An Automatic Integration of Infinite Range Integrals Involving Bessel Functions

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Abstract: - An efficient automatic quadrature procedure is developed for numerically computing the integrals $\int_0^\infty J_\nu(\omega t)f(t)dt$, where the function $f(t)$ is smooth and nonoscillatory at infinity and $J_\nu(\omega t)$ is the Bessel functions of order ν =1,0 and 1/4. The procedure involves the use of an automatic integration scheme of modified FFT used for evaluating Fourier integrals and product type integration, and the modified W-transformation used for computing oscillatory infinite integrals.

Key-words: - Automatic integration, infinite oscillatory integral, Bessel function, Hankel transform, modified W-transformation, Chebyshev expansion, FFT.

1 Introduction

Let the real functions $K(x)$ and $L(x)$ be such that

$$M(x) = K(x) + iL(x) = e^{ix} g(x), \qquad (1)$$

where the (in general complex) function $g(x)$ is infinitely differentiable for all large x and is nonoscillatory at infinity. In a recent paper by Hasegawa and Sidi [9] an automatic procedure for the fast and accurate computation of infinite-range integrals of the form

$$\int_a^\infty K(\omega t)f(t)dt \quad \text{or} \quad \int_a^\infty L(\omega t)f(t)dt \qquad a \geq 0,\ \omega > 0, \qquad (2)$$

was developed. Here the function $f(t)$ is real and infinitely differentiable for all large t and is nonoscillatory at infinity. This method combines the modified FFT of Hasegawa and Torii [8] and the (user-friendly) modified W-transformation(mW) of Sidi [20] with the approach of Hasegawa and Torii [6] for computing Fourier transforms.

The problem of evaluation of infinite-range oscillatory integrals of various sorts was considered earlier in the papers by Longman [11,12], Gabutt [3] and Sidi [17,19,20] whose methods can be used to compute also integrals of the form (2). Methods for computing the Hankel transform specifically have been considered by Piessens and Branders [15], Lund [14], Wieder [21] Piessens [16] and Sidi [17]. In addition, an automatic quadrature procedure for integrals of the form $\int_0^\infty e^{i\omega t} f(t)dt$ has been given by Hasegawa and Torii [6] which tries to minimize the number of function evaluations for a given required accuracy level.

In the present work we apply the method of Hasegawa and Sidi [9] to the computation of Hankel transforms

$$H_\nu[f;\omega] = \int_0^\infty tJ_\nu(\omega t)f(t)dt,$$

that form an important subclass of the class of integrals described above.

In the next sections we describe briefly the method [9] and apply it to the cases in which $\nu = 0, 1, 1/4$.

2 Description of the method

We shall restrict ourselves to the evaluation of the integral

$$Q(\omega) = \int_a^\infty K(\omega t) f(t) dt = \Re \int_a^\infty M(\omega t) f(t) dt. \quad (3)$$

by our assumption that $f(t)$ is a real function. For the functions $K(x)$ and $L(x)$ it turns out that a polynomial approximation is provided for them in a finite interval $[0,c]$ and, in the interval $[c,\infty]$ the function $g(x)$ of (1) is approximated by a polynomial in $1/x$. Normally these approximations are obtained by truncating the appropriate Chebyshev polynomial expansions. For example, for the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ of order ν, Luke [13, p.322, 342] gives the expansions:

$$J_\nu(x) = \left(\frac{31-28\nu}{24} x\right)^\nu \sum_{n=0}^\infty a_n^{(\nu)} T_{2n}\left(\tfrac{x}{8}\right), \quad |x| \le 8, \ \nu = 0, 1, 1/4,$$

and

$$J_\nu(x) + iY_\nu = e^{ix} \sqrt{\frac{2}{\pi x}}\, e^{-\frac{\pi}{4}(2\nu+1)i} \sum_{n=0}^\infty b_n^{(\nu)} T_n^*\left(\tfrac{5}{x}\right), \quad x \ge 5, \ \nu = 0, 1, 1/4,$$

where $T_k(x)$ and $T_k^*(x)$ are, respectively, the Chebyshev and shifted Chebyshev polynomials of order k.

