sub-Gaussian techniques in proving some strong limit theorems *

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Abstract: In this paper, we study some strong limit theorems for the sequence $\{\frac{1}{n^\beta}\sum_{k=1}^n X_n\}$, for each $\beta > 0$ and weighted sums $\sum_{k=1}^n a_{nk}X_k$ where $\{X_n,\ n \geq 1\}$ is a sequence of negative dependence Sub-Gaussian random variables and a_{nk} is an array of nonnegative real numbers.

Key Words: Negative Dependent,Sub-Gaussian, Strong Law of Large Numbers, Weighted Sums,Martingale.

1.Introduction

Some convergence theorems for weighted sums $\sum_{k=1}^n a_{nk}X_k$ has been studied by Chow [4] for the case where $\{X_n,\ n \geq 1\}$ is a sequence of independent, generalized Gaussian random variables. The case of m-dependent generalized Gaussian r.v.'s has been discussed by Ouy [6], and the strong law of large numbers for sequences of independent Sub-Gaussian random variables has been obtained by Taylor and Chung Hu [9]. In this paper, we extend some of these results and prove some strong limit theorems for the sequence of $\{\frac{1}{n^\beta}\sum_{k=1}^n X_n\}$, for each $\beta > 0$, and weighted sums $\sum_{k=1}^n a_{nk}X_k$ where $\{X_n, n \geq 1\}$ is a sequence of negative dependent Sub-Gaussian random variables and a_{nk} is an array of nonnegative real numbers. Also by sub-Gaussian techniques we prove that $\sum_{k=1}^\infty a_{nk}X_k$ converge with probability one for each n, where $E[X_n|\mathcal{F}_{n-1}] = 0$, $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ and $\sum_{j=k}^\infty a_{nj}^2 = O(k^{-\beta})$ for every $\beta > 0$.To prove these results we need to the following definitions, lemmas and theorems.

Definition 1. A symmetric random variable X is said to be Sub- Gaussian (SG) r.v. if there exist a nonnegative real number α such that for each real number t

$$Ee^{tX} \leq \exp[\frac{\alpha^2 t^2}{2}]. \qquad (1.1)$$

The number, $\tau(X) = \inf\{\alpha \geq 0 : E(e^{tX}) \leq \exp[\frac{\alpha^2t^2}{2}],\ t \in R\}$, will be called the Gaussian standard of the random variable X. It is evident that X will be a Sub-Gaussian random variable if and only if $\tau(X) < \infty$. Moreover

$$\tau(X) = \sup_{t\neq 0}\left[\frac{2\ln(E(e^{tX}))}{t^2}\right]^{1/2},$$

and inequality (1.1) holds for $\alpha = \tau(X)$.

Definition 2.A symmetric random variable X is strictly Sub-Gaussian if

$$E(X^2) = \tau^2(X).$$

Definition 3.The random variables $X_1, \cdots, X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^n (X_j \leq x_j)] \leq \prod_{j=1}^n P[X_j \leq x_j], \qquad (1.2)$$

and

$$P[\bigcap_{j=1}^n (X_j > x_j)] \leq \prod_{j=1}^n P[X_j > x_j], \qquad (1.3)$$

*This research supported by central research of statistics

for all $x_1, \cdots, x_n \in R$. An infinite sequence $\{X_n\ ,\ n \geq 1\}$ is said to be ND if every finite subset $X_1, \cdots, X_n$ is ND. The conditions (1.2) and (1.3) are equivalent for $n = 2$, but these do not agree for $n \geq 3$ (see [3],Pages 3-4).

The following Lemmas and Theorems which our work is based on can be found in Taylor, Chung Hu (1987),Buldygin, Kozachenko (1980) and Bozorgnia, Taylor (1996).

Theorem 1. ([10]) Let X be Sub-Gaussian random variable and a is a real number $(a \neq 0)$, then aX is sub-Gaussian random variable with $\tau(X) = |a|\tau(X)$.

Lemma 1. ([9]) If X is a Sub-Gaussian random variable with $\tau(X) \leq \alpha$, then

i) For every $t \in R$,

$$E[e^{t|X|}] \leq 2\exp[\frac{\alpha^2 t^2}{2}]. \qquad (1.4)$$

ii) For every $\varepsilon > 0$, we have

$$P[X > \varepsilon] \leq \exp[-\frac{\varepsilon^2}{2\alpha^2}] \qquad (1.5)$$

and

$$P[|X| > \varepsilon] \leq 2\exp[-\frac{\varepsilon^2}{2\alpha^2}]. \qquad (1.6)$$

Lemma 2. ([9]) If X is bounded $(|X| \leq M)$ and has zero mean $(E(X) = 0)$, then X is Sub-Gaussian random variable with $\tau(X) \leq \sqrt{2}M$.

