Domain Decomposition for a Singularly Perturbed Problem with Parabolic Layers

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Abstract:- This paper deals with multidomain domain decomposition algorithms applied to the solution of an advection-diffusion problem with parabolic layers. The finite difference approximations make use of special piecewise uniform meshes. Convergence properties of the algorithms are established. Numerical results for a test problem are presented.

Key-Words:- Singular perturbation, Convection-diffusion problem, Domain decomposition method, Uniform convergence, Parallel computing

1 Introduction

We are interested in iterative domain decomposition methods for solving the advection-diffusion problem with parabolic layers

$$\varepsilon \Delta u + b(P) u_x - c(P) u = f(P), \ P \in \Omega,$$

$$u = g \text{ on } \partial\Omega, \qquad (1)$$

where $\Omega = \{P = (x, y) : 0 < x < 1, 0 < y < 1\}$, ε is a small positive parameter, $b(P) \geq \beta_* > 0$, $c(P) \geq c_* > 0$ and $\partial\Omega$ is the boundary of Ω.

For $\varepsilon \ll 1$, problem (1) is singularly perturbed and characterized by an exponential layer of width $O(\varepsilon|\ln \varepsilon|)$ at $x = 0$ and by parabolic layers of width $O(\sqrt{\varepsilon}|\ln \varepsilon|)$ at $y = 0$ and $y = 1$ (see [1] for details).

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures for solving singularly perturbed problems have received much attention for their remarkable speed and parallelizability, see, for example, [2-4] and references cited there.

In [3], for solving the singularly perturbed advection-diffusion problem with the differential operator $-\varepsilon\Delta u + \mathbf{b}(P)\nabla u + c(P)u$, the classical Schwarz alternating method and some variants of it were analyzed. In the presence of only the elliptic boundary layers at $x = 1$ and $y = 1$ and in the case of domain decomposition into two subdomains, a convergence rate for the continuous problem (i.e. without resort to discretization in subdomains) as a function of the perturbation parameter ε and an amount of overlap between two subdomains was studied.

In this paper, we introduce a multidomain modification of the Schwarz alternating method proposed in [4] for solving singularly perturbed reaction-diffusion problems. In this approach, the domain is partitioned into many nonoverlapping subdomains with interface , . Small interfacial subdomains are introduced near the interface , , and approximate boundary values computed on , are used for solving problems on nonoverlapping subdomains.

We consider two domain decomposition algorithms: the first one is based on decomposition of the computational domain into nonoverlapping vertical subdomains, and second uses decomposition into nonoverlapping horizontal subdomains. We show that these algorithms converge uniformly in the perturbation parameter ε

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on the piecewise equidistant meshes of Shishkin-type [1]. The piecewise uniform meshes allow us to decompose the computational domain into subdomains outside boundary layers and inside them as well, and possess load balancing. This property is very important for implementation of the iterative algorithms on parallel computers, since it avoids loss of efficiency due to any processors being idle.

2 Undecomposed algorithm

Here for solving problem (1), we construct a difference scheme on piecewise uniform meshes which possesses uniform convergence in the perturbation parameter ε. We use the upwind difference scheme

$$\Lambda U(P) = f(P),\ P \in \Omega^h, \tag{2}$$

$$\Lambda \equiv \varepsilon(\delta_x^2 + \delta_y^2) + b\delta_x - c,$$

where δ_x^2, δ_y^2 and δ_x are the central difference and forward difference approximations to the second and first derivatives, respectively.