We now subdivide the interval in (3) in the form

$$Q(\omega) = Q_1(\omega) + Q_2(\omega), \quad (4)$$

where

$$Q_1(\omega) = \begin{cases} \int_a^{c/\omega} K(\omega t) f(t) dt & \text{if } a < c/\omega, \\ 0 & \text{if } a \ge c/\omega, \end{cases} \quad (5)$$

$$Q_2(\omega) = \Re \int_d^\infty M(\omega t) f(t) dt, \quad d = max(a, c/\omega). \quad (6)$$

2.1 Computation of $Q_1(\omega)$

For the computation of $Q_1(\omega)$, we only think when $a < c/\omega$.

For the case of order $\nu = 0,1$, we use the Clenshaw-Curtis (CC) method [1] along with a modified FFT due to Hasegawa et al. [7], since the integrand $K(\omega t) f(t)$ is smooth in the interval $[a, c/\omega]$. For values of ω that are very small the interval $[a, c/\omega]$ may become too large for the integral $Q_1(\omega)$ to be evaluated at once, we could break this interval into several smaller subintervals and apply the CC method to each subinterval separately.

For the case of order $\nu = 1/4$, we note that the term x^ν ($\nu = 1/4$) included in the Chebyshev polynomial expansion of $J_\nu(x)$ has a bad behavior near the origin. Namely, in the interval $[a, \xi]$ $(a < \xi \le c/\omega)$ around zero, it will be difficult to compute the integral with an ordinary numerical method. For this reason we use Hasegawa and Torii's automatic quadrature method to the so-called product type integration [8] in the interval $[a, \xi]$, where ξ is easily defined with a and our numerical experiments. Then, because the integrand $J_\nu(\omega t) f(t)$ becomes smooth in the interval $[\xi, c/\omega]$, we can keep on with our computation as that for the order $\nu = 0,1$. Thus we could compute the $Q_1(\omega)$ efficiently.

2.2 Computation of $Q_2(\omega)$

From (1) and (6) we have

$$Q_2(\omega) = \Re \int_d^\infty e^{i\omega t} g(\omega t) f(t) dt, \quad (7)$$

where $d = max(a, c/\omega)$. We now have to evaluate this Fourier integral efficiently. The method that we use for this purpose is the mW-transformation of Sidi [20].

We start by letting

$$x_0 = \frac{\pi}{\omega}\left(\left\lfloor \frac{\omega d}{\pi} \right\rfloor + 1\right), \ x_l = x_0 + \frac{l\pi}{\omega}, \quad l = 0,1,2,\ldots.$$

Here x_0 is simply the first zero of $sin(\omega t)$ that is greater than d, and x_l is the lth zero following x_0. We next compute numerically the finite range integrals $F(x)$, where

$$F(x_l) = \int_d^{x_l} e^{i\omega t} g(\omega t) f(t) dt, \quad l = 1,2,\ldots. \quad (8)$$

We finally compute a two-dimensional array of approximations $W_n^{(j)}$ to the integral by solving the linear system

$$F(x_l) = W_n^{(j)} + \psi(x_l) \sum_{i=0}^n \frac{\beta_i}{x_l^i}, \quad j \le l \le j+n+1, \quad (9)$$

where

$$\psi(x_l) = \int_{x_l}^{x_{l+1}} e^{i\omega t} g(\omega t) f(t) dt = F(x_{l+1}) - F(x_l), \quad l = 0,1,\ldots, \quad (10)$$

and the β_i serve as the additional unknowns. In addition, we define $F(x_{-1}) = 0$ in (8). The solution of the linear system in (9) can be achieved in a very efficient manner by the W-algorithm [18].

