Convergence of Weighted Sums of Random Variables *

M. Amini

Department of Mathematics,
Faculty of Sciences
Sistan and Baluchestan University,
Zahedan, Iran
Amini@hamoon.usb.ac.ir, Fax:05412446565

A. Bozorgnia

Department of Statistics
Faculty of Mathematical Sciences,
Ferdowsi University,
Mashhad, Iran
Bozorg@math.um.ac.ir, Fax:05118417749

Abstract: In this paper we study convergence almost sure of weighted sums $\sum_{j=1}^{n} a_{nj} X_j$*, where* $\{X_n,\ n \geq 1\}$ is a sequence of random variables and a_{nk} is an array of real numbers under *conditions on* a_{nj} *and sequence* $\{X_n, n \geq 1\}$.

Key Words: Negative Dependent, Strong Law of Large Numbers, Weighted Sums.

1. Introduction.

Convergence theorems for weighted sums have been obtained by Chow [5],Chow and Lai [6],Hanson and Koopman [7],Pruitt [8],Stout [10], Teicher [12] and Choi and Sung [4]. Several authors extended these theorem to sum of negatively dependent random variable,see for example Bozorgnia A., Patterson,R.F. and Taylor, R.L.[3],Taylor R.L.[11] and Amini M.,Azarnoosh H.A.and Bozorgnia A.[2]. In this paper, we present various condition on $\{a_{nj}\}$ and $\{X_n\}$ for which $\sum_{j=1}^{n} a_{nj} X_j$ converges almost sure. It is show that for stochastically bounded random variables the condition $E|X|^2 < \infty$ and $\sum_{j=1}^{n} a_{nj}^2 = O(n^{-\beta})$, $\beta > 1$, imply $\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0$, almost surely and for ND random variables with $EX_n = 0$, and $\sup|X_n| \leq c$ for some constant c and various condition on $\{a_{nj}\}$, we have $\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0$, almost surely.

Definition 1.The random variables $X_1, \cdots, X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^{n} (X_j \leq x_j)] \leq \prod_{j=1}^{n} P[X_j \leq x_j], \tag{1}$$

and

$$P[\bigcap_{j=1}^{n} (X_j > x_j)] \leq \prod_{j=1}^{n} P[X_j > x_j], \tag{2}$$

for all $x_1, \cdots, x_n \in R$. An infinite sequence $\{X_n,\ n \geq 1\}$ is said to be ND if every finite subset $X_1, \cdots, X_n$ is ND. The conditions (1) and (2) are equivalent for $n = 2$, but these do not agree for $n \geq 3$ (see Bozorgnia, [2]).

Definition 2. A sequence $\{X_n,\ n \geq 1\}$ of random variable is said to be stochastically bounded by random variable X if for all n and every real number $t > 0$,

$$P(|X_n| > t) \leq P(|X| > t).$$

Lemma 1(Serfling [8]) Let X be a r.v. with $E(X) = \mu$. If $P[a \leq X \leq b] = 1$. Then for every real number $t > 0$,

$$Ee^{t(X-\mu)} \leq e^{\frac{t^2(b-a)^2}{8}},$$

and

$$Ee^{t|X-\mu|} \leq 2e^{\frac{t^2(b-a)^2}{8}}.$$

Lemma 2 Let X be a r.v. with $E(X) = 0$. If $|X| \leq c$, W.P.1. for some constant $c < \infty$. Then for every real number t,

$$Ee^{tX} \leq \exp\{\frac{c^2 t^2}{2} \exp[ct]\}.$$

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Proof. From the following inequality

$$e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}, \quad \forall \ x \in R,$$

for every $t \geq 0$, we have

$$Ee^{tX} \leq 1 + E\{\frac{1}{2}t^2 X^2 e^{t|X|}\} \leq$$

$$1 + \frac{c^2t^2}{2}\exp[ct] \leq \exp\{\frac{c^2t^2}{2}\exp[ct]\}. \qquad \square$$

The next two lemmas will be needed in the proof of SLLN's, in the next section.

Lemma 3. (Bozorgnia, [2]) Let $X_1, \cdots, X_n$ be ND random variables and $f_1 \cdots, f_n$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing) , then $f_1(X_1), \cdots, f_n(X_n)$ are ND random variables.

Lemma 4. (Bozorgnia, [2]) Let $X_1, \cdots, X_n$ be ND nonnegative random variables, then

$$E[\prod_{j=1}^{n} X_j] \leq \prod_{j=1}^{n} E[X_j].$$

2. Weighted sums of stochastically bounded r.v.'s

In this section, we prove two theorems for weighted sums of stochastically bounded random variables under the condition on a_{nj}.

