Determination of All Stabilizing Constant Feedback Gains for Open-loop Unstable Second-order Systems with Time Delay

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Abstract: - In this paper, the problem of stabilizing an open-loop unstable second-order system with time delay is considered. A constant feedback gain is utilized to stabilize the system. Using a modified version of Hermite-Biehler Theorem, which is applicable to quasi-polynomials- the complete set of stabilizing constant feedback gains for open-loop unstable plants, is determined. The desired closed-loop performance, e.g., overshoot, settling time, rise time, etc. can be achieved by proper selection of the gain within the boundaries of obtained set of stabilizing gains. The results reported here will serve as a stepping stone for tackling the more complicated case of analytic control and stabilization of second order systems with time delay, using a PI or a PID controller. The proposed procedure is studied on a numerical example.

Key-Words: - Second-order systems, open-loop unstable plants, feedback stabilization, time delay, constant feedback gain.

1 Introduction

An important aspect of tuning a controller in process control industries is to develop a mathematical model that describes the behaviour of the process. Most of the models used to adjust the controller parameters are simple parametric models of first or second order which include a time delay term [1]. Despite the apparent simplicity of these models, the stability analysis of resulting closed-loop system is quite a complicated problem due to the presence of an infinite number of roots of the characteristic equation. In this paper, we will consider the problem of characterizing the set of all constant gains that stabilize a given second-order plant with time delay. It is interesting to point out that even though most of the tuning techniques based on second order models with time delay to provide satisfactory results, the range of stabilizing all constant gain values remain unknown to the best of the author's knowledge.

Recently, the problem of stabilization using fixed order and structure controllers has been tacked using the so-called Hermite-Biehler theorem [2,3]. In [3] a Generalization of this theorem was derived and then used to compute the set of all stabilizing P, PI and PID controllers for a given linear, time invariant plant described by a rational transfer function. Since the plants containing time delay, the synthesis results presented in [4], cannot be applied directly to such plants. In [5,6,7] using a version of the Hermite-Biehler theorem applicable to quasipolynomials, the whole stabilizing set of P, PI and PID gains for first order plants with time delay is obtained. In this paper, we will make use of these results to solve the problem of stabilizing a second order plant with time delay.

2 Preliminary Results on Analysis Systems with Time Delay

Many problems in process control engineering involve time delays. These time delays lead to dynamic models with characteristic equations of the form

$$\delta(s) = d(s) + e^{-sT_1} n_1(s) + e^{-sT_2} n_2(s) + ... + e^{-sT_m} n_m(s) \quad (1)$$

where $d(s)$, $n_i(s)$ for i=1,2,…,m, are polynomials with real coefficients. Characteristic equations of this form are also known as quasipolynomials. It can be shown that the so-called Hermite-Biehler Theorem for Hurwitz polynomials [8,9] does not carry over to arbitrary functions $f(s)$ of the complex variable s. However, Pontryagin [10] studied entire functions of the form $P(s,e^s)$, where $P(s,t)$ is a polynomial in two variables and is called a quasipolynomial. Based on Pontryagin's results, a suitable extension of the Hermite-Biehler Theorem can be developed [9,11] to study the stability of certain classes of quasipolynomials characterized as follows. In (1) we make the assumptions:

A1. $deg[d(s)]=n$ *and* $deg[n_i(s)]<1$ *for* $i=1,2,\ldots,m$;
A2. $0<T_1<T_2<\ldots<T_m$.

Instead of (1) we can consider the quasipolynomial

$$\delta^*(s) = e^{sT_m}\delta(s) = e^{sT_m}d(s) + e^{s(T_m-T_1)}n_1(s) + e^{s(T_m-T_2)}n_2(s) + ... + n_m(s). \quad (2)$$

Since e^{sT_m} does not have any finite zeros, the zeros of $\delta(s)$ are identical to those of $\delta^*(s)$. However, the quasipolynomial $\delta^*(s)$ has a principal term [11] since the coefficient of the term containing the highest powers of s and e^s is nonzero. It then follows that this quasipolynomial is either of the *delay* or of the *neutral* type [4]. This being the case, the stability of the system with characteristic equation (1) is equivalent to the condition that all the zeros of $\delta^*(s)$ be in the open left-half plane. We will say equivalently that $\delta^*(s)$ is Hurwitz or is stable. The following theorem gives necessary and sufficient conditions for the stability of $\delta^*(s)$ [11].

