Parameterization for Bivariate Nonseparable Wavelets

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Abstract: - In this paper, we give a complete and simple parameterization for bivariate non-separable compactly supported orthonormal wavelets based on the commonly used uniform dilation matrix $D = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Key-Words: - Wavelets, Nonseparable, Bivariate, Parameterization

1 Introduction

Univariate wavelets have found successful applications in signal processing. To apply wavelet theory to digital image processing, we have to construct bivariate wavelets. The most commonly used method is the tensor product of univariate wavelets, that is, the bivariate basis functions are separable. A separable wavelet basis is easy to construct and simple to study, for it inherits features of the corresponding wavelet basis in $\mathbb{R}^1$, such as smoothness and support size, and separable wavelet transforms are easy to implement. Nevertheless, separable bases have a number of drawbacks. Because they are so special, they have very little design freedom. Furthermore, separability imposes an unnecessary product structure on the plane, which is artificial for natural images. For example, the zero set of a separable scaling function contains horizontal and vertical lines. This "preferred directions" effect can create unpleasant artifacts that become obvious at high image compression ratios. Nonseparable wavelet bases offer the hope of a more isotropic analysis.

So, much effort has been spent on constructing non-separable bivariate wavelets in the last ten years. For example, in [1], Riemenschneider and Shen used bivariate box splines which are natural generalizations of B-splines to construct wavelets. These wavelets have infinite support, like the Battle-Lemarie wavelets. References [2~5] also discussed non-separable compactly supported biorthogonal wavelets and prewavelets. In [6], Cohen and Daubechies generalized the method in [7] to constructed non-separable bidimensional (discontinuous) compactly supported wavelets. Also, in [8], Kovacevic and Vetterli studied properties of multidimensional non-separable wavelets and numerically constructed examples of continuous non-separable compactly supported bivariate wavelets. Both Cohen-Daubechies' wavelets and Kovacevic-Vetterli's examples are based on the dilation matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. In [9], Belogay and Wang constructed a family of bivariate non-separable compactly supported wavelets based the dilation matrix $M^2 = \pm 2I$. All the above works hadn't obtain complete and simple parameterization express for bivariate non-separable compactly supported orthonormal wavelets. In [10] He and Lai constructed bivariate non-separable compactly supported orthonormal wavelets based on the uniform dilation matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and given a complete and simple parameterization express for their discussed wavelets.

In this paper, we give a complete and simple parameterization for bivariate non-separable compactly supported orthonormal wavelets based on the commonly used dilation matrix $D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

2 Main Results

To construct the wavelet, we start by constructing a compactly supported scaling function $\varphi$ which generates Multi-resolution analysis of $L^2(\mathbb{R}^2)$. Let $D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ be the sampling matrixes, and

$$H_\alpha(z_1, z_2) = \sum_{j,k,p,q} h(j, k)z_1^j z_2^k.$$
\[ z_1 = e^{i\omega}, \quad z_2 = e^{-i\omega}; \] be the scaling filter. Where \( h(j, k) \) be the coefficients of the scaling filter and \( p, q \) are some non-negative integers. According to the papers [5,8,10], we will construct \( H_\theta(z_1, z_2) \) which satisfies the following requirements:

(a). \( H_\theta(1,1) = \sqrt{2} \)

(b). \( \sum_{j=0}^{\infty} \left| H_\theta(e^{2\pi j/k}, z) \right|^2 = 2 \),

(c). \( H_\theta(e^{2\pi j/k}, h) = H_\theta(-1,1) = 0 \)

where \( z = (z_1, z_2) \), \( k_0 = (0,0), k_1 = (1,0) \)

Rewrite \( H_\theta(z_1, z_2) \) in its polyphase form:

\[ H_\theta(z_1, z_2) = \sum_{j} z_j \cdot H_\theta(z_1, z_2) + z_i H_\theta(z_2, z_2) \]

Where \( H_\theta(z_1, z_2) = a_i + b_i z_1 + c_i z_2 + d_i, z_1, z_2 \) be the poly-phase components of the scaling filter \( H_\theta(z_1, z_2) \).

We now give the main results in the present paper.