As has been shown in [20], the sequences $\{W_n^{(j)}\}_{n=0}^\infty$ for fixed j have very favorable convergence properties. Therefore, we choose to consider only the sequences $\{W_n^{(0)}\}_{n=0}^\infty$. Furthermore, $|W_{n+1}^0 - W_n^0|$ may provide an error estimate for the approximation $W_{n+1}^{(0)}$, which is probably the best of the approximations $W_p^{(j)}$, $j + p = n + 1$. For more details the reader is referred to the original papers [18,20]. Once the best $W_p^{(j)}$

has been computed, $Q_2(\omega)$ is approximated by $\Re W_p^{(0)}$.

The problem that remains is that of computing the $\psi(x_l)$ that were defined in (10) to the level of accuracy prescribed by the user.

2.3 Computation of the finite oscillatory integrals

As mentioned in the previous section, the mW-transformation requires the sequence of the finite integrals $\psi(x_l)$, $l=-1,0,1,\ldots$, given in (10) or, equivalently, the integrals $F(x_l)$ defined by (9). The computation of the $\psi(x_l)$ can be performed accurately by an appropriate quadrature rule such as Gaussian formula. If the integrand function $f(t)$ is smooth, however, it might be more efficient to devise a quadrature method for computing several, say r, integrals $\psi(x_{s+l})$, … , $\psi(x_{s+r})$, or, the indefinite integral $\int_{x_s}^{x} e^{i\omega t} g(\omega t)f(t)dt$ where we take $x=x_{s+i}$ $(i=1,2,\ldots,r)$, at a time for an arbitrary integer s.

Indeed, for a positive integer m and a non-negative integer μ, define $s=m+\mu r$ and subdivide the integration interval $[c/\omega, x_{s+l}]$ for the integral $F(x_{s+l})$, $0<l\le r$, into $\mu+1$ subintervals K_q $(q=-1,0,1,\ldots,\mu-1)$, and extra one $(x_s, x_{s+l}]$ as follows:

$$[c/\omega, x_{s+l}] = \left(\bigcup_{q=-1}^{\mu-1} K_q\right) \cup \left(x_s, x_{s+l}\right],$$

where we take $K_q=(x_{m+qr}, x_{m+qr+r}]$ $(q=0,1,\ldots)$, and in particular $K_{-1}=[c/\omega, x_m]$; the appropriate values of m and r are determined later. Then, we have

$$F(x_{s+l}) = \sum_{q=-1}^{\mu-1} F(K_q) + \int_{x_s}^{x_{s+l}} e^{i\omega t} g(\omega t)f(t)dt,$$
$$1\le l\le r,\ s=m+\mu r,\ \mu=0,1,\ldots, \qquad (11)$$

where $F(K_q)$ is defined by

$$F(K_q) = \int_{t\in K_q} e^{i\omega t} g(\omega t)f(t)dt = \int_{x_{m+qr}}^{x_{m+qr+r}} e^{i\omega t} g(\omega t)f(t)dt.$$

The knowledge of the indefinite integrals $\int_{x_s}^{x} e^{i\omega t} g(\omega t)f(t)dt$, where $s=m+\mu r$ and $x\in K_\mu$ $(\mu=0,1,\ldots)$, could enable the efficient evaluation of each integral in the right-hand side of (11).

In the following we briefly describe how to approximate the integrals $\int_{x_s}^{x} e^{i\omega t} g(\omega t)f(t)dt$. Here, for example, we set $\alpha=x_s$ and $\beta=x_{s+r}$ for obtaining the integral on $[x_s, x_{s+l}]$, then to approximate the indefinite integral $\int_{\alpha}^{x} e^{i\omega t} g(\omega t)f(t)dt$, for $\alpha\le x\le\beta$, let $\phi:[\alpha,\beta]\to[-1,1]$ be a linear function defined by

$$\phi(t) = (2t-\beta-\alpha)/(\beta-\alpha),$$
$$\phi(\alpha)=-1,\quad \phi(\beta)=1, \qquad (12)$$

and approximate the non-oscillatory part $g(\omega t)f(t)$ in the integral $\int_{\alpha}^{x} e^{i\omega t} g(\omega t)f(t)dt$ by a sum $P_N(t)$ of the Chebyshev polynomials $T_k(t)$:

$$P_N(t) = p_N(\phi(t)) \equiv {\sum_{k=0}^{N}}' a_k^N T_k(\phi(t)),$$
$$\alpha\le t\le\beta, \qquad (13)$$

where the prime denotes the summation whose first term is halved. Then, defining $W=(\beta-\alpha)/2$ and $T=(\beta+\alpha)/2$ we have

$$\int_{\alpha}^{x} e^{i\omega t} g(\omega t)f(t)dt \sim \int_{\alpha}^{x} e^{i\omega t} P_N(t)dt$$
$$= W\,exp(i\omega T) I(\omega W, \phi(x); p_N), \qquad (14)$$

where $I(\omega, x; p)$ is defined by

$$I(\omega, x; p) = \int_{-1}^{x} e^{i\omega t} p(t)dt, \quad -1\le x\le 1. \qquad (15)$$

To evaluate the indefinite integral in the right of (14) or $I(\omega, x; p)$ given by (15) with $p(t)$ replaced by $p_N(t)$ in (13) efficiently, we used the automatic quadrature scheme for indefinite integration of oscillatory functions by the Chebyshev series expansion incorporated with modified FFT [6,7]. One can find details from these papers.

It is efficient to choose r mentioned in the head of this section to be a larger positive integer so long as $f(t)$ is a sufficiently smooth function on the interval $[\alpha,\beta]$, whence one can expect that the truncated Chebyshev series (13) converges rapidly as N increases, since $g(\omega t)$ is a smooth function, too. Several numerical experiments suggest that the near optimum choice of the integer r depends on the tolerance ε_2 for the integral $Q_2(\omega)$ (6) to minimize the total number of function evaluations required to satisfy ε_2. Let $M=[log_{10}\varepsilon_2]$, then in view of the observation that the mW-transformation converges so rapidly for slowly convergent integrals that $M+2$ finite integrals $\psi(x_i), -1\le i\le M$, might be sufficient to achieve the accuracy ε_2, we determine empirically for (11) that $m=2$ and $r=3+0.7M$. It remains an open problem to determine the optimum values of m and r depending on the required accuracy ε_2 and the class of the given function $f(t)$.

3 Chebyshev series expansion and the truncation errors

We here describe how to construct the sequence of the Chebyshev interpolation polynomials $\{p_N\}$ using a modified FFT [7] that is efficiently used in CC method and the automatic quadrature method to the so-called product type [8] for evaluating the integral $Q_1(\omega)$ (see section 2.1), and used in approximating the indefinite integral of oscillatory functions [6] to evaluate $I(\omega,x;f)$ (15) for computing the integrals $Q_2(\omega)$ (5) (see section 2.3) to the required accuracy. It suffices to consider only, for example the indefinite integral $I(\omega,x;f)$ (15), only on the interval $[-1,1]$ since an arbitrary finite interval can be easily transformed into $[-1,1]$ by a linear function such as $\phi(t)$ (12).

Here and henceforth we assume that N is a power of 2, $2^n (n=2,3,...)$, unless otherwise stated. Let $t_j^N = cos(\pi j/N)\ (0 \le j \le N)$ be the zeros of the polynomial $w_{N+1}(t)$ defined by

$$\begin{aligned} w_{N+1}(t) &= T_{N+1}(t) - T_{N-1}(t) \\ &= 2(t^2-1)U_{N-1}(t), \end{aligned} \tag{16}$$

where $U_{N-1}(t)$ denotes the Chebyshev polynomial of the second kind defined by $U_{N-1}(t) = sin(N\theta)/sin\theta$, $t = cos\theta$. Then, the coefficients a_k^N of $p_N(t)$ (13) are determined [1] so that $p_N(t)$ interpolates $f(t)$ at the abscissae t_j^N, and consequently a_k^N is represented in the form:

$$a_k^N = \frac{2}{N}\sum_{j=0}^{N}{}'' f\left(cos\frac{\pi j}{N}\right) cos\frac{\pi kj}{N}, \quad 0 \le k \le N. \tag{17}$$

We know that the right-hand side of (17) can be efficiently computed by means of the FFT for real data [4].