Theorem 1. *Let* $\delta^*(s)$ *be given by (2), and write* $\delta^*(jw) = \delta_r(w) + j\delta_i(w)$, *where* $\delta_r(w)$ *and* $\delta_i(w)$ *represent, respectively, the real and imaginary parts of* $\delta^*(jw)$. *Under assumption (A1) and (A2),* $\delta^*(s)$ *is stable if and only if*

(1) $\delta_r(w)$ *and* $\delta_i(w)$ *have only simple real roots and these interlace,*

(2) $\delta_i'(w_o)\delta_r(w_o) - \delta_i(w_o)\delta_r'(w_o) > 0$, *for some* w_o *in* $(-\infty, \infty)$.

Where $\delta_r'(w)$ *and* $\delta_i'(w)$ *denotes the first derivative with respect to w of* $\delta_r(w)$ *and* $\delta_i(w)$, *respectively.*

Theorem 2. *Let M and N denote the highest powers of s and* e^s, *respectively, in* $\delta^*(s)$. *Let* η *be an appropriate constant such that the coefficients of terms of highest degree in* $\delta_r(w)$ *and* $\delta_i(w)$ *do not vanish at* $w = \eta$. *Then for the equations* $\delta_r(w) = 0 / (\delta_i(w) = 0)$ *to have only real roots, it is necessary and sufficient that in the interval* $-2l\pi + \eta \le w \le 2l\pi + \eta$, $\delta_r(w) / (\delta_i(w))$ *has exactly 4lN+M real roots starting with a sufficiently large l..*

3 Open-loop Second-order Systems with Time Delay

The second-order plants with time delay can be mathematically described by

$$G(s) = \frac{k}{s^2 + a_1 s + a_o} e^{-Ls} \quad (3)$$

where k represent the steady state gain, L time delay, a_1 and a_o plant parameters.

Consider the feedback control system shown in Fig. 1 where r is the command signal, y is the output, *G(s)* is the plant, and *C(s)* is the controller. Here, $C(s)=k_p$. Our objective is to analytically determine the values of the controller parameter in which the closed-loop system is stable.

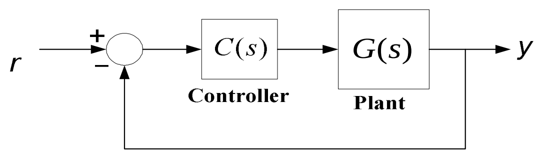

Fig. 1. Feedback control system.

If the open-loop system is unstable, it means that $a_1 > 0$ and $a_o < 0$. We assume that $k > 0$ and $L > 0$. It is clear that when the time delay of the model is zero, *i.e.*, the closed-loop characteristic equation of the system is given by

$$\delta(s) = s^2 + a_1 s + \left(a_o + kk_p\right)$$

If we assume that the steady-state gain k of the plant is positive, the conditions for the stability of the system are

$$a_1 > 0 \quad \text{and} \quad k_p > -\frac{a_o}{k} \quad (4)$$

Now, we will consider the case where the time delay is different from zero. In this case, the closed-loop characteristic equation of the plant is given by

$$\delta(s) = kk_p e^{-Ls} + s^2 + a_1 s + a_o \quad (5)$$

Due to the presence of e^{-Ls}, the number of zeros of $\delta(s)$ is infinite and this make the stability check so difficult. For this, we consider the quasipolynomial $\delta^*(s)$ defined by

$$\delta(s) = e^{Ls}\delta(s) = kk_p + (s^2 + a_1 s + a_o)e^{Ls} \quad (6)$$

Substituting $s = jw$, we have

$$\delta^*(jw) = \delta_r(w) + j\delta_i(w)$$

where

$$\delta_r(w) = kk_p + (a_o - w^2)\cos(Lw) - a_1 w \sin(Lw),$$

$$\delta_i(w) = (a_o - w^2)\sin(Lw) + a_1 w \cos(Lw)$$

Substituting $z=Lw$ leads to

$$\delta_r(z) = kk_c + \left(a_o - \frac{z^2}{L^2}\right)\cos(z) - \frac{a_1}{L} z\sin(z),$$
$$\delta_i(z) = \left(a_o - \frac{z^2}{L^2}\right)\sin(z) + \frac{a_1}{L} z\cos(z) \quad (7)$$

4 Stabilization Using a Constant Feedback Gain

In this section, we present a Theorem that gives a closed form solution to the constant gain stabilization problem for an open-loop unstable plant. From (4), it is seen that an unstable open-loop plant can be stabilized using a constant gain only if it has a single unstable pole. This means that an unstable but stabilizable plant must necessarily have $a_1 > 0$ and $a_o < 0$. As before, let us assume that $k > 0$ and $L > 0$.