**Theorem 1:** For the sampling matrix \( D \),

\[ H_\theta(z_1, z_2) = \sum_{j} z_j \cdot H_\theta(z_1, z_2) + P(z)S(z) \]

\[ P(z) = \frac{1 + z_1}{2} \]

\[ S(z) = a_0 + a_0 z_1 + a_0 z_2 + b_0 z_1 + b_0 z_2 + b_2 z_1 z_2 \]

\[ a_0 = \sqrt{2} + \frac{1}{2} \left( \cos \alpha + \cos \beta \right) \]

\[ a_0 = \frac{1}{2} \left( \sin \alpha - \cos \alpha \right) + \frac{1}{4} \left( \sin \beta - \cos \beta \right) \]

\[ a_0 = \frac{\sqrt{2}}{2} - \frac{1}{2} \left( \sin \alpha + \sin \beta \right) \]

\[ b_0 = \frac{1}{2} \left( \cos \beta - \cos \alpha \right) \]

\[ b_1 = \frac{1}{2} \left( \sin \beta + \cos \alpha \right) - \frac{1}{2} \left( \cos \beta + \sin \alpha \right) \]

\[ b_2 = \frac{1}{2} \left( \sin \beta - \sin \alpha \right) \]

Suppose that \( \alpha, \beta \) satisfy

\[ \sin \left( \frac{\pi}{4} + \alpha \right) = \sin \left( \frac{\pi}{4} + \beta \right), \]

then \( H_\theta(z_1, z_2) \) is an orthogonal scaling filter.

Proof. According to the requirement (b),

\[ \left| H_\theta(z_1, z_2) \right|^2 + \left| H_\theta(z_1, z_2) \right|^2 = 1, \]

we have:

\[ \sum_{i=0}^{\infty} (a_i + b_i + c_i d_i) = 0, \quad \sum_{i=0}^{\infty} (a_i c_i + b_i d_i) = 0 \]

(1)

\[ \sum_{i=0}^{\infty} a_i d_i = 0, \quad \sum_{i=0}^{\infty} b_i c_i = 0 \]

(2)

\[ \sum_{i=0}^{\infty} (a_i^2 + b_i^2 + c_i^2 + d_i^2) = 1 \]

(3)

So, we have:

\[ \sum_{i=0}^{\infty} (a_i - b_i)^2 + (c_i - d_i)^2 = 1 \]

and \( \sum_{i=0}^{\infty} (a_i - b_i)(c_i - d_i) = 0 \)

So,

\[ \sum_{i=0}^{\infty} (a_i - b_i + c_i - d_i)^2 = 1 \]

(4)

notice that,

\[ H_\theta(1,1) = a_0 + b_0 + c_0 + d_0 + a_1 + b_1 + c_1 + d_1 = \sqrt{2} \]

\[ H_\theta(-1,1) = a_0 + b_0 + c_0 + d_0 - (a_1 + b_1 + c_1 + d_1) = 0 \]

so,

\[ a_0 + b_0 + c_0 + d_0 = a_1 + b_1 + c_1 + d_1 = \frac{\sqrt{2}}{2} \]

so the expression (4) is equivalent to:

\[ \sum_{i=0}^{\infty} (2(a_i + c_i) - \frac{\sqrt{2}}{2})^2 = 1 \]

(5)

similarly, we have:

\[ \sum_{i=0}^{\infty} (2(a_i + b_i) - \frac{\sqrt{2}}{2})^2 = 1 \]

(6)

Because \( H_\theta(z_1, z_2) = P(z)S(z) \) and \( P(z) = \frac{1 + z_1}{2} \),

so \( H_\theta(-1, z_2) = 0 \), i.e.

\[ a_0 + c_0 = a_1 + c_1 \]

\[ b_0 + d_0 = b_1 + d_1 \]

(7)

From (5) and (6), we easily obtain that

\[ b_0 = \frac{\sqrt{2}}{4} \left( \cos \beta - a_0 \right) \]

\[ b_1 = \frac{\sqrt{2}}{4} \left( \sin \beta - a_1 \right) \]

(8)

\[ c_1 = \frac{\sqrt{2}}{2} - a_i, \quad d_i = -b_i, \quad i = 0, 1 \]

(9)

