# Spectral Theory for Systems of Matrices


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*Abstract:* - The *joint spectrum* of a $d$-tuple $\mathbf{A}$ of $(n\times n)$ matrices $A_1,\ldots, A_d$ with real eigenvalues is interpreted to be the smallest set $\gamma(\mathbf{A})$ in $\mathbf{R}^d$ about which a suitably smooth function $f$ must be defined in order to form a function $f(A_1,\ldots, A_d)$ of the matrices $A_1,\ldots, A_d$. The paper makes this idea more precise and connects the joint spectrum with the support of a symmetric hyperbolic system of PDE. A geometric characterisation of the joint spectrum for two hermitian matrices due to Bazer and Yen is given using the methods of Clifford analysis.



*Key-Words:* - eigenvalue, joint spectrum, functional calculus, symmetric hyperbolic system, Clifford analysis


## 1 Introduction

The problem of forming functions of matrices arises naturally in the analysis of the equations of mathematical physics. The solution of the linear system $y'=Ay$, $y(0) = y_0$ of ordinary differential equations can be expressed as $y(t) = e^{At}y_0$ for $t$  0 so that we have to form the exponential $e^{At}$ of the matrix $A$. This is done by working out the eigenvalues of $A$ and putting it in its Jordan canonical form. Other functions $f(A)$ of the matrix $A$ are found by appealing to the Cauchy-Riesz formula

(1) $$f(A)=\frac{1}{2\pi i}\int_C(\lambda I-A)^{-1}f(\lambda)\,d\lambda$$

for functions $f$ that are analytic in a neighbourhood of the collection $\sigma(A)$ of eigenvalues of $A$ and $C$ being a simple closed contour surrounding $\sigma(A)$.

There are two ways of looking at the eigenvalues of a matrix $A$. One is the usual algebraic definition: $\lambda$ is an eigenvalue if there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. The other view is more analytic and so is more closely attuned to formula (1): $\lambda$ is an eigenvalue if it is a singularity of the matrix valued resolvent function $x\mapsto(xI-A)^{-1}$. It is the latter approach that needs to be considered when $A$ is a linear operator acting on a function space rather than a matrix: the set $\sigma(A)$ of such singularities is known as the *spectrum* of $A$. In the infinite dimensional case we need to distinguish between the point spectrum and the continuous part of the spectrum.

Similar considerations apply to solving the symmetric hyperbolic system

(2) $$\frac{\partial u}{\partial t}+A_1\frac{\partial u}{\partial x_1}+\cdots+A_d\frac{\partial u}{\partial x_d}=0,\qquad u(\cdot,0)=u_0.$$

Here $u$ is a vector valued function of $(x,t)\in \mathbf{R}^d\times\mathbf{R}_+$ with values in $\mathbf{R}^n$ and $A_1,\ldots, A_d$ are $(n\times n)$ matrices. Systems of equations of this form arise from the linearised equations of magnetohydrodynamics.

Under suitable assumptions on the matrices (for example, they are hermitian), the fundamental solution of (2) at time $t = 1$ is a matrix valued distribution $W_{\mathbf{A}}=\left(e^{i(x_1A_1+\cdots+x_dA_d)}\right)^{\wedge}$.

Here the Fourier transform $(\cdot)^{\wedge}$ is taken with respect to the variables $x\in\mathbf{R}^d$ in the sense of L. Schwartz's distribution theory. It turns out that the matrix valued distribution $W_{\mathbf{A}}$ has compact support $K$ in $\mathbf{R}^d$ and we can think of the mapping $f\mapsto W_{\mathbf{A}}(f)$ as a *functional calculus* like the Cauchy-Riesz calculus defined by formula (1), except that now we only require $f$ to be smooth in a neighbourhood of the support $K$ of $W_{\mathbf{A}}$. More suggestively, the matrix $W_{\mathbf{A}}(f)$ can be written as $f(A_1,\ldots,A_d)$.

In forming the function $f(A_1,\ldots,A_d)$ of the matrices $A_1,\ldots,A_d$, it follows from the way we have defined it that all possible matrix products are equally weighted, for we have allowed the possibility that the matrices $A_1,\ldots,A_d$ do not commute with each other. The set $K$ serves as a type of *joint spectrum* of the matrices $A_1,\ldots,A_d$ in a similar way that the spectrum $\sigma(A)$ of a single matrix $A$ is associated with the Cauchy-Riesz functional calculus by formula (1).