Theorem 3 *Under the above assumption on k and L, a necessary condition for a gain k_p to simultaneously stabilize the delay-free and the plant with delay is $\left|\frac{a_1}{a_o}\right| > L$. If this necessary condition is satisfied, then the set of all stabilizing gains k_p for a given open-loop unstable plant with transfer function G(s) as in (3) is given by*

$$-\frac{a_o}{k} < k_p < \frac{a_1 z_1}{kL\sin(z_1)} \quad (8)$$

where z_1 is the solution of the equation

$$\cot(z) = \frac{z^2 - L^2 a_o}{La_1 z} \quad (9)$$

in the interval $(0,\pi)$.

Proof. According to Theorem 1, we need to check two conditions to ensure the stability of the quasipolynomial $\delta^*(s)$:

First, the condition 2 of Theorem 1 is checked: $E(w_o) = \delta_i'(w_o)\delta_r(w_o) - \delta_r'(w_o)\delta_i(w_o) > 0$ for some w_o in $(-\infty,\infty)$.

Let us take $w_o=z_o=0$. Thus $\delta_i(z_o) = 0$ and $\delta_r(z_o) = kk_p + a_o$. We also have

$$\delta_i'(z) = \left(a_o - \frac{z^2}{L^2} + \frac{a_1}{L}\right)\cos(z)$$
$$-\left(\frac{2}{L^2} z + \frac{a_1}{L} z\right)\sin(z)$$
$$\Rightarrow E(w_o) = \left(a_o + \frac{a_1}{L}\right)\left(kk_p + a_o\right)$$

From (4), it is clear that from the closed-loop stability of the delay-free system, we have $(kk_p + a_o) > 0$. Hence, to have $E(z_o)>0$, we must have $a_o + \frac{a_1}{L} > 0$ or $-\frac{a_1}{a_o} > L$,

$$\Rightarrow \left|\frac{a_1}{a_o}\right| > L.$$

We now check condition 1 of Theorem 1: the interlacing of the roots of $\delta_r(z)$ and $\delta_i(z)$. We can compute the roots of imaginary part, i.e., $\delta_i(z) = 0$. This gives the following equation

$$\left(a_o - \frac{z^2}{L^2}\right)\sin(z) + \frac{a_1}{L} z\cos(z) = 0 \quad (10)$$

From the above equation we can see that $z_o=0$ is a root of the imaginary part. Also, it is clear that $l\pi, l = 1,2,...,$ are not roots of the imaginary part. Thus for $z \neq 0$, we can rewrite the previous equation as

$$\cot(z) = \frac{z^2 - L^2 a_o}{La_1 z} \quad (11)$$

An analytic solution of (11) is difficult to find. However, we can plot the two terms involved in the equation, i.e., $\cot(z)$ and $\frac{z^2 - L^2 a_0}{La_1 z}$ to study the nature of real solutions. Let us denote the positive real roots of (11) by $z_j, j = 1,2,,...,$ arranged in increasing order of magnitude. Clearly the non-negative real roots of the imaginary part satisfy

