Periodic Oscillatory Solution in Delayed Competitive-cooperative Neural Networks: A Decomposition Approach*

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Abstract: In this paper, the problems of exponential convergence and the exponential stability of the periodic solution for a general class of non-autonomous competitive-cooperative neural networks are analyzed via the decomposition approach. The idea is to divide the connection weights into inhibitory or excitatory types and thereby to embed a competitive-cooperative delayed neural network into an augmented cooperative delay system through a symmetric transformation.Some simple necessary and sufficient conditions are derived to ensure the componentwise exponential convergence and the exponential stability of the periodic solution of the considered neural networks. These results generalize and improve the previous works, and they are easy to check and apply in practice.

Keywords: Competitive-cooperative neural networks; Time delay; Trapping region; Exponential convergence; Periodic solution.

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1 Introduction

Competitive and cooperative mechanisms arise from biological networks. The excitatory-inhibitory connectivity structure of neural networks is similar to the competitive-cooperative mechanism of biological networks. By competitive connection we mean the way in which a neuron's firing inhibits the firing the other neurons. Conversely, cooperative connection refers to the way in which a neuron's firing excites the firing of others. In most cases, the activation of a neuron is characterized by a sigmoid function. The competitive-cooperative connection pattern can thus be recognized by the sign of the weights: positive weights are due to excitatory synapses, negative weights are due to inhibitory synapses, while a zero weight indicates no neural connection at all.

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These mechanisms play an important role in the collective dynamics in neural networks. The stability analysis of such systems has raised great interest [1-8]. Recently, several efforts have been devoted to the study of general competitive-cooperative neural networks using monotone dynamical system theory [7-9]. In particular, a decomposition approach has been proposed in [7-9], that consists of dividing the connectivity of a neural network into an augmented cooperative dynamical system. Using this method, several necessary and sufficient conditions on guaranteed component exponential convergence have been established for competitive-cooperative neural networks with delay in [9]. In the same way, some sufficient and necessary conditions on componentwise exponential convergence have been established for discrete-time neural networks in [7]. A similar approach that embeds a competitive-cooperative neural networks into a larger cooperative system has also been presented in [8] to confirm the stabilization effect of inhibitory self-connections on general delayed neural networks.

Studies on neural dynamical systems not only involve discussion of stability property, but also involve many dynamics behavior such as periodic oscillatory, bifurcation and chaos. In many applications, the property of periodic solutions is of great interest. To the best of our knowledge, few authors study the periodic solution by the discussion of componentwise exponential convergence and the application of contraction mapping principle. We will extend the studies in [9] and give some sufficient and necessary conditions on componentwise exponential convergence for the more general class of non-autonomous competitive-cooperative neural networks. In this paper, we do not require the activation functions to be bounded, differentiable and global Lipschitz continuous; also we do not assume that the considered model has any equilibriums. Specially, we give conditions on the global exponential stability and the existence of the periodic solution of delayed neural networks by the method using in [10-13,18]. In addition, one example is given to illustrate the results.

2 Preliminaries

2.1 *Model description*

The delay neural networks we consider are modelled by the following nonlinear functional equation

$$\frac{dx}{dt} = -Ax(t) + Bf(x(t)) + Cg(x(t-\tau)) + I(t), \tag{1}$$

where $x(t) \in R^n$ is the neural state vector at time t, and $\tau > 0$ is the time delay in the networks; $A = diag[a_1, a_2, \cdots, a_n]$ with every $a_i > 0$ is the relax matrix, the entries of B and C may be positive (excitatory synapses) or negative (inhibitory synapses); the last term $I(t)$ is the bounded external input function to the networks; $f(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^T$ and $g(x) = [g_1(x_1), g_2(x_2), \cdots, g_n(x_n)]^T$ are vector-valued output functions which possess the following properties:

(H_1) f_i, g_i are continuous and monotone nondecreasing , $i = 1, 2, \cdots, n$;

(H_2) $f_i(r_1) \leq \alpha_i r_1, g_i(r_1) \leq \beta_i r_1$, for any $r_1 \in R^+$, where $\alpha_i > 0, \beta_i > 0$ are constants,

$i = 1, 2, \cdots, n$.
For convenience, the sector nonlinear function class $\digamma$ is defined by the functions which satisfy (H_1) and (H_2).