Knowledge of the support $K$ of $W_{\mathbf{A}}$ is obviously important, because the cone in $\mathbf{R}^d\times\mathbf{R}_+$ through $K\times\{1\}$ describes how the solution of (2) propagates with time.

In this paper, I will describe how information about $K$ can be obtained by identifying the singularities of a function with values in an algebra over the matrices, just like we can identify $\sigma(A)$ with the set of singularities of the resolvent of $A$. This will lead us to the main result, which gives a purely algebraic characterisation of $K$ in the case $d = 2$. The characterisation is, roughly speaking, that the complement of the compact set $K$ is the set of all complex numbers $\boldsymbol{x}$ such that all eigenvalues of the matrix $(\bar{y}I-A^*)(\boldsymbol{y}I-A)^{-1}$ have modulus one for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$. Here we have identified $\mathbf{R}^2$ with $\mathbf{C}$ and we have set $A = A_1+iA_2$.

The result is essentially due to Bazer and Yen [3],[4] in their study of symmetric hyperbolic systems (2). However, the connection with R. Kippenhahn's study [11] of the numerical range of matrices, essential for the complete result given in Section 5 is missing from [3],[4]. A more general theory is presented in [1],[2] but the results there are not as fine in the present special setting of symmetric *systems* of equations. One of the purposes of the present paper is to emphasise the connection with the spectral theory and numerical range of a single matrix.

Section 2 gives a brief outline of Clifford analysis—the higher dimensional analogue of complex analysis for functions with values in a Clifford algebra. In Section 3, the main ideas involved in developing a 'functional calculus' for a system of matrices or linear operators based on Clifford analysis is described. The key idea of the plane wave decomposition of the Cauchy kernel is given in Section 4, where attention is restricted to the case $d = 2$. The main result characterising the *joint spectrum* $g(\mathbf{A}) := K$ of a pair $\mathbf{A} = (A_1,A_2)$ of $(n\times n)$ hermitian matrices is given in Section 5. If $A_1$ and $A_2$ actually commute, then $g(\mathbf{A})$ is just the set of eigenvalues of the matrix $A = A_1+iA_2$, otherwise $g(\mathbf{A})$ is a subset of the plane with nonempty interior.

## 2 Clifford Analysis

The key idea behind the Cauchy-Riesz formula (1) for forming functions of matrices is the Cauchy integral formula

$$(3)\qquad f(z) = \frac{1}{2pi}\int_C \frac{f(z)}{z-z}dz\,,$$

valid for all functions $f$ analytic in a neighbourhood of the simple closed contour $C$ together with its interior and for all complex numbers $z$ inside $C$. Suitably interpreted, we can simply replace the complex number $z$ by a matrix $A$ to obtain (1). Because we are working with $d$ matrices $A_1,\dots,A_d$, we would like a $d$-dimensional version of (3) so that we could simply replace the parameters $z_1,\dots,z_d$ with the matrices $A_1,\dots,A_d$.

There are many ways to generalise formula (3) to higher dimensions. One way which reflects the possibility that the matrices $A_1,\dots,A_d$ may not commute with each other is to allow our functions $f$ to take values in a noncommutative algebra: a Clifford algebra $\mathbf{C}_{(d)}$ over $d$ generators $e_1,\dots,e_d$ satisfying $e_je_k + e_ke_j = 2d_{jk}$ for $j,k = 1,\dots,d$ and with a unit $e_0$ and complex scalars. A Clifford algebra also has a natural conjugation $u \mapsto \bar{u}$ and we can think of $\mathbf{R}^{d+1}$ as being inside $\mathbf{C}_{(d)}$ by virtue of the basis vectors $e_0,e_1,\dots,e_d$ of $\mathbf{R}^{d+1}$. More details may be found in [5], but a brief review follows.