$$z_1 \in (0,\pi), z_2 \in (\pi,2\pi), z_3 \in (2\pi,3\pi),... \quad (12)$$

Let us now use Theorem 2 to check if $\delta_i(w)$ has only real roots. Substituting $s_1=Ls$ in the expression of $\delta^*(s)$, we can see that for the new quasipolynomials in s_1, $M=2$ and $N=1$. Next we choose $\eta = \pi/4$ to satisfy the requirement of $\delta_i(z)$ does not vanish at $w = \eta$. It can be easily seen that in the interval $[0,2\pi - \pi/4] = [0,7\pi/4]$, $\delta_i(z) = 0$ has three real roots including a real root at origin. Since $\delta_i(z)$ is an odd function it follows that in the interval $[-7\pi/4, 7\pi/4]$, $\delta_i(z)=0$ will have 5 real roots also $\delta_i(z)=0$ has one real root in $(7\pi/4, 9\pi/4]$. Thus, $\delta_i(z)$ has $4N+M=6$ real roots in the interval $[-2\pi+\pi/4, 2\pi+\pi/4]$. Moreover, $\delta_i(z)$ has two real roots in each of the intervals $[-2l\pi+\pi/4, 2(l+1)\pi+\pi/4]$ and $[-2(l+1)\pi+\pi/4, -2l\pi+\pi/4]$ for $l=1,2,....$ Hence, it follows that $\delta_i(z)$ has exactly $4lN+M$ real roots in $[-2l\pi+\pi/4, 2l\pi+\pi/4]$, which by Theorem 2 implies that $\delta_i(z)$ has only real roots.

We now evaluate $\delta_r(z)$ at the roots of the imaginary part $\delta_i(z)$. For $z_o=0$ using (7) we obtain

$$\delta_r(z_o) = kk_p + a_o. \tag{13}$$

For z_j, $j=1,2,,...$ using (7) and (11) we obtain

$$\delta_r(z_j) = kk_p - \frac{a_1 z_j}{L\sin(z_j)} \tag{14}$$

Thus, we obtain

$$\delta_r(z_j) = k\left[k_p - M(z_j)\right] \tag{15}$$

where

$$M(z) = \frac{a_1 z}{kL\sin(z)}. \tag{16}$$

It was stated that $k_p > -\frac{a_o}{k}$. Thus, from (13) we see that $\delta_r(z_o) > 0$. Then, interlacing the roots of $\delta_r(z)$ and $\delta_i(z)$ is equivalent to $\delta_r(z_1)<0, \delta_r(z_2)>0, \delta_r(z_3)<0,$ and so on. Using this fact and equations (13) and (15) we obtain

$$\begin{aligned}
&\delta_r(z_o) > 0 \Rightarrow k_p > -\frac{a_o}{k}\\
&\delta_r(z_1) < 0 \Rightarrow k_p < M(z_1) =: M_1\\
&\delta_r(z_2) > 0 \Rightarrow k_p > M(z_2) =: M_2\\
&\quad\vdots
\end{aligned} \tag{17}$$

From (12) we can see that z_j for odd values of *j*, are either in the first or second quarter. Thus for odd values of *j*, $\sin(z_j)>0$ and from (16), we conclude that $M(z_j)>0$ for odd values of the parameter *j*. Similarly, we can see that $M(z_j)<0$ for even values of parameter *j*. thus the inequalities (17) can be rewritten as

$$k_p > -\frac{a_o}{k}$$

and

$$\max_{j=2,4,6,...}\left(M_j\right) < k_p < \min_{j=1,3,5,...}\left(M_j\right) \tag{18}$$

Using (11), we have

$$\begin{aligned}
&M(z_j) = \pm\frac{1}{kL^2}\sqrt{(z_j^2 - L^2 a_o)^2 + L^2 a_1^2 z_j^2}\\
&\Rightarrow M(z_j) = \pm\frac{1}{kL^2}\sqrt{z_j^4 + L^2(a_1^2 - 2a_o)z_j^2 + L^4 a_o^2}
\end{aligned} \tag{19}$$

where the plus sign (+) is used for odd values of *j*, and the minus sign (-) is used for even values of *j*. Notice that since $a_o < 0$, we have $a_1^2 \geq 2a_o$. Thus, (19) we can see that $M(z_j)$ is a monotonically increasing function for odd values of *j* and it is a monotonically decreasing function for even values of *j*. Moreover, it is seen that $M(0) = -\frac{a_o}{k}$. Using these observations, the bounds of k_p can be expressed as

$$-\frac{a_o}{k} < k_p < \frac{a_1 z_1}{kL\sin(z_1)}. \tag{20}$$

Note that for values of k_p in the above range, the interlacing property and the fact that the roots of $\delta_i(z)$ are all real can be used in Theorem 3 to guarantee that $\delta_r(z)$ also has only real roots. Thus all the conditions of Theorem 1 are satisfied and this completes the proof.

5 Numerical example

We now present to illustrate the applications of Theorem 3.