Lemma 3.([3]) Let $X_1, \cdots, X_n$ be ND random variables and $f_1\ \cdots, f_n$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing) , then $f_1(X_1), \cdots, f_n(X_n)$ are ND random variables.

Lemmas 4.([3]) Let $X_1, \cdots, X_n$ be ND non-negative random variables. Then

$$E[\prod_{j=1}^{n} X_j] \leq \prod_{j=1}^{n} E[X_j].$$

Examples .

1) Let X has uniform distribution in (a, b) interval, then $Y = X - \frac{a+b}{2}$ is Sub-Gaussian random variable with $\tau(Y) \leq \sqrt{2}(b-a)$.

2) Let X be a continuous random variable with d.f. $F(x)$, then $Y = F(X) - \frac{1}{2}$ is Sub-Gaussian with $\tau(Y) \leq \sqrt{2}$.

3) Let X be a random variable with d.f. Normal and $E(X) = 0,\ \ Var(X) = \sigma^2$, then X is strictly Sub-Gaussian with $\tau^2(X) = \sigma^2$.

2. Strong Limit Theorems

In this section we obtain some strong limit theorems for sequence $\{\frac{1}{n^\beta}\sum_{k=1}^{n} X_k\}$ for each $\beta > 0$, where $\{X_n, n \geq 1\}$ is a sequence of negative dependent Sub-Gaussian random variables with $\tau(X_n) \leq \alpha_n$, for every $n \geq 1$, under the conditions on $\sum_{k=1}^{n} \alpha_k^2$.

Theorem 2. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND Sub-Gaussian random variables with $\tau(X_n) \leq \alpha_n$.

i) $S_n = \sum_{k=1}^{n} X_k$ is a Sub-Gaussian r.v. with $\alpha^2 = \sum_{i=1}^{n} \alpha_i^2$.

ii) If $\sum_{i=1}^{n} \alpha_i^2 = O(n^{2-\beta})$ for every $\beta > 0$, then

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = 0 \quad W.P.1.$$

iii) If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then for some $\beta > \frac{1}{2}$,

$$\lim_{n\longrightarrow\infty} \frac{1}{n^\beta}\sum_{k=1}^{n} X_k = 0 \quad W.P.1.$$

Proof.

i) By Lemmas 1,3,4 and Theorems 1 we have

$$E[e^{tS_n}] \leq \prod_{k=1}^{n} E[e^{tX_k}] \leq \exp[\frac{\alpha^2 t^2}{2}],$$

hence S_n is a Sub-Gaussian r.v. with $\alpha^2 = \sum_{i=1}^{n} \alpha_i^2$.

ii) For each $\varepsilon > 0$ by part i and Lemma (1.ii), we have

$$\sum_{n=1}^{\infty} P[|\sum_{k=1}^{n} X_k| > n\varepsilon] \leq 2\sum_{n=1}^{\infty} \exp[-\frac{\varepsilon^2 n^{(2-\beta)}}{2c}] < \infty,$$

iii) and also

$$\sum_{n=1}^{\infty} P[|\frac{1}{n^{\beta}}\sum_{k=1}^{n} X_k| > \varepsilon] \leq$$

$$2\sum_{n=1}^{\infty} \exp[-\frac{\varepsilon^2 n^{2\beta-1}}{2\alpha^2}] < \infty.$$

which these complete the proof. □.

Theorem 3. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND random variables satisfying $P[a \leq X_i \leq b] = 1$, for each i where $a < b$, then for every $\beta > \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} (X_k - E(X_k)) = 0, \qquad W.P.1.$$

Proof. Define $Y_k = X_k - E(X_k)$, $k = 1, 2, \cdots, n$, then, $E(Y_k) = 0$ and $|Y_k| \leq (b-a)$, W.P.1, hence by Lemma 2 $\{Y_k,\ k \geq 1\}$ be a sequence of Sub-Gaussian random variables with $\tau(Y_k) \leq \sqrt{2}(b-a)$. Thus by Theorem (3.iii), for every $\beta > \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} (X_k - E(X_k)) = 0, \qquad W.P.1.$$

Corollary 1. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND identically distributed random variables with $E(X_1) = 0$, $Var(X_1) = 1$ and $E[X_1^k] < \infty \quad k \geq 1$, then $\frac{S_n}{\sqrt{n}}$ is an asymptotically Sub-Gaussian random variable, when $n \longrightarrow \infty$ with $\tau(\frac{S_n}{\sqrt{n}}) \leq 1$.