Assume that the nonlinear system (1) is supplemented with initial values of the type $x(t) = \phi(t), t \in [-\tau, 0]$. It is usually assumed that the given n-vector function ϕ is continuous though it need only be measurable for Eq.(1) to be well defined. Here, we also assume a bounded and piecewise continuous initial function with finite discontinuity points.

To characterize the dynamical behavior of model (1), we consider two bounded, continuous and differentiable functions $\xi(t)$, $\zeta(t) : [-\tau, \infty) \to R^n$, with $\xi(t) > 0$, $\zeta(t) > 0$ and define the time-variant set

$$\Omega_{\xi,\zeta}(t) = \{x \in R^n : -\xi(t) \le x(t) \le \zeta(t)\},$$

where and throughout inequalities between vectors are in componentwise sense.

Definition 1. If for every $f, g \in \digamma$, and for any $t_0 > 0$, the solution of Eq.(1) satisfies $x(t) \in \Omega_{\xi,\zeta}(t)$ for $t \ge t_0$ whenever $x(t_0 + \theta) \in \Omega_{\xi,\zeta}(t_0 + \theta)$ for $\theta \in [-\tau, 0]$, we call the set $\Omega_{\xi,\zeta}(t)$ a guaranteed trapping region of model (1). That is, $\Omega_{\xi,\zeta}(t)$ is such a set in R^n where solutions, once they enter, can not leave as time increases.

Definition 2. If $\Omega_{\xi,\zeta}(t)$ is a guaranteed trapping region, we further impose certain restriction on the set $\Omega_{\xi,\zeta}(t)$ by letting

$$\xi(t) = \alpha e^{-\sigma t}, \quad \zeta(t) = \beta e^{-\sigma t}, \tag{2}$$

for some scalar $\sigma > 0$ and two constant vectors $\alpha, \beta \in R^n$ with $\alpha, \beta > 0$, then the system is said to be guaranteed componentwise exponentially convergent (GCEC).

We will establish necessary and sufficient conditions for the guaranteed trapping region. This is done by using the decomposition approach to be developed below.

2.2 *A decomposition approach*

We split the connection matrices B and C into two parts, respectively:

$$B = B^+ - B^-, \quad C = C^+ - C^-,$$

with $b^+_{ij} = \max\{b_{ij}, 0\}$, $c^+_{ij} = \max\{c_{ij}, 0\}$ signifying the excitatory weights and $bij^- = \max\{-b_{ij}, 0\}$, $c^-_{ij} = \max\{-c_{ij}, 0\}$ the inhibitory weights. The system (1) can be rewritten as

$$\frac{dx}{dt} = -Ax(t) + (B^+ - B^-)f(x(t)) + (C^+ - C^-)g(x(t-\tau)) + I(t). \tag{3}$$

Now take the symmetric transformation $y = -x$ from Eq.(3), it follows that

$$\frac{dy}{dt} = -Ay(t) + B^+\tilde{f}(x(t)) + B^-f(x(t)) + C^+\tilde{g}(x(t-\tau)) + C^-g(y(t-\tau)) - I(t), \tag{4}$$

$$\frac{dx}{dt} = -Ax(t) + B^-\tilde{f}(x(t)) + B^+f(x(t)) + C^-\tilde{g}(x(t-\tau)) + C^+g(y(t-\tau)) + I(t), \tag{5}$$

where $\tilde{f}(u) = -f(-u)$, $\tilde{g}(u) = -g(-u)$. Obviously, $\tilde{f}$, $\tilde{g} \in \mathcal{F}$. Accordingly, we introduce the following augmented system:

$$\dot{d}(t) = -\Lambda d(t) + \Pi_1 h_1(d(t)) + \Pi_2 h_2(d(t-\tau)) + \tilde{I}(t), \tag{6}$$

where

$$d(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad \Lambda = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} C^+ & C^- \\ C^- & C^+ \end{bmatrix},$$

$$h_1(d(t)) = \begin{bmatrix} \tilde{f}(p(t)) \\ f(q(t)) \end{bmatrix}, \quad h_2(d(t)) = \begin{bmatrix} \tilde{g}(p(t)) \\ g(q(t)) \end{bmatrix}, \quad \tilde{I}(t) = \begin{bmatrix} -I(t) \\ I(t) \end{bmatrix}.$$

Noticing the (element-wise) non-negativity of Π_i (i=1, 2), system (6) itself is cooperative and hence possesses the following important order-preserving property.