Example Consider the constant gain stabilization problem for a plant with parameters as $k=0.442, L=3.6\,\text{sec}, a_1=1.2148, a_o=-0.151$ [1]. The transfer function of this plant is as follows

$$G(s) = \frac{0.442}{s^2 + 1.2148s - 0.151}e^{-3.6s}$$

Since the plant is open-loop unstable we will use Theorem 3 to obtain the set of stabilizing gains. According to (11), we compute $z_1 \in (0,\pi)$ satisfying

$$\cot(z) = \frac{z^2 + 1.95696}{4.37328z}$$

Solving the above equation we obtain $z_1 = 0.97179$. Thus, from (20) the set of stabilizing gains is given by

$$0.34163 < k_p < 0.97179.$$

Open-loop system response of the above system is depicted in Fig. 2. As it can be seen the system is open-loop unstable. The closed-loop system response with a k_p of the above range, $k_p = 0.52$, is shown in the Fig. 2, which is stable.

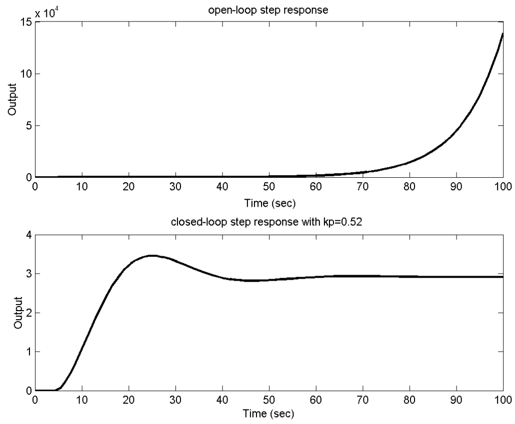

Fig. 2. Open-loop and closed-loop responses.

6 Conclusion

In this paper, we have obtained a characterization of the complete set of stabilizing constant feedback gains for a given open-loop unstable second-order plus time delay system. This result is based on an extension of the Hermite-Biehler Theorem to quasipolynomials, due to Pontryagin, and opens up the possibility of designing feedback gains to optimize given performance criteria. The characterization also provides us with a tool to understand the relationship between the time delay exhibited by a system and its stabilization using a feedback gain. Using the obtained feedback gains, the desired closed-loop performance such as settling time, rise time, overshoot, etc. can be received. This research is preparation to analytically calculate of all stabilizing PI and PID case for second-order systems with time delay, which is a challenging problem.

References:

[1] Astrom, K., and Hagglund, T. *PID Controllers: theory, design, and tuning*. Research Triangle Park: Instrument Society of America, 1995.

[2] Ho, M. T., Datta, A., and Bhattacharyya, S. P. Control system design using low order controllers: Constant gain, PI and PID. *In Proc. American Control Conference (ACC).* Albuquerque, NM, pp.571-578, 1995.

[3] Ho, M.T., Datta, A., and Bhattacharyya, S. P. A linear programming characterization of all stabilizing PID controllers. *In Proc. American Control Conference (ACC)* Albuquerque, NM, pp.3922-3928, 1997.

[4] Kharitonov, V. L., and Zhabko, A. P. Robust stability of time delay systems. *IEEE Transactions on Automatic Control*, *39*(12), pp.2388-2397, 1994.

[5] Silva, G.J., Datta, A., Bhattacharyya, S.P. PI Stabilization of first-order systems with time delay, *Automatica*, 37, pp.2025-2031, 2001.

[6] Silva, G.J., Datta, A., Bhattacharyya, S.P. Stabilization of first-order systems with time delay using the PID controller, *in Proc. American control Conference (ACC)*, Arlington, VA, pp.4650-4655, 2001.

[7] Silva, G.J., Datta, A., Bhattacharyya, S.P. New results on the synthesis of PID controllers, *IEEE Trans. Automatic Control*, 47(2), pp.241-252, 2002.

[8] Gantmacher, F. R. *The theory of matrices*. New York: Chelsea Publishing Company, 1959.

[9] Bhattacharyya, S. P., Chapellat, H., and Keel, L. H. *Robust control: The parametric approach*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[10] Pontryagin, L. S. On the zeros of some elementary transcendental function (English Translation). *American Mathematical Society* T*ranslation*, *2*, 95-110, 1955.

[11] Bellman, R., & Cooke, K. L. *Differential-difference equations*. London: Academic Press Inc, 1963.