Now introduce piecewise equidistant meshes of Shishkin-type from [1] that are adapted to the singularly perturbed behavior of the exact solution. For the exponential layer, we divide interval $\bar{\Omega}^x = [0,1]$ into two parts $[0,\sigma_x]$, $[\sigma_x, 1]$, and in each part we use a uniform grid with $N_x/2+1$ mesh points. The transition point σ_x from [1] is determined by $\sigma_x = \min\{2^{-1}, \varepsilon\tau^{-1}\ln N_x\}$, where $\tau = O(\beta_*), \tau > 0$. If $\sigma_x = 1/2$, then N_x^{-1} is very small relative to ε. This is unlikely in practice, and in this case the difference scheme (2) can be analysed using standard techniques. We therefore assume that

$$\sigma_x = \varepsilon\tau^{-1}\ln N_x. \tag{3}$$

This defines the piecewise equidistant mesh in the x-direction condensed in the boundary layer at $x = 0$. A piecewise uniform mesh in the y-direction is formed by dividing the interval $[0,1]$ into the three parts $[0,\sigma_y]$, $[\sigma_y, 1-\sigma_y]$ and $[1-\sigma_y, 1]$, where

$$\sigma_y = \kappa\sqrt{\varepsilon}\ln N_y \tag{4}$$

with any positive constant κ independent of ε. Assuming that N_y is divisible by 4, we use equidistant meshes on each of these intervals, with $N_y/4+1$ points in $[0,\sigma_y]$ and $[1-\sigma_y, 1]$ and $N_y/2+1$ points in $[\sigma_y, 1-\sigma_y]$.

Theorem 1 *The difference scheme (2) on the piecewise uniform mesh Ω^h from (3),(4) converges ε-uniformly to the solution of (1):*

$$\max_{P\in\bar{\Omega}^h} |U(P) - u(P)| \le C(N^{-1}\ln N)^p,$$

where $N = \min\{N_x, N_y\}$, constant C is independent of ε, N and $p = 1/18$.

The proof of the theorem can be found in [1].

Remark 1 *We mention that the numerical estimates given in [5] for the rates of ε-uniform convergence are considerably better, which suggests that this theoretical result is not as sharp as possible.*

3 Domain decomposition algorithm in the exponential layer

We consider decomposition of the domain $\bar{\Omega}$ into M nonoverlapping subdomains (vertical strips) $\bar{\Omega}_m, m = 1, \ldots, M$:

$$\Omega_m = (x_{m-1}, x_m) \times (0,1),$$

$$,_m = \{x = x_m, 0 \le y \le 1\},$$

where $\bar{\Omega}_m \cap \bar{\Omega}_{m+1} = \ ,_m$. Thus, we can write down the boundary of Ω_m as

$$\partial\Omega_m = \ ,_m^0 \cup \ ,_{m-1} \cup \ ,_m,$$

where $,_m^0 = \partial\Omega \cap \partial\Omega_m$. Additionally, we consider $(M-1)$ interfacial subdomains $\omega_m, m = 1, \ldots, M-1$:

$$\omega_m = (x_m^b, x_m^e) \times (0,1),\ \omega_{m-1} \cap \omega_m = \emptyset,$$

where $x_m^b < x_m < x_m^e$, $m = 1, \ldots, M-1$. The boundaries of ω_m are denoted by

$$\gamma_m^b = \{P : x = x_m^b, 0 \le y \le 1\},$$

$$\gamma_m^e = \{P : x = x_m^e, 0 \le y \le 1\},$$

$$\gamma_m^0 = \partial\Omega \cap \partial\omega_m.$$

On $\bar{\Omega}_m, m = 1, \ldots, M$ and $\bar{\omega}_m, m = 1, \ldots, M-1$ we introduce meshes $\bar{\Omega}_m^h$ and $\bar{\omega}_m^h$, respectively, and suppose that $\bar{\Omega}^h = \bigcup \bar{\Omega}_m^h$, and the mesh points in $\bar{\omega}_m^h, m = 1, \ldots, M-1$ coincide with the mesh points of $\bar{\Omega}^h$.

We consider the following iterative domain decomposition algorithm for solving problem (2). On each iterative step , firstly, we solve problems on the nonoverlapping subdomains $\bar{\Omega}_m^h, m = 1, \ldots, M$ with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\bar{\omega}_m^h, m = 1, \ldots, M-1$, and problems on the interfacial subdomains are computed. Finally, we impose continuity for piecing the solutions on the subdomains together.