The analogue in Clifford analysis of the Cauchy-Riemann operator $\frac{¶}{¶\bar{z}}$ in complex analysis is the differential operator $D = \sum_{j=0}^{d} e_j \frac{¶}{¶x_j}$. A function $f(x_0,x_1,\dots,x_d)$ of the real variables $(x_0,x_1,\dots,x_d) \in \mathbf{R}^{d+1}$ and with values in the Clifford algebra $\mathbf{C}_{(d)}$ is called *left monogenic* if $Df = 0$. Then we obtain a higher dimensional version

$$(4)\qquad f(x) = \int_{¶\Omega} G_w(x)n(w)f(w)dm(w)\,,\quad x\in\Omega$$

of (3). Here $\Omega$ is an open subset of $\mathbf{R}^{d+1}$ such that the boundary $\partial\Omega$ of $\Omega$ is a smooth oriented $d$-dimensional manifold. The $\mathbf{C}_{(d)}$-valued function $f$ is defined and left monogenic in a neighbourhood of $\bar{\Omega}$, $n(w)$ is the outward unit normal of $\partial\Omega$ at the point $w \in \partial\Omega$ and $m$ is the volume measure of $\partial\Omega$. The *Cauchy kernel*

$$(5)\qquad G_w(x) = \frac{1}{s_d}\frac{\bar{w}-\bar{x}}{|w-x|^{d+1}}\,,\ w,x \in \mathbf{R}^{d+1},\ w \neq x$$

plays the role of $1/2\pi(z-z)^{-1}$ in the Cauchy integral formula (3). The normalisation factor $s_d$ in (5) is the volume of the unit $d$-sphere in $\mathbf{R}^{d+1}$.

## 3 The Functional Calculus

By analogy with formula (1), one would like to define

$$f(\mathbf{A}) = \int_{¶\Omega} G_w(\mathbf{A})n(w)f(w)dm(w)$$

for the $d$-tuple $\mathbf{A} = (A_1,\dots,A_d)$ of $(n\times n)$ matrices, which we suppose to have real eigenvalues. A difficulty occurs in making sense of the Cauchy kernel $w \mapsto G_w(\mathbf{A})$, a function that should be defined and monogenic for all $w$ off a nonempty closed subset $g(\mathbf{A})$ of $\mathbf{R}^d$ inside $\Omega$, and taking values in the tensor product $\mathbf{M}_n \otimes \mathbf{C}_{(d)}$ of the space $\mathbf{M}_n$ of $(n\times n)$ matrices with $\mathbf{C}_{(d)}$. Here $\mathbf{M}_n \otimes \mathbf{C}_{(d)}$ is a type of "Clifford algebra" with matrix coefficients. The function $w \mapsto G_w(\mathbf{A})$ is discontinuous as $w \in \mathbf{R}^{d+1}$ approaches points in $g(\mathbf{A})$ from above ($w_0 \to 0^+$) or below ($w_0 \to 0^-$).

In the Cauchy-Riesz functional calculus for a single matrix $A$, the set of singularities of the resolvent function $x \mapsto (xI - A)^{-1}$ is precisely the spectrum $\sigma(A)$ of $A$, so the set $g(\mathbf{A})$ may be interpreted as a higher-dimensional analogue of the spectrum of a single operator. The set $\partial\Omega$ can be smoothly varied in the region where $w \mapsto G_w(\mathbf{A})$ is monogenic.

In the special case that $d$ is odd and $A_1,\dots,A_d$ are *commuting* matrices, it turns out [14],[15] that $g(\mathbf{A})$ is the complement of all $l \in \mathbf{R}^d$ such that the matrix $\sum_{j=1}^{d}(l_j I - A_j)^2$ is invertible and

$$G_w(\mathbf{A}) = \frac{1}{s_d}(\overline{w}I + \mathbf{A})\left(w_0^2 I + \sum_{j=1}^{d}(w_j I - A_j)^2\right)^{-\frac{d+1}{2}}$$

for all $w \in \mathbf{R}^{d+1}$ off $g(\mathbf{A})$.

If $d$ is even or if the matrices $A_1,\dots,A_d$ do not commute with each other, then $G_w(\mathbf{A})$ may be defined by $W_{\mathbf{A}}(G_w)$ for all $w \in \mathbf{R}^{d+1}$ outside the support of the distribution $W_{\mathbf{A}}$ considered in the Introduction, see [8]. For $w \in \mathbf{R}^{d+1}$ with large absolute value, $G_w(\mathbf{A})$ is defined in [12], [13] using a power series expansion.