Substitute \( d_i \) of (9) into \( \sum_{i=0}^{\infty} a_i d_i = 0 \), we obtain:

\[ \left[ a_0 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \cos \beta \right) \right]^2 + \left[ a_1 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \sin \beta \right) \right]^2 \]

\[ = \frac{1}{16} \left( \cos \beta + \sin \beta \right)^2 \]

(10)

Notice that \( \sum_{i=0}^{\infty} b_i c_i = 0 \) and \( \sum_{i=0}^{\infty} a_i d_i = 0 \),

following (19) we have
\[
\left[ a_0 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \cos \beta \right) \right] + \left[ a_1 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \sin \beta \right) \right] = \frac{\sqrt{2}}{4} + \frac{1}{4} (\cos \beta + \sin \beta) \tag{11}
\]

set
\[
\tilde{a}_0 = a_0 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \cos \beta \right),
\]
\[
\tilde{a}_1 = a_1 - \left( \frac{\sqrt{2}}{8} + \frac{1}{4} \sin \beta \right)
\]

From (10) and (11), we obtain:
\[
(\tilde{a}_0 - \sqrt{\frac{2}{8}})^2 + (\tilde{a}_1 - \sqrt{\frac{2}{8}})^2 = \frac{1}{16}
\]
\[a_0 = \frac{\sqrt{2}}{4} + \frac{1}{4} (\cos \alpha + \cos \beta)\]
\[a_1 = \frac{\sqrt{2}}{4} + \frac{1}{4} (\sin \alpha + \sin \beta)\] (12)

Substitute (12) into (11), we have the constrained equation: $\sin(\frac{\pi}{4} + \alpha) = \sin(\frac{\pi}{4} + \beta)$

The proof of Theory 1 is complete.

**Theorem 2:** Suppose that \( H_0 (z) \) is a scaling filter given by Theorem 1, let \( \varphi \) is the scaling function defined in terms of Fourier transform as follows:

\[
\varphi = \lim_{k \to \infty} \varphi_k
\]

\[
\varphi_k (\omega) = \varphi(0) \prod_{j=1}^{k} \frac{H_0 ((D^{-T})^j \omega)}{\sqrt{M}}
\]

if \( |S(z)| \leq \sqrt{2} \), then \( \varphi \) generated by (12) is continuous.

Proof. Because of \( H_0 (1,1) = \sqrt{2} \) and \( P(1,1) = 1 \), so \( S(1,1) = \sqrt{2} \).

For \( \forall \omega = (\omega_1, \omega_2) \), \( \exists n \in \mathbb{Z}^2 \), such that for all \( k \geq n \), one has \( |\omega|_{2^k} < 1 \).

Take the Taylor expansion for \( S(\omega) = S(e^{i\omega}) \) at \( \omega = (0,0) \), we get:
\[
\left| 1 - \frac{S(e^{i\omega})}{\sqrt{2}} \right| = O\left( \frac{|\omega|^2 + |\omega|^2}{2^k} \right) = O\left( \frac{|\omega|}{2^k} \right)
\]

\((k \to \infty)\)
Hence,
\[
\phi(\omega) = \hat{\phi}(0) \prod_{j=1}^{\infty} \frac{H_0((D^{-T})^j \omega)}{\sqrt{2}} \leq C\left(1 + |\omega_1|\right)^{-1-\varepsilon_1} \left(1 + |\omega_2|\right)^{-1-\varepsilon_2}
\]
Thus, the scaling function \( \phi \) generated by (12) is continuous. This completes the proof of Theorem 2.

3 Example and Conclusion

3.1 Example
We choose the parameters in the scaling filter \( h_0(z_1, z_2) \) as: \( \alpha = \frac{\pi}{12}, \beta = \frac{11}{12} \pi, k = 8 \) iteration according to (12), we easily obtain the scaling function described as Figure 1.

![Figure 1. The scaling function after 8 iteration](image)

3.2 Conclusion
Univariate wavelets have found successful applications in signal processing. To apply wavelet theory to digital image processing, we have to construct bivariate wavelets. Up to now, much effort has been spent on constructing non-separable bivariate wavelets in the last ten years. In this paper, we give a complete and simple parameterization for bivariate non-separable compactly supported orthonormal wavelets based on the commonly used dilation matrix \( D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \).

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