On subdomains $\bar{\Omega}_m^h, m = 1, \ldots, M$, introduce mesh functions $v_m^{(n)}(P), m = 1, \ldots, M$ (here the index n stands for a number of iterative steps) satisfying the following difference schemes

$$\Lambda v_m^{(n)}(P) = f(P),\ P \in \Omega_m^h, \qquad (5a)$$

$$v_m^{(n)}(P) = g(P),\ P \in ,{}_m^{h0},$$

$$v_m^{(n)}(P) = V^{(n-1)}(P),\ P \in ,{}_{m-1}^{h} \cup ,{}_m^{h}.$$

On the interfacial subdomains $\bar{\omega}_m^h, m = 1, \ldots, M-1$, we determine the following difference problems

$$\Lambda z_m^{(n)}(P) = f(P),\ P \in \omega_m^h, \qquad (5b)$$

$$z_m^{(n)}(P) = \begin{cases} g(P), & P \in \gamma_m^{h0}, \\ v_m^{(n)}(P), & P \in \gamma_m^{hb}, \\ v_{m+1}^{(n)}(P), & P \in \gamma_m^{he}. \end{cases}$$

The mesh function $V^{(n)}(P)$ is determined in the form

$$V^{(n)}(P) = \begin{cases} v_m^{(n)}(P), & P \in \bar{\Omega}_m^{h*}; \\ z_m^{(n)}(P), & P \in \overline{\omega}_m^h, \end{cases} \qquad (5c)$$

where $\bar{\Omega}_m^{h*} = \overline{\Omega_m^h \setminus (\omega_{m-1}^h \cup \omega_m^h)}$. Initial guesses $V^{(0)}(P),\ P \in ,{}_m^h,\ m = 1, \ldots, M-1$ must be prescribed. Algorithm (5) can be carried out by parallel processing, since on each iterative step n the M problems (5a) for $v_m^{(n)}(P), m = 1, \ldots, M$ and the $(M-1)$ problems (5b) for $z_m^{(n)}(P), m = 1, \ldots, M-1$ can be implemented concurrently.

On $\bar{\Omega}_m^h$ introduce the following difference problems:

$$\Lambda_* \phi_m^{1,2}(P) - c_* \phi_m^{1,2}(P) = 0,\ P \in \Omega_m^h, \qquad (6)$$

$$\phi_m^1(P) = \begin{cases} 1, & P \in ,{}_{m-1}^h, \\ 0, & P \in \partial\Omega_m^h \setminus ,{}_{m-1}^h, \end{cases}$$

$$\phi_m^2(P) = \begin{cases} 1, & P \in ,{}_m^h, \\ 0, & P \in \partial\Omega_m^h \setminus ,{}_m^h, \end{cases}$$

where the difference operator Λ_* is defined by

$$\Lambda_* = \varepsilon(\delta_x^2 + \delta_y^2) + b\delta_x.$$

Introduce the notations

$$\rho_m^b = \|\phi_m^1(P) + \phi_m^2(P)\|_{\gamma_m^{hb}},$$

$$\rho_m^e = \|\phi_{m+1}^1(P) + \phi_{m+1}^2(P)\|_{\gamma_m^{he}},$$

$$\rho = \max_{1 \le m \le M-1} (\rho_m^b, \rho_m^e).$$

The following theorem holds true.

Theorem 2 *Algorithm (5) on the piecewise uniform mesh (3), (4) converges to the solution $u(P)$ of (1) with the following rate*

$$\max_{P \in \bar{\Omega}^h} |V^{(n)}(P) - u(P)| \le C[(N^{-1} \ln N)^p + \rho^n],$$

where $V^{(n)}(P)$ from (5c), $p = 1/18$, the contraction coefficient $\rho \in (0,1)$ and constant C are independent of ε, N_x, N_y.