Theorem 1. Let $\{X_n, \ n \geq 1\}$ be a sequence of stochastically bounded by random variable X, with $EX^2 < \infty$ and let $\{a_{nj}, \ 1 \leq j \leq n\}$, for each $n \geq 1$ be a triangular array of real numbers with $\sum_{j=1}^{n} a_{nj}^2 = O(n^{-\beta}), \quad \beta > 1$. Then

$$\sum_{j=1}^{n} a_{nj}X_j \longrightarrow 0 \qquad a.s.$$

Proof. From Cauchy Schwartz inequality we have

$$|\sum_{j=1}^{n} a_{nj}X_j|^2 \leq (\sum_{j=1}^{n} a_{nj}^2)(\sum_{j=1}^{n} X_j^2) \leq cn^{-\beta}\sum_{j=1}^{n} X_j^2.$$

For fixed $n \geq 1$, choose an integer k such that $2^{k-1} \leq n < 2^k$. Then

$$\limsup |\sum_{j=1}^{n} a_{nj}X_j|^2 \leq c \limsup \frac{1}{(2^{k-1})^{\beta}} \sum_{j=1}^{2^k} X_j^2, \qquad (3)$$

we have

$$\sum_{k=1}^{\infty} P((2^{k-1})^{-\beta}\sum_{j=1}^{2^k} X_j^2 > \varepsilon) \leq \frac{1}{\varepsilon}\sum_{k=1}^{\infty}(2^{k-1})^{-\beta}\sum_{j=1}^{2^k} EX_j^2$$

$$= \frac{1}{\varepsilon}\sum_{j=1}^{\infty} EX_j^2 \sum_{\{k:2^k \geq j\}}^{\infty} 2^{-k\beta} = O(1)\sum_{j=1}^{\infty}\frac{EX_j^2}{j^{\beta}} <$$

$$O(1)\sum_{j=1}^{\infty}\frac{EX^2}{j^{\beta}} < \infty.$$

Thus by Borel Cantelli lemma we have

$$\limsup_k 2^{-\beta(k-1)}\sum_{j=1}^{2^k} X_j^2 = 0. \quad W.P.1$$

and by inequality (3)

$$\limsup_n |\sum_{j=1}^{n} a_{nj}X_j| = 0. \quad W.P.1$$

this complete the proof. $\square$

Theorem 2. Let $\{X_n, \ n \geq 1\}$ be a sequence of stochastically bounded independent random variables with $EX_n = 0$ and $EX^2 < \infty$. Let $\{a_{nj}, 1 \leq j \leq n, n \geq 1\}$ be a triangular array of real numbers with

$$\sum_{j=1}^{n} |a_{nj} - a_{n(j+1)}| = O(n^{-\beta}), \quad \beta > 1,$$

where $a_{n(n+1)} = 0$. Then

$$\sum_{j=1}^{n} a_{nj}X_j \longrightarrow 0, \qquad a.s.$$

Proof. By the Following inequality

$$|\sum_{j=1}^{n} a_{nj}X_j| \leq \max_{1 \leq i \leq n}|\sum_{j=1}^{i} X_j|(\sum_{j=1}^{n}|a_{nj} - a_{n(j+1)}|)$$

$$\leq cn^{-\beta}\max_{1 \leq i \leq n}|\sum_{j=1}^{i} X_j|, \qquad (4)$$

and the Kolmogorov maximal inequality for any $\varepsilon > 0$, we obtain

$$\sum_{n=1}^{\infty} P(n^{-\beta} \max_{1\le i\le n} |S_i| > \varepsilon) \le \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} n^{-2\beta} \sum_{j=1}^{n} EX_j^2$$

$$= \frac{1}{\varepsilon^2} \sum_{j=1}^{\infty} EX_j^2 \sum_{n=j}^{\infty} n^{-2\beta} = O(1) \sum_{j=1}^{\infty} \frac{EX_j^2}{j^{2\beta-1}}$$

$$< O(1) \sum_{j=1}^{\infty} \frac{EX^2}{j^{2\beta-1}} < \infty.$$

Thus by (4) we have

$$\sum_{n=1}^{\infty} P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \le$$

$$\sum_{n=1}^{\infty} P(n^{-\beta} \max_{1\le i\le n} |S_n| > \varepsilon) < \infty,$$

hence by Borel Cantelli lemma we have

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s. \quad \square$$

3. Weighted sums negatively dependent r.v.'s

In this section, we obtain some strong limit theorems for weighted sums $\sum_{j=1}^{n} a_{nj} X_j$ where $\{X_n,\ n \ge 1\}$ is a sequence of negative dependent random variables and a_{nk} is a triangular array of real numbers under the conditions on X_n and a_{nj}.

