarly, if we replace q^i in (17) by some $q^i \in C^1(\bar\Omega)$, such that $q^i(x)\nu^i(x) \ge \kappa/d > 0$ for some $\kappa > 0$ and for every i, then

$$|z|^2 \le K_T\|(v_0, v_1)\|_S \qquad (32)$$

for some constant $K_T > 0$. This last inequality also implies that the output z of the dual system belongs to L^2. $\square$

Now, we use the above lemma to prove the following proposition.

Proposition 1 *If* (A1) *to* (A3) *in Lemma 1 are satisfied, then the system* $\mathcal{M}$ *is exact controllable.*

Proof. To prove the exact controllability of $\mathcal{M}$, we apply the Hilbert Uniqueness Method, introduced by Lions in [5]. First, we define a linear map $\Lambda_T : H_0^1 \times L^2 \to L^2 \times H^{-1}$. Let $(v_0, v_1) \in H_0^1 \times L^2$, then one can compute the output $z = z(x,t)$ from the system $\mathcal{M}$ described in (5) - (9). Using this output z, one can find the initial condition $(w(x,0), w_t(x,0))$ by solving the following backward value problem

$$w_{tt} - \Delta w + \alpha(t) w = 0 \text{ in } \Omega \times (0,T), \tag{33}$$

$$w(x,t) = z(x,t)\chi_{\omega\times(0,T)} \text{ on } \partial\Omega \times (0,T), \tag{34}$$

$$w(x,T) = 0 \text{ and } w_t(x,T) = 0. \tag{35}$$

We define $\Lambda_T(v_0, v_1) = (-w_t(x,0), w(x,0))$. This relation says that the initial condition $(w(x,0), w_t(x,0))$ can be driven to the equilibrium position in time T by the control z computed from the dual system $\mathcal{M}'$ whose initial condition is given by (v_0, v_1). Hence, we can prove the exact controllability of $\mathcal{M}$, if we can show that λ_T is surjective.

After integrating $\int_{\Omega\times(0,T)} L(w)v \, dx \, dt$ by parts, we obtain

$$\begin{aligned} 0 &= \int_{\Omega\times(0,T)} (L(w)v - L(v)w) \, dx \, dt \\ &= \langle\langle(-w_t(x,0), w(x,0)), (v_0, v_1)\rangle\rangle \\ &\quad - \int_{\omega\times(0,T)} |z(x,t)|^2 ds \, dt. \end{aligned} \tag{36}$$

Here, $\langle\langle\cdot,\cdot\rangle\rangle$ is a nondegenerate bilinear form

$$\begin{aligned} \langle\langle(-w_t(x,0), w(x,0)), (v_0, v_1)\rangle\rangle &= \langle -w_t(x,0), v_0\rangle \\ &\quad + (w(x,0), v_1)\,. \end{aligned} \tag{37}$$

The first term on the right hand side denotes the duality pairing of H^{-1} and H_0^1. The second term denotes the L^2-inner product. Hence, (36) becomes

$$\langle\langle\Lambda_T(v_0, v_1), (v_0, v_1)\rangle\rangle = \int_{\omega\times(0,T)} |z|^2 ds \, dt. \tag{38}$$

By the continuity of the bilinear form and the map Λ_T, one may use (13) and the Lax-Milgram Lemma to conclude that Λ_T is surjective. In other words, the system $\mathcal{M}$ described by (1) to (4) is exact controllable. □

If α is zero, the equation becomes a wave equation. The results on this type of equation have been quite complete. Some results can be found in [5] and [11]. Some sharper results are found through microlocal analysis. See [3] for detailed results. And, if α is constant, we are able to make a numerical scheme to compute the exact control numerically, see [9].

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