Actually an automatic quadrature of non-adaptive type is generally constructed from the sequence of the approximations $\{I_N\}$, where $I_N \equiv I(\omega,x;p_N)$, converging to the integral, having an adequate method of error estimation, until a stopping criterion is satisfied. It is an usual and simple way to double the degree N of $p_N(t)$ (13) for generating the sequence $\{I_N\}$. In order to make an automatic quadrature efficient, however, it is advantageous to have more chances of checking the stopping criterion than doubling N. Hasegawa et al. [7] showed an iterative procedure for computing the sequence of the truncated Chebyshev series, $\{p_N, p_{5N/4}, p_{3N/2}\}$, $N=2^n$, $n=2,3,...$, until a stopping criterion described in section 4 is satisfied.

For integer $\sigma = 2$ and 4, let $\{v_j^{N/\sigma}\}$ $(0 \le j \le N/\sigma)$ be a subset of the zeros of $T_N(t)$, in particular, be chosen to agree with a set consisting of the N/σ zeros of $T_{N/\sigma} - cos 3\pi/(2\sigma)$. Then, represent the polynomials $p_{N+N/\sigma}(t)$ ($\sigma = 2,4$) interpolating $f(t)$ at the node $\{v_j^{N/\sigma}\}$, $0 \le j \le N/\sigma$ $(\sigma = 2,4)$, as well as at the zeros of $w_{N+1}(t)$ (16) in the Newton form:

$$\begin{aligned} p_{N+N/\sigma}(t) - p_N(t) &= -w_{n+1}(t)\sum_{k=1}^{N/\sigma} B_k^{N/\sigma} U_{k-1}(t) \\ &= \sum_{k=1}^{N/\sigma} B_k^{N/\sigma}\{T_{N-k}(t) - T_{N+k}(t)\} \end{aligned} \tag{18}$$

The coefficients $\{B_k^{N/\sigma}\}$ are determined to satisfy the condition

$$f\left(v_j^{N/\sigma}\right) = p_{N+N/\sigma}\left(v_j^{N/\sigma}\right), \quad 0 \le j \le N/\sigma,\ \sigma = 2,4,$$

and the FFT[7] is used for efficiently evaluating the coefficients $B_k^{N/\sigma}$. Thus we can construct the Chebyshev series expansion more moderately as follows:

$$\{p_N, p_{5N/4}, p_{3N/2}\}, \quad N = 2^N.$$

Let $f(z)$ be a meromorphic function inside and on an ellipse in the complex plane. Then we can find in [11] that the estimate R_N of the truncation error of integrals such as $I(\omega,x;p_N)$ (15) and $R_{N+N/\sigma}$ of the truncation errors of integrals such as $I(\omega,x;p_{N+N/\sigma})$ $(\sigma = 2,4)$ as follows

$$R_N = \Omega_k^N(x)\frac{(|a_N^N|/2)r}{(r-1)^2}, \tag{19}$$

$$R_{N+N/\sigma} = 2\Omega_k^N(x)\frac{\left(1+|cos(3\pi/(2\sigma))|\right)|B_{N/\sigma}^{N/\sigma}|r}{(r-1)^2}, \tag{20}$$

where $\Omega_k^N(x)$ is defined by

$$\Omega_k^N(x) = \int_{-1}^{x} e^{i\omega t} w_{N+1}(t)T_k(t)dt, \tag{21}$$

and further $\Omega_k^N(x)$ can be bounded by $|\Omega_k^N(x)| \le 4$, independently of N, k, ω and x for $|x| \le 1$. The constant r may be estimated from the asymptotic behavior of a_k^N [1].

The relations (19) and (20) indicate that the errors are estimated independently of the value of ω. Thus, the errors for the quadrature rules $|I(0,1;p_{N+mN/4})|$ $(m=0,1,2)$, to the non-oscillatory integral $I(0,1;f) = \int_{-1}^{1} f(t)dt$ can also be estimated by (19) and (20), respectively. In the next section we will make use of the error estimations (19) and (20) to derive the stopping criterion in the automatic quadrature for $Q(\omega)$.