Lemma 1. Let $u(t)$ and $v(t)$ be the solutions of Eq.(6). Then $u(t_0+\theta) \leq v(t_0+\theta)$ for $\theta \in [-\tau, 0]$ implies $u(t) \leq v(t)$ for $t \geq t_0 \geq 0$. Moreover, if $u(t)$ satisfies $\dot{\omega}(t) \geq -\Lambda\omega(t) + \Pi_1 h_1(\omega_1(t)) + \Pi_2 h_2(\omega(t-\tau)) + \tilde{I}(t)$ for $t \geq t_0 \geq 0$, then $u(t_0+\theta) \leq \omega(t_0+\theta)$ for $\theta \in [-\tau, 0]$ implies $u(t) \leq \omega(t)$ for $t \geq t_0 \geq 0$.

This is a specialization of general results (e.g., [14,17]) on monotone dynamics of cooperative delay differential systems to Eq.(6). It indicates that the sates of a cooperative system will retain for all time their initial relationship, a partial ordering induced by the subset of non-negative state vectors of the state space. In the literature, such results are also referred to as comparison principles for delay systems [16].

Lemma 2. Assume for Eqs.(3) and (6) that the initial condition $-p(t_0+\theta) \leq x(t_0+\theta) \leq q(t_0+\theta)$ holds for $\theta \in [-\tau, 0]$, then $-p(t) \leq x(t) \leq q(t)$ for $t \geq t_0 \geq 0$.

Proof. Since $x(t)$ is the solution of Eq.(3),

$$u(t) = \begin{bmatrix} -x(t) \\ x(t) \end{bmatrix}$$

is the solution of Eq.(6).

Then,

$$u(t) = \begin{bmatrix} -x(t) \\ x(t) \end{bmatrix} \text{ and } v(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$$

are solutions of Eq.(6). From $-p(t_0+\theta) \leq x(t_0+\theta) \leq q(t_0+\theta)$, we have

$$u(t_0+\theta) = \begin{bmatrix} -x(t_0+\theta) \\ x(t_0+\theta) \end{bmatrix} \leq \begin{bmatrix} -p(t_0+\theta) \\ q(t_0+\theta) \end{bmatrix} = v(t_0+\theta).$$

By Lemma 1, we can deduce $u(t) \leq v(t)$, that is $-p(t) \leq x(t) \leq q(t)$. This completes the proof. □

3 Main results

3.1 *Componentwise exponential convergence*

We present here necessary and sufficient conditions for trapping regions of model (1).
Theorem 1. The set $\Omega_{\xi,\zeta}(t)$ is a guaranteed trapping region for model (1) if and only if

$$\dot{\Gamma}(t) \geq (-\Lambda + \Pi_1\Sigma_1)\Gamma(t) + \Pi_2\Sigma_2\Gamma(t-\tau) + \tilde{I}(t), \quad t \geq 0, \tag{7}$$

where $\Gamma(t) = [\xi(t)^T\zeta(t)^T]^T$, $\Sigma_1 = diag[\alpha_1, \cdots, \alpha_n, \alpha_1, \cdots, \alpha_n]$, $\Sigma_2 = diag[\beta_1, \cdots, \beta_n, \beta_1, \cdots, \beta_n]$.
Proof. We first proceed to show the efficiency of condition (7). From the definition of the class $\digamma$ it is easy to see that $h_1(\Gamma(t)) \leq \Sigma_1\Gamma(t),\ h_2(\Gamma(t-\tau)) \leq \Sigma_2\Gamma(t-\tau)$. Then by noticing the non-negativity of the entries of matrices Π_1 and Π_2, it follows that $\Pi_1 h_1(\Gamma(t)) \leq \Pi_1\Sigma_1\Gamma(t),\ \Pi_2 h_2(\Gamma(t-\tau)) \leq \Pi_1\Sigma_1\Gamma(t-\tau)$. Hence, if condition (7) holds, then we have

$$\dot{\Gamma}(t) \geq (-\Lambda + \Pi_1 h_1(\Gamma(t)) + \Pi_2 h_2(\Gamma(t-\tau)) + \tilde{I}(t), \quad t \geq 0. \tag{8}$$