It turns out that the distribution $W_{\mathbf{A}}$ is defined provided that the eigenvalues of the matrix $x_1A_1 + \cdots + x_dA_d$ are real for all $x \in \mathbf{R}^d$ [7]. This is automatically true if the matrices $A_1,\dots,A_d$ are hermitian or are simultaneously upper triangularisable with real eigenvalues.

Although it can be shown that the support of the distribution $W_{\mathbf{A}}$ is equal to the set $g(\mathbf{A})$ of discontinuities of the Cauchy kernel $w \mapsto G_w(\mathbf{A})$ [8, Theorem 6.2], we need a way of computing the *joint spectrum* $g(\mathbf{A})$ of the matrices $\mathbf{A} = (A_1,\dots,A_d)$.

## 4 Plane Wave Decomposition

I shall concentrate now on the case $d = 2$ treated in [10] and following the general case in [9]. Then the Cauchy kernel (5) has the representation

$$G_w(x) = -\frac{\operatorname{sgn}(y_0)}{8p^2}\int_{\mathbf{T}} \left(\langle y - x, s\rangle e_0 - y_0 s I\right)^{-2} dn(s)$$

for $w = y_0e_0 + y$, $y_0 \neq 0$, $x,y \in \mathbf{R}^2$ and $n$ is arclength measure on the unit circle $\mathbf{T}$. The advantage of this representation is that we can simply replace $x \in \mathbf{R}^2$ by the pair $\mathbf{A} = (A_1,A_2)$ of matrices to obtain(6)

$$G_w(\mathbf{A}) = -\frac{\operatorname{sgn}(y_0)}{8p^2}\int_{\mathbf{T}} \left(\langle yI - \mathbf{A}, s\rangle e_0 - y_0 s I\right)^{-2} dn(s).$$

For a pair $T = (T_1,T_2)$ of $(n\times n)$ matrices, the matrix $x_1T_1 + x_2T_2$ is written as $\langle T,x\rangle$ for all $x \in \mathbf{R}^2$. The identity matrix is written as $I$ and $yI$ denotes that pair $(y_1I,y_2I)$ of matrices. The expression $\left(\langle yI - \mathbf{A}, s\rangle e_0 - y_0 sI\right)^{-2}$ is interpreted in the space $\mathbf{M}_n \otimes \mathbf{C}_{(2)}$: it exists in $\mathbf{M}_n \otimes \mathbf{C}_{(2)}$ provided that the eigenvalues of the matrix $\langle \mathbf{A}, x\rangle = x_1A_1 + x_2A_2$ are real for all $x \in \mathbf{R}^2$ [9, Lemma 2.4] and this is what is needed for the distribution $W_{\mathbf{A}}$ to exist. Then (6) equals $W_{\mathbf{A}}(G_w)$ for all $w \in \mathbf{R}^{d+1}$ outside the support of $W_{\mathbf{A}}$ [8, Theorem 6.2] and [9, Theorem 3.6].

## 5 The Joint Spectrum

Using formula (6), we are now in a position to calculate the *joint spectrum* $g(\mathbf{A})$ of a pair $\mathbf{A} = (A_1,A_2)$ of $(n\times n)$ hermitian matrices. As mentioned above, this is identical to the support of the distribution $W_{\mathbf{A}}$ and so describes the propagation of the solution of the system of linear PDE (2). The subset $g(\mathbf{A})$ of $\mathbf{R}^2$ (identified with $\mathbf{C}$) differs from the numerical range of the matrix $A = A_1 + iA_2$ by virtue of the existence gaps or *lacunas*, see the figures below. If $x \in \mathbf{R}^2$, denote $x_1 + ix_2 \in \mathbf{C}$ by $\boldsymbol{x}$.