Theorem 3. Let $\{X_n,\ n \ge 1\}$ be a sequence of ND random variables with $EX_j = 0$ and $\sup |X_n| \le c$, for some finite constant c. Let $\{a_{nj},\ 1 \le j \le n,\ n \ge 1\}$ be a triangular array of real number with $\max |a_{nj}| = O(n^{-1})$. Then

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s.$$

Proof. By Lemmas 2, 3 and 4 we have

$$E \exp\{t \sum_{j=1}^{n} a_{nj} X_j\} \le \prod_{j=1}^{n} E \exp\{t a_{nj} X_j\}$$

$$\le \exp\{\frac{c^2 t^2}{n} \exp[\frac{ct}{n}]\}.$$

By Markove inequality and for $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \le$$

$$\sum_{n=1}^{\infty} e^{-t\varepsilon} E \exp\{t \sum_{j=1}^{n} a_{nj} X_j\} \le$$

$$\sum_{n=1}^{\infty} \exp\{-t\varepsilon + \frac{c^2 t^2}{n} \exp[\frac{ct}{n}]\}$$

Thus by putting $t = \frac{2}{\varepsilon} \ln(n)$, we have

$$\sum_{n=1}^{\infty} P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) =$$

$$O(1) \sum_{n=1}^{\infty} \exp\{-2 \ln(n)\} < \infty,$$

and Borel Cantelli lemma complete the proof. $\square$

Corollary 1. (Choi and Sung, [3]) Let $\{X_n,\ n \ge 1\}$ be a sequence of independent random variables with $EX_j = 0$ and $\sup |X_n| \le c$, for some finite constant c.
Let $\{a_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of real number with $\max |a_{nj}| = O(n^{-1})$. Then

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s.$$

Theorem 4 Let $\{X_n,\ n \ge 1\}$ be a sequence of ND random variables with $EX_j = 0$, and $|X_n| \le c$, W.P.1. for some finite constants. Let $\{a_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of real number with $\max |a_{nj}| = O(n^{-\beta})$ for every $\beta > (1/2)$. Then

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s.$$

Proof. By Markove inequality,and Lemmas 1,3 and 4 for every $\varepsilon > 0$ and $t > 0$, we have

$$P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \le e^{-t\varepsilon} E \exp\{t |\sum_{j=1}^{n} a_{nj} X_j|\}$$

$$\le 2 \exp\{-t\varepsilon + \frac{t^2 c^2}{8} \sum_{j=1}^{n} a_{nj}^2\}.$$

By putting $t = \frac{4\varepsilon}{c^2 \sum_{j=1}^{n} a_{nj}^2}$,for some $0 < K < \infty$,

$$P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \leq 2\exp\{-\frac{2\varepsilon^2}{c^2 \sum_{j=1}^{n} a_{nj}^2}\} \leq$$

$$2\exp\{-K^2 n^{2\beta-1}\},$$

hence

$$\sum_{n=1}^{\infty} P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \leq 2\sum_{n=1}^{\infty} \exp\{-K^2 n^{2\beta-1}\} < \infty.$$

Now Borel Cantelli Lemma complete the proof. □.

Theorem 5. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND random variables with $EX_j = 0$ and $\sigma_j^2 = EX_j^2 < \infty$. Let

$$c_n = \max\{esssup(\frac{|X_j|}{B_n}),\ 1 \leq j \leq n\}, \quad B_n^2 = \sum_{j=1}^{n} \sigma_j^2.$$

If $\sum_{n=1}^{\infty} \exp\{-\frac{2\varepsilon^2}{c_n^2 B_n^2 \sum_{j=1}^{n} a_{nj}^2}\} < \infty$, for every $\varepsilon > 0$, then

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s.$$

Proof. By Markove inequality,Lemma 1, 3 and 4 for every $\varepsilon > 0$, and $t > 0$,

$$P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) = P(\frac{|\sum_{j=1}^{n} a_{nj} X_j|}{B_n} > \frac{\varepsilon}{B_n})$$

$$\leq \exp\{\frac{-t\varepsilon}{B_n}\} E\exp\{t\frac{|\sum_{j=1}^{n} a_{nj} X_j|}{B_n}\} \leq$$

$$2\exp\{-\frac{t\varepsilon}{B_n} + \frac{t^2 c_n^2}{8}\sum_{j=1}^{n} a_{nj}^2\},$$

for $t = \frac{4\varepsilon}{c_n^2 \sum_{j=1}^{n} a_{nj}^2}$, we have

$$\sum_{n=1}^{\infty} P(|\sum_{j=1}^{n} a_{nj} X_j| > \varepsilon) \leq$$

$$\sum_{n=1}^{\infty} 2\exp\{-\frac{2\varepsilon^2}{c_n^2 B_n^2 \sum_{j=1}^{n} a_{nj}^2}\} < \infty,$$

now Borel Cantelli Lemma complete the proof. □

Corollary 2.If $\max|a_{nj}| = O(n^{-\beta}(c_n B_n)^{-1})$ for some $\beta > (1/2)$, then

$$\sum_{j=1}^{n} a_{nj} X_j \longrightarrow 0, \quad a.s. \quad \square$$

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