Remark 2 *Theorem 2 guarantees us that the domain decomposition algorithm (5) converges for any initial guesses.*

Now, estimate the contraction coefficient ρ in Theorem 2. Consider algorithm (5) with the interfacial subdomains $\omega_m, m = 1, \ldots, M-1$ located in the x-direction outside the exponential boundary layer. In this case, ρ satisfies the following estimate

$$\rho < (M-1)\frac{2\varepsilon}{\beta_* h_x},$$

where h_x is the step-size of the piecewise equidistant mesh outside the exponential boundary layer. Thus, for sufficiently small values of ε the inequality $\varepsilon \ll h_x$ holds, and as follows from Theorem 2 the order of convergence of algorithm (5) is defined by N but not by the contraction coefficient ρ of domain decomposition.

Consider decomposition of the computational domain where the interfacial subdomains are localized inside the exponential layer. We decompose the boundary layer $[0, \sigma_x]$ and the region outside the layer $[\sigma_x, 1]$ into $M/2$ equal subdomains, respectively, such that each subdomain $\bar{\Omega}_m^{hx}, m = 1, \ldots, M$ contains the same number of mesh points $2I + 1, I = N_x/(2M)$. Here $\bar{\Omega}_m^{hx}$ is the mesh on $[x_{m-1}, x_m]$. The interfacial subdomains $\bar{\omega}_m^{hx}, m = 1, \ldots, M-1$ contain the same number of mesh points $2I_\omega + 1,\ I_\omega \leq I$, and the center of the discrete interval $\bar{\omega}_m^{hx}$ is located at x_m, where $\bar{\omega}_m^{hx}$ is the mesh on $[x_m^b, x_m^e]$. In the case of the maximal size of the interfacial subdomains $I_\omega = I$, it can be proved the following estimate

$$\rho \leq M N_x^{-s/M},\ s = \beta_*/\tau,$$

where β_* and τ from (1) and (3), respectively. From here and Theorem 2, we conclude that the domain decomposition algorithm converges uniformly. Consider the limiting case of this decomposition, where only the first subdomain $\bar{\Omega}_1^h$ lies in the boundary layer (the unbalanced decomposition), i.e. region $[\sigma_x, 1]$ outside the layer is decomposed into $M-1$ equal subdomains and all subdomains $\bar{\Omega}_m^h,\ m = 2, \ldots, M$ contain the same number of mesh points. In the case of the maximal size of $\bar{\omega}_1^{hx}$ and $\varepsilon \ll h_x$, we have the estimate

$$\rho \leq N_x^{-s/2}.$$

Note here that getting the better convergence property of algorithm (5) on the unbalanced decomposition, we have lost load balancing, since the sizes of domains $\bar{\Omega}_1^h$ and $\bar{\omega}_1^h$ for large values of M are sufficiently bigger then others. To keep load balancing for algorithm (5) on the unbalanced decomposition, we need to use the second level of parallelization for solving discrete systems on these two subdomains.

4 Domain decomposition algorithm in the parabolic layers

We consider decomposition of domain $\bar{\Omega}$ into L nonoverlapping subdomains (horizontal strips) $\bar{\Omega}_l, l = 1, \ldots, L$:

$$\Omega_l = (0,1) \times (y_{l-1}, y_l),$$

$$,\ _l = \{P : 0 \leq x \leq 1, y = y_l\},$$

where $\bar{\Omega}_l \cap \bar{\Omega}_{l+1} = ,\ _l$. Thus, the boundary of Ω_l can be written in the form

$$\partial\Omega_l = ,\ _l^0 \cup ,\ _{l-1} \cup ,\ _l,$$

where $,\ _l^0 = \partial\Omega \cap \partial\Omega_l$. Additionally, we consider $(L-1)$ interfacial subdomains $\omega_l, l = 1, \ldots, L-1$:

$$\omega_l = (0,1) \times (y_l^b, y_l^e), \quad \omega_{l-1} \cap \omega_l = \emptyset,$$

where $y_l^b < y_l < y_l^e,\ l = 1, \ldots, L-1$, with the boundaries

$$\gamma_l^b = \{P : 0 \leq x \leq 1, y = y_l^b\},$$

$$\gamma_l^e = \{P : 0 \leq x \leq 1, y = y_l^e\},\ \gamma_l^0 = \partial\Omega \cap \partial\omega_l.$$