4 Stopping criterion

The efficiency of an automatic quadrature scheme depends on the adequate stopping criterion based on the error estimates as well as on the use of appropriate quadrature rules.

We remember that the integral $Q(\omega)$ (4) is divided into the two integrals $Q_1(\omega)$ on $[a, c/\omega]$ and $Q_2(\omega)$ on $[c/\omega, \infty]$. We want to approximate both integrals to assigned tolerances ε_1 and ε_2, respectively, so as to attain the overall accuracy $\varepsilon = \varepsilon_1 + \varepsilon_2$ for $Q(\omega)$ by using the CC method and its extension described in sections 2.2, 2.3 and 3, with as small number of function evaluations as possible. Now, we have to determine the adequate values of ε_1 and ε_2 for the integrals $Q_1(\omega)$ and $Q_2(\omega)$, respectively. The result of numerical experiments suggests to choose $\varepsilon_1 = \varepsilon/20$ and

$\varepsilon_2 = 19\varepsilon/20$, see [2] and [10, p.173] for a detailed discussion on a more general topic, the software interface problem.

Further, we have seen that the infinite integral $Q_2(\omega)$ (6) can be efficiently approximated by using the approximations to the finite integrals $F(x_i)\ (i = -1,0,1,...)$ (8) or $F(x_{s+l})$ in (11) along with the mW-transformation. The next question is how to assign the tolerance to each $F(x_q)\ (q = -1,0,1,...)$ in (11) on the interval K_q. It may in general be difficult to know at the outset how many integrals $F(K_q)\ (q = -1,0,1,...)$ are required in the mW-transformation to attain the assigned accuracy ε_2 for $Q_2(\omega)$. Numerical experiments, however, suggest that since the mW-transformation can transform a large class of convergent infinite oscillatory integrals into very quickly convergent ones, two or (at most) three intervals $K_q\ \ (q = -1,0,\text{or}\ 1)$ are sufficient to obtain the tolerance ε_2.

From the observation above we empirically determine the tolerance to each integral $F(K_q)$ on the $K_q\ (q = -1,0,1,...)$ as follows. Assume that $f(x)$ in $Q_2(\omega)$ is a smooth function of slow convergence at infinity, and that only three intervals $K_q\ (q = -1,0,1)$ are enough. Then we assign the tolerance $\varepsilon_2/3$ to each integral $F(K_q)$ $(q = -1,0,1)$.

5 Numerical examples

Here we compute the following integrals [5, pp. 667, 681 and 707] of $J_0(\omega x), J_1(\omega x)$ and $J_{1/4}(\omega x)$, having a parameter a for a variety of ω-value to illustrate the performance of the present automatic quadrature,

$$\int_0^\infty J_0(\omega x)\frac{x}{(x^2+a^2)^{1/2}}dx = \frac{e^{-a\omega}}{\omega}, a = 1, 1/8, \quad \text{(A)}$$

$$\int_0^\infty J_1(\omega x)\frac{x^2}{(x^2+a^2)^{3/2}}dx = e^{-a\omega}, a = 1, 1/8, \quad \text{(B)}$$

Table 1: Performances of the present quadrature scheme for the integrals $\int_0^\infty J_\nu(\omega t) f(t)dt\ (\nu = 0,1,1/4)$