Proof. For $t \in R$ we have

$$E[e^{\frac{t}{\sqrt{n}}S_n}] \leq \prod_{k=1}^{n} E[e^{\frac{t}{\sqrt{n}}X_k}] =$$

$$[1 + \frac{t^2}{2n} + \circ(\frac{1}{n})]^n \longrightarrow e^{\frac{t^2}{2}},$$

when $n \longrightarrow \infty$.

2.Some Strong Limit Theorems for weighted sums

In this section, we obtain some strong limit theorems for weighted sums $T_n = \sum_{k=1}^{\infty} a_{nk}X_k$ and $\sum_{k=1}^{n} a_{nk}X_k$, where $\{X_n,\ n \geq 1\}$ is a sequence of negative dependence Sub-Gaussian random variables and a_{nk} is an array of nonnegative real numbers.Also we prove $T_n = \sum_{k=1}^{\infty} a_{nk}X_k$ convergence W.P.1. under the condition that $E[X_n|\mathcal{F}_{n-1}] = 0$, $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ and $\sum_{j=k}^{\infty} a_{nj}^2 = O(k^{-\beta})$ for every $\beta > 0$ and $n \geq 1$.

Lemma 5. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND Sub-Gaussian random variables with $\tau(X_k) \leq \alpha$. Then

i) T_n is a Sub-Gaussian random variable with $\tau(T_n) \leq \alpha\sqrt{A_n}$ for all n.

ii) For every $\varepsilon > 0$

$$P[|T_n| > \varepsilon] \leq 2\exp[-\frac{\varepsilon^2}{2\alpha^2 A_n}].$$

Where $A_n = \sum_{k=1}^{\infty} a_{nk}^2$.

Proof.

i) By Lemmas 1,3,4 and Theorems 1, for every $h \in R$ we have

$$E[e^{hT_{nm}}] \leq \prod_{k=1}^{m} E[e^{ha_{nk}X_k}] \leq$$

$$\exp[\frac{h^2\alpha^2\sum_{k=1}^{m} a_{nk}^2}{2}] \leq \exp[\frac{h^2\alpha^2 A_n}{2}].$$

Hence, by Fatou's Lemma

$$E[e^{hT_n}] \leq \exp[\frac{h^2\alpha^2 A_n}{2}].$$

ii) This follows by part i and Lemma 1. □

Corollary 2.

i) If $\sum_{n=1}^{\infty} \exp[-\frac{\varepsilon^2}{2\alpha^2 A_n}] < \infty$, then

$$\lim_{n\longrightarrow\infty} \sum_{k=1}^{n} a_{nk}X_k = 0 \quad W.P.1. \qquad (2.1)$$

In particular if $A_n = \circ(\ln^{-1}(n))$, then (2.1) holds.

ii) If $S_n = \sum_{k=1}^{n} X_k$ and $\beta > 0$ then

$$\lim_{n\longrightarrow\infty} n^{-1/2}(\ln^{-(1+\beta)/2}(n))S_n = 0 \quad W.P.1.$$

Theorem 4. Let $\{X_n,\ n\geq 1\}$ be a sequence of ND Sub-Gaussian r.v.'s

i) If $\lim_{n\to\infty}\sum_{k=1}^{n} a_{nk}^2 = l \neq 0 < \infty$, then for every $\beta > 0$

$$\lim_{n\longrightarrow\infty} n^{-\beta}\sum_{k=1}^{n} a_{nk}X_k = 0 \quad W.P.1. \tag{2.2}$$

ii) If $a_{nk} = O(n^{-\beta})$ for some $k \leq n$ and $\beta > \frac{1}{2}$, then

$$\lim_{n\longrightarrow\infty} \sum_{k=1}^{n} a_{nk}X_k = 0 \quad W.P.1. \tag{2.3}$$

iii) If $\sum_{k=1}^{n} a_{nk}^2 = O(n^{-\beta})$ for some $\beta > 0$, then

$$\lim_{n\longrightarrow\infty} \sum_{k=1}^{n} a_{nk}X_k = 0 \quad W.P.1. \tag{2.4}$$