On $\bar{\Omega}_l, l = 1, \ldots, L$ and $\bar{\omega}_l, l = 1, \ldots, L-1$ we introduce meshes $\bar{\Omega}_l^h$ and $\bar{\omega}_l^h$, respectively, and suppose that $\bar{\Omega}^h = \bigcup \bar{\Omega}_l^h$, and the mesh points in $\bar{\omega}_l^h, l = 1, \ldots, L-1$ coincide with the mesh points of $\bar{\Omega}^h$.

Similarly to (5), we consider the following iterative domain decomposition algorithm for solving problem (2). On subdomains $\bar{\Omega}_l^h, l = 1, \ldots, L$, introduce mesh functions $v_l^{(n)}(P), l = 1, \ldots, L$ satisfying the following difference schemes

$$\Lambda v_l^{(n)}(P) = f(P),\ P \in \Omega_l^h, \qquad (7a)$$

$$v_l^{(n)}(P) = g(P),\ P \in ,\ _l^{h0},$$

$$v_l^{(n)}(P) = V^{(n-1)}(P),\ P \in ,\ _{l-1}^h \cup ,\ _l^h.$$

On the interfacial subdomains $\bar{\omega}_l^h, l = 1, \ldots, L-1$, we determine the following difference problems

$$\Lambda z_l^{(n)}(P) = f(P),\ P \in \omega_l^h, \qquad (7b)$$

$$z_l^{(n)}(P) = \begin{cases} g(P), & P \in \gamma_l^{h0}, \\ v_l^{(n)}(P), & P \in \gamma_l^{hb}, \\ v_{l+1}^{(n)}(P), & P \in \gamma_l^{he}. \end{cases}$$

The mesh function $V^{(n)}(P)$ is determined in the form

$$V^{(n)}(P) = \begin{cases} v_l^{(n)}(P), & P \in \bar{\Omega}_l^{h*}; \\ z_l^{(n)}(P), & P \in \overline{\omega}_l^h, \end{cases} \qquad (7c)$$

where $\bar{\Omega}_l^{h*} = \overline{\Omega_l^h \setminus (\omega_{l-1}^h \cup \omega_l^h)}$. Initial guesses $V^{(0)}(P),\ P \in ,\ _l^h,\ l = 1, \ldots, L-1$ must be prescribed. Algorithm (7) can be carried out by parallel processing, since on each iterative step n the L problems (7a) and the $(L-1)$ problems (7b) can be implemented concurrently.

Similar to (6), on $\bar{\Omega}_l^h$ introduce the following difference problems:

$$\Lambda_* \varphi_l^{1,2}(P) - c_* \varphi_l^{1,2}(P) = 0, \ P \in \Omega_l^h,$$

$$\varphi_l^1(P) = \begin{cases} 1, & P \in \, {}_{l-1}^{h}, \\ 0, & P \in \partial\Omega_l^h \setminus \, {}_{l-1}^{h}, \end{cases}$$

$$\varphi_l^2(P) = \begin{cases} 1, & P \in \, {}_{l}^{h}, \\ 0, & P \in \partial\Omega_l^h \setminus \, {}_{l}^{h}, \end{cases}$$

and introduce the notations

$$\varrho_l^b = \|\varphi_l^1(P) + \varphi_l^2(P)\|_{\gamma_l^{hb}},$$

$$\varrho_l^e = \|\varphi_{l+1}^1(P) + \varphi_{l+1}^2(P)\|_{\gamma_l^{he}},$$

$$\varrho = \max_{1 \le l \le L-1} (\varrho_l^b, \varrho_l^e).$$

Similar to Theorem 2, we can get the following result.