**Theorem.** *Let* $\mathbf{A} = (A_1,A_2)$ *be a pair of* $(n\times n)$ *hermitian matrices and set* $A = A_1 + iA_2$. *Then the joint spectrum* $g(\mathbf{A})$ *of* $\mathbf{A}$ *is the complement of the set of all points* $x \in \mathbf{R}^2$ *such that for all* $y \in \mathbf{R}^2$ *in a neighbourhood of* $x$, *the matrix* $\boldsymbol{y}I - A$ *is invertible and every eigenvalue of the matrix*

(7) $\qquad (\bar{\boldsymbol{y}}I - A^*)(\boldsymbol{y}I - A)^{-1}$

*has modulus one.*

If $A_1$ and $A_2$ commute, then (7) is a unitary matrix if $\boldsymbol{y}I - A$ is invertible, so necessarily $g(\mathbf{A}) = s(A)$, the set of eigenvalues of $A$, otherwise $g(\mathbf{A})$ must have nonempty interior (if the matrix (7) has eigenvalues inside $\mathbf{T}$ for $y = x$, then the same is true for all $y$ in a neighbourhood of $x$).

The following figures for two pairs of $(3\times 3)$ hermitian matrices illustrate the lacunas mentioned above.

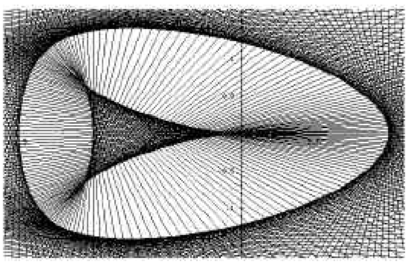

*Figure 1*

$$\mathbf{A} = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right)$$

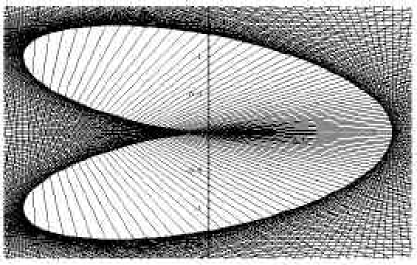

*Figure 2*

$$\mathbf{A} = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right)$$

In the lightly shaded region in each figure in which only one line passes through each point $y$, two of the eigenvalues of the $(3\times3)$ matrix (6) have modulus not equal to one. The other eigenvalue represents the normal to the line passing through $y$. Hence $g(\mathbf{A})$ is precisely the lightly shaded region. The dark region in the centre of Figure 1 is a lacuna. The convex hull of the light region in each figure is exactly the numerical range

$$W(A) = \{\langle Au,u\rangle : u \in \mathbf{C}^n,\ |u| = 1\}$$

of the $(n\times n)$ matrix $A$. Here $\langle u,v\rangle$ denotes the inner product of $u,v \in \mathbf{C}^n$.

The numerical range of the matrix $A = A_1 + iA_2$ is the convex hull of the finite union of algebraic curves studied by R. Kippenhahn [11]. These form the boundaries of the light regions in Figures 1 and 2. The numerical range of matrices is studied from the point of view of numerical analysis in [6].

The theorem above is a purely algebraic characterisation of the support of the distribution $W_{\mathbf{A}}$ and hence, of the propagation cone of the symmetric hyperbolic system (2).

*Sketch of the Clifford Analysis Proof of the Theorem*. According to the definition of $g(\mathbf{A})$ mentioned in Section 3, we need to look at the discontinuities of the Cauchy kernel $w \mapsto G_w(\mathbf{A})$ as $w_0 \to 0$. We examine the individual terms in the plane-wave formula (6) for $G_{y+y_0e_0}(\mathbf{A})$.

In the usual manner, we convert an integral with respect to arclength measure $n$ on $\mathbf{T}$ into a contour integral over $\mathbf{T}$.