Proof. By Lemma 5 for some $0 < B < \infty$, and $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|n^{-\beta}\sum_{k=1}^{n} a_{nk}X_k| > \varepsilon] \leq$$

$$2\sum_{n=1}^{\infty} \exp[-\frac{n^{2\beta}\varepsilon^2}{2\alpha^2\sum_{k=1}^{n} a_{nk}^2}] < \infty,$$

and

$$\sum_{n=1}^{\infty} P[|\sum_{k=1}^{n} a_{nk}X_k| > \varepsilon] \leq \sum_{n=1}^{\infty} 2\exp[-\frac{\varepsilon^2}{2\alpha^2\sum_{k=1}^{n} a_{nk}^2}]$$

$$\leq \sum_{n=1}^{\infty} 2\exp[-\frac{\varepsilon^2 n^{2\beta-1}}{2\alpha^2 B^2}] < \infty.$$

Now (2.2) and (2.3) follow from the Borel Cantelli Lemma, and (2.4) follows from part (ii). □

Theorem 5. Let $\{X_n,\ n\geq 1\}$ be a sequence of ND Sub-Gaussian r.v.'s. Then for every $x \in R$

$$P[\max_{j\leq m}|T_{nj}| \geq x] \leq 2exp[-\frac{x^2}{2\alpha^2 A_n}].$$

Proof. By Lemmas 1,3,4 and Theorems 1 for every $h \in R$ we have

$$Ee^{h|T_{nm}|} \leq Ee^{hT_{nm}} + Ee^{-hT_{nm}} \leq 2\exp[\frac{h^2\alpha^2 A_n}{2}].$$

Since $\{T_{nm}, \mathcal{F}_m,\ m \geq 1\}$ is a martingale and $\{|T_{nm}|, \mathcal{F}_m,\ m\geq 1\}$ is submartingale and $\varphi(t) = e^{th}$ for each $h \geq 0$ is increasing and convex function, then by the submartingale inequality

$$P[\max_{j\leq m}|T_{nj}| \geq x] = P[\max_{j\leq m}\varphi(|T_{nj}|) \geq \varphi(x)] \leq$$

$$\frac{E[\varphi(|T_{nm}|)]}{\varphi(x)} \leq 2\exp[-hx + \frac{h^2\alpha^2 A_n}{2}].$$

For $h = \frac{x}{\alpha^2 A_n}$ we have

$$P[\max_{j\leq m}|T_{nj}| \geq x] \leq 2\exp[-\frac{x^2}{2\alpha^2 A_n}]. \quad \square$$

Theorem 6. Under the assumptions of Theorem 5

i) If $\{T_{nm},\ m\geq 1\}$ converges in probability for every n, then it converges W.P.1.

ii) $T_n = \sum_{k=1}^{\infty} a_{nk}X_k$ converges W.P.1 for each n.

Proof.

i) Let $T_{nm} \longrightarrow l_n$ in probability for every n, then there exist a subsequence $\{m_k,\ k\geq 1\}$ such that $T_{nm_k} \longrightarrow l_n$ W.P.1. We define

$$S_{nk} = \max_{m_k < m \leq m_{k+1}} |T_{nm} - T_{nm_k}|.$$

By Theorem 5

$$P[S_{nk} > \varepsilon] \leq 2\exp[-\frac{\varepsilon^2}{2\alpha^2\sum_{j=m_k+1}^{\infty} a_{nj}^2}].$$

Hence by the Borel Cantelli Lemma $S_{nk} \longrightarrow 0$ a.e., when $k \longrightarrow \infty$.Thus

$$|T_{nm} - l_n| \leq S_{nk} + |T_{nm_k} - l_n| \longrightarrow 0 \quad W.P.1.$$

ii) For every $N > m$ by Lemma 4

$$P[|T_{nN} - T_{nm}| > \varepsilon] \leq 2\exp[-\frac{\varepsilon^2}{2\alpha^2 \sum_{j=m+1}^{\infty} a_{nj}^2}].$$

If $m \longrightarrow \infty$, the left hand side of above inequality tends to zero. Hence, $\{T_{nm},\ m \geq 1\}$ converges in probability by the Cauchy criterion. Now part i shows that T_n converges W.P.1. $\square$

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent Sub-Gaussian r.v.'s with $\tau(X_n) \leq \alpha$, for every n, then the assumption
$E[X_n|\mathcal{F}_{n-1}] = 0$ can be replaced by
$E(X_n) = 0$, (i.e.$E(X_n) = E[X_n|\mathcal{F}_{n-1}] = 0$).
Thus all the above Theorems, Lemmas , and Corollaries are true in this case.

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