Now consider an arbitrary $f, g \in \digamma$ and let $x(t)$ be the solution of the corresponding Eq.(1) with the initial value satisfying $-\xi(\theta) \leq x(\theta) \leq \zeta(\theta)$ for $\theta \in [-\tau, 0]$. Take in Eq.(6) $P(\theta) = \xi(\theta),\ q(\theta) = \zeta(\theta)$ and without loss of generality, let the initial time $t_0 = 0$, then by Lemma 2,

$$-p(t) \leq x(t) \leq q(t), \quad t \geq 0. \tag{9}$$

Meanwhile, let $u(\theta) = [p(\theta)^T q(\theta)^T]^T = \Gamma(\theta)$. From condition (8) and Lemma 1,

$$u(t) \leq \Gamma(t), \quad t \geq 0.$$

This together with condition (9) yields

$$-\xi(t) \leq x(t) \leq \zeta(t), \quad t \geq 0.$$

Therefore, condition (7) is sufficient for $\Omega_{\xi,\zeta}t$ to be a guaranteed trapping region for the system.

To see the necessity of the condition, let us suppose that $\Omega_{\xi,\zeta}t$ is a guaranteed trapping region for system (1), but the condition (7) is false. Then there should exist an index $i \in \{1, \cdots, n\}$ and a time $t_1 > 0$, such that

$$\dot{\xi}_i(t_1) < -a_i\xi_i(t_1) + B_i^+K_1\xi(t_1) + B_i^-K_1\zeta(t_1) + C_i^+K_2\xi(t_1-\tau) + C_i^-K_2\zeta(t_1-\tau) - I_i(t_1) \tag{10}$$

or

$$\dot{\zeta}_i(t_1) < -a_i\zeta_i(t_1) + B_i^+K_1\xi(t_1) + B_i^+K_1\zeta(t_1) + C_i^-K_2\xi(t_1-\tau) + C_i^+K_2\zeta(t_1-\tau) + I_i(t_1), \tag{11}$$

where B_i^+, B_i^-, C_i^+ and C_i^- are the ith row vectors of matrices B^+, B^-, C^+ and C^-, respectively, and $K_1 = diag[\alpha_1, \cdots, \alpha_n]$, $K_2 = diag[\beta_1, \cdots, \beta_n]$.
Next pick for Eq.(1) particular activation functions $f(x)$, $g(x)$ defined by $f(x) = \frac{1}{2}(|K_1(x + \delta_1)| -$

$|K_1(x-\delta_1)|)$ and $g(x)=\frac{1}{2}(|K_2(x+\delta_2)|-|K_2(x-\delta_2)|)$ with $\delta_1=\xi(t_1)+\zeta(t_1)$ and $\delta_2=\xi(t_1-\tau)+\zeta(t_1-\tau)$.

Clearly $f(x)$, $g(x)\in \mathcal{F}$ and

$$f(-\xi(t_1))=-k_1\xi(t_1),\quad f(\zeta(t_1))=k_1\zeta(t_1), \tag{12}$$

$$g(-\xi(t_1-\tau))=-k_2\xi(t_1-\tau),\quad g(\zeta(t_1-\tau))=k_2\zeta(t_1-\tau). \tag{13}$$

Then, consider the solution $x(t)$ with $x(t_1+\theta)\in\Omega_{\xi,\zeta}(t_1+\theta)$ for $\theta\in[-\tau,0]$ defined by if $b_{ii}\geq 0$

$$x_i(t_1+\theta)=\begin{cases}-\xi_i(t_1), & \text{if } \theta=0,\\ -\xi_i(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ii}\geq 0,\\ -\zeta_i(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ii}<0,\end{cases}$$

and for $j\neq i, j=1,\cdots,n$

$$x_j(t_1+\theta)=\begin{cases}-\xi_j(t_1), & \text{if } \theta=0,\ b_{ij}\geq 0,\\ \zeta_j(t_1+\theta), & \text{if } \theta=0,\ b_{ij}<0,\\ -\xi_j(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ij}\geq 0,\\ \zeta_j(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ij}<0.\end{cases}$$