First we look at the integrand $\left(\langle yI-\mathbf{A},s\rangle e_0 - y_0 sI\right)^{-2}$ of the plane-wave formula (6). It is an element of the space $\mathbf{M}_n\otimes\mathbf{C}_{(2)}$ for each $s \in \mathbf{T}$, nonzero $y_0 \in \mathbf{R}$ and $y \in \mathbf{R}$. It is equal to

$$\left(\langle yI-\mathbf{A},s\rangle + y_0 sI\right)^2\left(\langle yI-\mathbf{A},s\rangle^2 + y_0^2 I\right)^{-2}$$
$$= \left[\left(\langle yI-\mathbf{A},s\rangle^2 - y_0^2 I\right) + 2sy_0\langle yI-\mathbf{A},s\rangle\right]$$
$$\times\left(\langle yI-\mathbf{A},s\rangle^2 + y_0^2 I\right)^{-2}$$

Then the limit

$$\lim_{y_0\to0}|y_0|\int_{\mathbf{T}} s\langle yI-\mathbf{A},s\rangle\left(\langle yI-\mathbf{A},s\rangle^2 + y_0^2 I\right)^{-2} dn(s)$$

only 'sees' singularities of the integrand converging to points on $\mathbf{T}$, because of the factor $|y_0|$.

A calculation shows that the limit exists uniformly for all $y$ in compact subsets of $\mathbf{R}^2$ disjoint from the Kippenhahn curves mentioned above. The calculation appeals to the fact that the matrices $A_1$ and $A_2$ are assumed to be hermitian.

Discontinuities of $G_{y+y_0e_0}(\mathbf{A})$ are exhibited as *jumps* in the $e_0$-component of $G_{y+y_0e_0}(\mathbf{A})$ as $y_0 \to 0^+$ and $y_0 \to 0^-$. Because of the appearance of $\mathrm{sgn}(y_0)$ in formula (6), a jump occurs if

(8)
$$\lim_{y_0\to0}\int_{\mathbf{T}}\left(\langle yI-\mathbf{A},s\rangle^2 - y_0^2 I\right)\left(\langle yI-\mathbf{A},s\rangle^2 + y_0^2 I\right)^{-2} dn(s)$$

is a nonzero matrix.

If $y$ is not an eigenvalue of $A$ and there is the maximal number possible of solutions $s \in \mathbf{T}$

(9) $$\det\langle yI-\mathbf{A},s\rangle = 0,$$

corresponding to the situation where every eigenvalue of the matrix (7) has modulus one, it follows that $G_w(\mathbf{A})$ is continuous at $w = (0,y)$, so that $y$ belongs to $g(\mathbf{A})^c$.

On the other hand, if singularities of the integrand of (6) occur inside $\mathbf{T}$ in a neighbourhood of $y = x$, then analytic continuation in $y \in \mathbf{R}^2$ (embedded in $\mathbf{C}^2$) along a curve in the plane shows that (8) cannot be zero in a neighbourhood of $x$, that is, $G_{y+y_0e_0}(\mathbf{A})$ has a discontinuity at $y = x$ as $y_0 \to 0^+$ and $y_0 \to 0^-$. Consequently, $x$ belongs to $g(\mathbf{A})$.

## 5 Conclusion

The notion of the spectral theory or the *spectrum* of a system $\mathbf{A} = (A_1,...,A_d)$ of $(n\times n)$ matrices is an imprecise one, especially if the matrices $A_1,...,A_d$ do not commute with each other. The point of view of this paper is to tie these notions in with applications in mathematical physics, such as magnetohydrodynamics.

In the commuting case, at least if each of the matrices $A_1,...,A_d$ have real eigenvalues, the joint spectrum $g(\mathbf{A})$ of $\mathbf{A}$ may be thought of as the complement of the set of all $l \in \mathbf{R}^d$ where $lI-(A_1e_1+ \cdots +A_de_d)$ is invertible in the Clifford algebra $\mathbf{M}_n\otimes\mathbf{C}_{(d)}$ with matrix coefficients, and when $d = 2$, the joint spectrum $g(\mathbf{A})$ is precisely the set $s(A)$ of eigenvalues of the matrix $A = A_1+iA_2$.

In the noncommuting case, $g(\mathbf{A})$ is interpreted as the set of singularities of the Cauchy kernel $w \mapsto G_w(\mathbf{A})$, just as the eigenvalues of a single matrix $A$ are the singularities of the resolvent $l \mapsto (lI-A)^{-1}$ of $A$. In this way we obtain a geometric characterisation of $g(\mathbf{A})$ and so a description of the propagation of the symmetric hyperbolic system of PDE (2), at least in the case $d = 2$. In higher dimensions, the situation is more complicated and involves algebraic geometry, see [1], [2].

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