Theorem 3 *Algorithm (7) on the piecewise uniform mesh (3), (4) converges to the solution $u(P)$ of (1) with the following rate*

$$\max_{P \in \bar{\Omega}^h} |V^{(n)}(P) - u(P)| \le C[(N^{-1} \ln N)^p + \varrho^n],$$

where $V^{(n)}(P)$ from (7c), $p = 1/18$, coefficient $\varrho \in (0,1)$, and constant C is independent of ε, N_x, N_y, and ϱ.

Remark 3 *Theorem 3 guarantees us that the domain decomposition algorithm (7) converges for any initial guesses.*

Now we estimate coefficient ϱ from Theorem 3. Firstly, consider algorithm (7) with the interfacial subdomains $\omega_l, l = 1, \dots, L-1$ located in the y-direction outside the parabolic boundary layers. In this case the following estimate on ϱ holds

$$\varrho < 4\varepsilon(c_* h_y^2)^{-1},$$

where h_y is the step-size of the piecewise equidistant mesh outside the parabolic boundary layers. In practice, $\varepsilon \ll N_y^{-1}$, then as follows from Theorem 3, the order of convergence of algorithm (7) is defined by N_x, N_y, but not by coefficient ϱ of domain decomposition.

Now we estimate coefficient ϱ from Theorem 3 in the case where some of the interfacial subdomains are located inside the parabolic boundary layers. Consider a balanced decomposition of the computational domain. We decompose each of the parabolic boundary layers $[0, \sigma_y]$ and $[1-\sigma_y, 1]$ into $L/4$ equal subdomains, and the interval $[\sigma_y, 1-\sigma_y]$ into $L/2$ equal subdomains. We note that each of the subdomains $\bar{\Omega}_l^{hy}, l = 1, \dots, L$ contains the same number of mesh points $2J+1, J = N_y/(2L)$, where $\bar{\Omega}_l^{hy}$ is the mesh on $[y_{l-1}, y_l]$. The interfacial subdomains $\bar{\omega}_l^{hy}, l = 1, \dots, L-1$ inside and outside the boundary layers contain the same number of mesh points $2J_\omega + 1$, $J_\omega \le J$, and the center of the discrete interval $\bar{\omega}_l^{hy}$ is located at y_l. In the case of the maximal size of the interfacial subdomains $J_\omega = J$, it can be proved the following estimate

$$\varrho \le 2N_y^{-s/L}.$$

Remark 4 *In the context of parallel computing, the balanced domain decomposition guarantees us load balancing of a multi-processor computer, since subdomains $\bar{\Omega}_l^h, l = 1, \dots, L$ and the interfacial subdomains $\bar{\omega}_l^h, l = 1, \dots, L-1$ contain the same number of mesh points $(N_x+1)(N_y L^{-1}+1)$ and $(N_x+1)(2J_\omega+1)$, respectively.*

Consider the limiting case of this decomposition, where only the first and the last subdomains $\bar{\Omega}_1^{hy}$ and $\bar{\Omega}_L^{hy}$ lie in the boundary layers (the unbalanced decomposition), i.e. the region outside the layer $[\sigma_y, 1-\sigma_y]$ is decomposed into $L-2$ equal subdomains and all subdomains $\bar{\Omega}_l^{hy}$, $l = 2, \dots, L-1$ contain the same number of mesh points $N_y(2(L-2))^{-1} + 1$. In the case of the maximal size of $\bar{\omega}_1^{hy}$ and $\bar{\omega}_{L-1}^{hy}$, we have

$$\varrho \le 2N_y^{-s/4},$$

which independent of ε and L. It should be noted that improving convergence property of algorithm (7) on the unbalanced decomposition, we have lost load balancing, since the sizes of domains $\bar{\Omega}_1^h$, $\bar{\Omega}_L^h$, $\bar{\omega}_1^h$ and $\bar{\omega}_{L-1}^h$ for large values of L are sufficiently bigger then others. To retain load balancing for algorithm (7) on this decomposition, we need to use the second level of parallelization for solving discrete systems on these four subdomains.