From Eqs. (3), (10), (12) and (13), the ith component of the vector $x(t_1)$ satisfies

$$\begin{aligned}\dot{x}_i(t_1) &= -a_ix_i(t_i)+(B_i^+-B_i^-)f(x(t_1))+(C_i^+-C_i^-)g(x(t_1-\tau))+I_i(t_1)\\ &= a_i\xi_i(t_1)xi(t_1))+B_i^+f(-\xi(t_1))-B_i^-f(\zeta(t_1))+C_i^+g((-\xi(t_1-\tau))\\ &\quad -C_i^-g(\zeta(t_1-\tau))+I_i(t_1)\\ &= a_i\xi_i(t_1)xi(t_1))-B_i^+K_1(\xi(t_1)-B_i^-K_1(\zeta(t_1))-C_i^+K_2((\xi(t_1-\tau))\\ &\quad -C_i^-K_2(\zeta(t_1-\tau))+I_i(t_1)\\ &< -\dot{\xi}_i(t_1).\end{aligned}$$

By noting $x_i(t_1)=-\xi_i(t_1)$ as defined above, this implies $x_i(t_1+\Delta)<-\xi(t_1+\Delta)$ for sufficiently small $\Delta>0$,

if $b_{ii}<0$

$$x_i(t_1+\theta)=\begin{cases}\zeta_i(t_1), & \text{if } \theta=0,\\ \zeta_i(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ii}\geq 0,\\ -\xi_i(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ii}<0,\end{cases}$$

and for $j\neq i, j=1,\cdots,n$

$$x_j(t_1+\theta)=\begin{cases}\zeta_j(t_1), & \text{if } \theta=0,\ b_{ij}\geq 0,\\ -\xi_j(t_1+\theta), & \text{if } \theta=0,\ b_{ij}<0,\\ \zeta_j(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ij}\geq 0,\\ -\xi_j(t_1+\theta), & \text{if } \theta\in[-\tau,0),\ c_{ij}<0.\end{cases}$$

From Eqs. (3), (11), (12) and (13), the ith component of the vector $x(t_1)$ satisfies

$$\begin{aligned}
\dot{x}_i(t_1) &= -a_i x_i(t_i) + (B_i^+ - B_i^-) f(x(t_1)) + (C_i^+ - C_i^-) g(x(t_1 - \tau)) + I_i(t_1) \\
&= -a_i \zeta_i(t_1) xi(t_1)) + B_i^+ f(\zeta(t_1)) - B_i^- f(-\xi(t_1)) + C_i^+ g((\zeta(t_1 - \tau)) \\
&\quad -C_i^- g(-\xi(t_1 - \tau)) + I_i(t_1) \\
&= -a_i \zeta_i(t_1) xi(t_1)) + B_i^+ K_1(\zeta(t_1) + B_i^- K_1(\xi(t_1)) - C_i^+ K_2((\zeta(t_1 - \tau)) \\
&\quad +C_i^- K_2(\xi(t_1 - \tau)) + I_i(t_1) \\
&> \dot{\zeta}_i(t_1).
\end{aligned}$$

By noting $x_i(t_1) = -\zeta_i(t_1)$ as defined above, this implies $x_i(t_1 + \Delta) > \zeta_i(t_1 + \Delta)$ for sufficiently small $\Delta > 0$.
Hence, $\Omega_{\xi,\zeta}(t_1)$ could not be a guaranteed trapping region for system (1), showing the necessity of condition (7). This completes the proof. □

By taking $\xi(t)$ and $\zeta(t)$ to be two constant vectors, we obtain a special guaranteed trapping region.

Corollary 1. For two constant vectors $\alpha, \beta \in R^n$ with $\alpha > 0$ and $\beta > 0$, the set $\Omega_{\alpha,\beta} = \{x \in R^n : -\alpha \leq x \leq \beta\}$ is a guaranteed trapping region for system (1) if and only if

$$(\Lambda - \Pi_1 \Sigma_1 - \Pi_2 \Sigma_2)\eta \geq \tilde{I}(t), \tag{14}$$

where $\eta = [\alpha^T \beta^T]^T$.