Int.	a	ω	Exact integral	$\varepsilon_a = 10^{-6}$ N	Error	M	$\varepsilon_a = 10^{-12}$ N	Error	M
(A)	1	1	$3.67879441171442\times10^{-1}$	37	7×10^{-8}	7	87	1×10^{-13}	11
		5	$1.34758939981709\times10^{-3}$	39	2×10^{-8}	7	71	5×10^{-15}	12
		9	$1.37122004540755\times10^{-5}$	33	1×10^{-8}	7	59	2×10^{-16}	13
	1/8	1	$8.82496902584596\times10^{-1}$	83	1×10^{-9}	7	171	4×10^{-14}	11
		5	$1.07052285703798\times10^{-1}$	51	1×10^{-8}	6	83	5×10^{-14}	10
		9	$3.60724963731500\times10^{-2}$	35	2×10^{-8}	6	83	7×10^{-14}	10
(B)	1	1	$3.67879441171442\times10^{-1}$	55	3×10^{-8}	6	95	2×10^{-15}	11
		5	$6.73794699908547\times10^{-3}$	39	2×10^{-8}	7	71	1×10^{-14}	12
		9	$1.23409804086680\times10^{-4}$	37	5×10^{-8}	7	67	2×10^{-14}	13
	1/8	1	$8.82496902584596\times10^{-1}$	89	2×10^{-8}	6	215	1×10^{-15}	11
		5	$5.35261428518990\times10^{-1}$	57	3×10^{-8}	6	99	1×10^{-13}	11
		9	$3.24652467358350\times10^{-1}$	47	3×10^{-8}	6	87	4×10^{-14}	11
(C)	2	1	$5.09472479361313\times10^{-1}$	50	5×10^{-9}	6	96	2×10^{-14}	11
		4	$1.26039144211853\times10^{-1}$	42	6×10^{-8}	6	76	2×10^{-14}	12
		16	$3.12644007425545\times10^{-2}$	35	3×10^{-8}	6	62	2×10^{-14}	14
	1/8	1	2.05862733674139	82	1×10^{-8}	6	152	1×10^{-14}	11
		4	1.29926010861454	66	2×10^{-8}	6	116	1×10^{-14}	11
		16	$5.09472479361313\times10^{-1}$	47	5×10^{-9}	6	95	2×10^{-14}	11
(D)	2	1	$3.11726890645208\times10^{-1}$	46	8×10^{-12}	4	96	5×10^{-15}	6
		4	$1.98261365091314\times10^{-1}$	46	8×10^{-10}	5	80	1×10^{-15}	8
		16	$6.01141513214857\times10^{-2}$	35	1×10^{-8}	5	49	2×10^{-14}	9
	1/8	1	$9.61826421143769\times10^{-1}$	46	5×10^{-9}	6	68	2×10^{-14}	10
		4	$2.47933767975759\times10^{-1}$	42	7×10^{-9}	6	56	2×10^{-14}	10
		16	$6.23761465186444\times10^{-2}$	35	7×10^{-9}	6	39	1×10^{-14}	10

$$\int_0^\infty J_{1/4}(\omega x)\frac{1}{(x^2+a^2)^{1/2}}dx=\frac{e^{-a\omega}}{\omega},$$
$$a=2,1/8, \qquad \text{(C)}$$

$$\int_0^\infty J_{1/4}(\omega x)exp(-ax)=\frac{\omega^{-1/4}(\sqrt{a^2+\omega^2})^{1/4}}{\sqrt{a^2+\omega^2}},$$
$$a=2,1/8, \qquad \text{(D)}$$

In the Table 1 we show the numbers N of function evaluations, required to achieve the requested accuracies, $\varepsilon_a=10^{-6}$ and 10^{-12}, and the actual errors, for the integrals (A)-(D) of $J_\nu(\omega x)$ $(\nu=0,1,1/4)$. The numbers of the integrals $\psi(x_l)$ (10) on the half periods of the oscillation in the interval $[5/\omega,\infty]$, used in the mW-transformation due to Sidi, are also listed in the columns headed "M". The Table 1 experimentally verify the note given in section 2.3, i.e., the mW-transformation converges so rapidly that $[log_{10}\varepsilon_2]+2$ integrals $\psi(x_l)$ are sufficient to obtain the required accuracy $\varepsilon_a(=20\varepsilon_2/19)$.

To the best of our knowledge, there is no automatic quadrature scheme that treats the integrals as the one described in this paper and we also have shown that the mW-transformation is a very efficient and user-friendly method for coping with oscillatory infinite range integrals mentioned here.

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