5 Numerical results

As a test problem, consider the following problem

$$\varepsilon\Delta u + (1 + x^2 + y^2)u_x = 0, \ \Omega = (0,1) \times (0,1),$$

$$u(x,0) = x^3, \ u(x,1) = x^2, \ x \in [0,1],$$

$$u(0,y) = 0, \ u(1,y) = 1, \ y \in [0,1].$$

where $\beta_* = 1, c_* = 0$ in (1). The assumption $c(P) \geq c_* > 0$ in (1) can always be obtained via a change of variables $u(P) = \bar{u}(P)\exp(-dx), d = \text{const} > 0$.

M	$n_1;\ n_2,\ N = 32$		$n_1;\ n_2,\ N = 128$	
2	3; 2	3; 2	3; 2	3; 2
4	9; 7	9; 7	9; 6	10; 6
8	32; 21	32; 21	32; 17	33; 18
16	n.a.	n.a.	119; 60	123; 62
ε	10^{-3}	10^{-5}	10^{-3}	10^{-5}

Table 1

In all our numerical experiments, we choose $N_x = N_y = N$. Let n_s be a number of iterations on mesh (3), (4) with $s = \beta_*/\tau$ and $\kappa = 1$. In Table 1, we give the numbers of iterations $n_s, s = 1,2$ for algorithm (5) on the balanced domain decomposition in the elliptic layer with the maximal size of the interfacial subdomains at $N = 32, 128$. Our numerical results show, that for N, M fixed, n_s is independent of ε. The uniform convergent result confirms our theoretical estimates. For M fixed, the number of iterations $n_s(N)$ depends almost not at all on N. The number of iterations as a function of the parameter s is decreasing one. This result means that if we increase the transition point σ_x, the number of iterations decreases that is in agreement with the theoretical estimates.

M	$n_1; n_2,\ N = 32$		$n_1; n_2,\ N = 128$	
3	6; 5	6; 5	6; 4	7; 4
5	13; 10	13; 10	13; 9	14; 9
9	na.	n.a.	27; 18	29; 19
ε	10^{-3}	10^{-5}	10^{-3}	10^{-5}

Table 2

Table 2 represents the numbers of iterations $n_s, s = 1,2$ for algorithm (5) on the unbalanced domain decomposition with the maximal size of the interfacial subdomains at $N = 32, 128$. The main features of algorithm (5) on the balanced domain decomposition highlighted from Table 1 hold true on the unbalanced domain decomposition, where only the first subdmain $\bar{\Omega}_1^h$ lies in the elliptic layer. As we can see from Tables 1 and 2, algorithm(5) on the unbalanced decomposition converges sufficiently faster then on the balanced decomposition, comparing $M = 4(2+2)$ from Table 1 with $M = 3(1+2)$ from Table 2, and so on.

L	$n_1;\ n_2,\ N = 32$		$n_1;\ n_2,\ N = 128$	
4	2; 1	2; 1	2; 1	2; 1
8	3; 2	3; 2	2; 2	2; 2
16	n.a.	n.a.	5; 4	5; 3
ε	10^{-3}	10^{-5}	10^{-3}	10^{-5}

Table 3

In Tables 3, we present numerical results for algorithm (7) with the balanced domain decomposition in the parabolic layers. Let n_κ be a number of iterations with $\tau = \beta_*$ $(s = 1)$ and $\kappa = 1,2$. From the data in Table 3, it follows that for N, L fixed, n_κ is independent of ε, which confirms our theoretical estimates. We note that in the contrast to the algorithm (5) with domain decomposition in the elliptic layer, the rate of convergence of algorithm (7) on the unbalanced decomposition is only slightly higher then on the balanced one.

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