Remark 1. The above results depend only on $\alpha_i, \beta_i (i = 1, \cdots, n)$ and thus are applicable to the whole set of $\mathcal{F}$. For a neural network (1) with given f and g, one can similarly conclude that the set $\Omega_{\xi,\zeta}(t)$ is a trapping region if and only if

$$\dot{\Gamma}(t) \geq -\Lambda \Gamma(t) + \Pi_1 h_1(\Gamma(t)) + \Pi_2 h_2(\Gamma(t - \tau)) + \tilde{I}(t), \quad t \geq 0, \tag{15}$$

where $h_1(\Gamma(t))$, $h_2(\Gamma(t))$ and $\Gamma(t)$ are specified as in Eqs.(6) and (7). Also for the set $\Omega_{\alpha,\beta}$ in Corollary 1, condition (14) reads

$$\Lambda \eta - \Pi_1 h_1(\eta) - \Pi_2 h_2(\eta) \geq \tilde{I}(t).$$

If h_1 and h_2 are bounded, one can pick a constant vector $\eta > 0$ to fulfill this condition. This indicates that a delay neural network with bounded activation function always has a trapping region. (see also [15])

By further assuming restriction of the set $\Omega_{\xi,\zeta}(t)$ in Theorem 1, we obtain the following componentwise convergence result. Particularly, inserting special $\xi(t)$ and $\zeta(t)$ specified by Eq.(2) into condition (7) leads the following necessary and sufficient condition.

Theorem 2. Model (1) is GCEC if and only if there are two constant vectors $\alpha, \beta \in R^n$ with $\alpha, \beta > 0$ and a scalar $\sigma > 0$ such that

$$(\sigma I - \Lambda + \Pi_1 \Sigma_1 + e^{\sigma\tau} \Pi_2 \Sigma_2)\eta + e^{-\sigma t} \tilde{I}(t) \leq 0, \quad t \geq 0, \tag{16}$$

where $\eta = [\alpha^T \beta^T]^T$ and I is an identity matrix with appropriate dimensions.

Corollary 2. Model (1) is GCEC if and only if there are two constant vectors $\alpha, \beta \in R^n$ with $\alpha, \beta > 0$ and a scalar $\sigma > 0$ such that

$$(\sigma I - A + |B|K_1 + e^{\sigma\tau}|C|K_2)\rho \le 0, \tag{17}$$

where $\rho = \alpha + \beta > 0$, $|B| = [\ |b_{ij}|\ ] = B^+ + B^-$ and $|C| = [\ |c_{ij}|\ ] = C^+ + C^-$. **Proof.** Rewrite condition (16) as

$$(\sigma I - A + B^+K_1 + e^{\sigma\tau}C^+K_2)\alpha + (B^-K_1 + e^{\sigma\tau}C^-K_2)\beta - e^{-\sigma t}I(t) \le 0,$$

$$(\sigma I - A + B^+K_1 + e^{\sigma\tau}C^+K_2)\beta + (B^-K_1 + e^{\sigma\tau}C^-K_2\alpha + e^{-\sigma t}I(t) \le 0.$$

Adding them we have

$$(\sigma I - A + |B|K_1 + e^{\sigma\tau}|C|K_2)\rho \le 0.$$

This completes the proof. □

Next, we make some comments on the symmetrical case $\alpha = \beta$. In this case, condition (17) reduce to

$$(\sigma I - A + |B|K_1 + e^{\sigma\tau}|C|K_2)\alpha \le 0. \tag{18}$$

Clearly, it is equivalent to the existence of a constant vector $\alpha > 0$ such that

$$(A - |B|K_1 - |C|K_2)\alpha > 0. \tag{19}$$

Noticing the non-positivity of every off-diagonal entries of matrix $A-|B|K_1-|C|K_2$, condition (19) is in turn tantamount to saying that the matrix $A - |B|K_1 - |C|K_2$ is an M matrix [9]. Further, by the properties of M matrices, it is also equivalent to

$$\begin{vmatrix} h_{11} & \cdots & h_{1i} \\ \vdots & \ddots & \vdots \\ h_{i1} & \cdots & h_{ii} \end{vmatrix} > 0, \quad i = 1, \cdots, n, \tag{20}$$

where

$$h_{ij} = \begin{cases} a_i - \alpha_i|b_{ii}| - \beta_i|c_{ii}|, & \text{for } i = j, \\ -\alpha_j|b_{ij}| - \beta_j|c_{ij}|, & \text{for } i \ne j. \end{cases}$$

Remark 2. Observe that, although conditions (17),(18),(19) and (20) are all necessary and sufficient for GCEC of the system, the trajectory behavior that they can yield for a network may be quite different. The first two conditions can guarantee a network to be convergent with a prescribed exponential decay rate and trajectory bounds, described respectively by σ and α, β, while the last two only ensure exponential convergence in a network, saying nothing about nothing decay rate explicitly (condition (18) also provides an estimate of the trajectory bound). On the other hand, it should be noted that conditions (18) and (19) are delay independent. This is of practical significance in the case where time delays exist but their magnitudes could not be evaluated accurately.

3.2 *The existence and exponential stability of periodic solution*

Theorem 3. Under the assumption of (H_1) and (H_2), the system

$$\frac{dx}{dt} = -Ax(t) + Bf(x(t)) + Cg(x(t-\tau)) + I(t), \tag{21}$$

where $I(t+\omega) = I(t)$, that is, $I(t)$ is a periodic function, there exists exactly one ω-periodic solution of (21) and all other solutions of Eq.(21) converge exponentially to it as $t \to +\infty$ if any one of the following conditions holds:

(I) $(\sigma I - \Lambda + \Pi_1\Sigma_1 + e^{\sigma\tau}\Pi_2\Sigma_2)\eta + e^{-\sigma t}\tilde{I}(t) \leq 0, \quad t \geq 0,$

(II) $(\sigma I - A + |B|K_1 + e^{\sigma\tau}|C|K_2)\rho \leq 0,$

(III) there exists a positive vector α such that$(A - |B|K_1 - |C|K_2)\alpha > 0,$

(IV) $A - |B|K_1 - |C|K_2$ is an M matrix,

where notations denote as above.

Proof. Let $C = C([-\tau, 0], R^n)$ be the Banach space of continuous functions which map $[-\tau, 0]$ into R^n with the topology of uniform convergence. For any $\phi \in C$, we define $\|\phi\| = \sup_{\theta\in[-\tau,0]} |\phi(\theta)|$ in which $|\phi(\theta)| = \sum\limits_{i=1}^{n}[\phi_i(\theta)]^2$. For any $\phi, \varphi \in C$, we denote the solutions of Eq.(21) by $(0, \phi)$ and $(0, \varphi)$ by $x(t, \phi) = [x_1(t, \phi)), \cdots, x_n(t, \phi)]^T$, $x(t, \varphi) = [x_1(t, \varphi)), \cdots, x_n(t, \varphi)]^T$, respectively. Define $x_t(\phi) = x(t + \theta, \phi), \theta \in [-\tau, 0], t \geq 0$.

From Theorem 2, we have $\|x_t(\phi)\| \leq \max\{\|\alpha\|, \|\beta\|\}e^{-\sigma t}$ and $\|x_t(\varphi)\| \leq \max\{\|\alpha\|, \|\beta\|\}e^{-\sigma t}$. Let $M = \frac{2\max\{\|\alpha\|,\|\beta\|\}}{\|\phi-\varphi\|}$, clearly,

$$\|x_t(\phi) - x_t(\varphi)\| \leq \|x_t(\phi)\| + \|x_t(\varphi)\| \leq M\|\phi - \varphi\|e^{-\sigma t}. \tag{22}$$

We can choose a positive integer m such that $Me^{-\sigma m\omega} \leq \frac{1}{9}$.

Define a Poincare mapping $P : C \to C$ by $P\phi = x_\omega(\phi)$, we can derive from (22) that $\|P^m\phi - P^m\varphi\| \leq \frac{1}{9}\|\phi - \varphi\|$.

This implies that P^m is a contraction mapping, hence there exists a unique fixed point $\phi^* \in C$ such that $P^m\phi^* = \phi^*$. Note that $P^m(P\phi^*) = P(P^m\phi^*) = P\phi^*$.

This shows that $P\phi^* \in C$ is also a fixed point of P^m, so $P\phi^* = \phi^*$, ie, $x_\omega(\phi^*) = \phi^*$.

Let $x(t, \phi^*)$ be the solution of (21) through $(0, \phi^*)$. Obviously, $x(t+\omega, \phi^*)$ is also a solution of (21), and $x_{t+\omega}(\phi^*) = x_t(x_\omega(\phi^*)) = x_t(\phi^*)$ for $t \geq 0$, therefore $x(t + \omega, \phi^*) = x(t, \phi^*)$ for $t \geq 0$, showing that $x(t, \phi^*)$ is exactly an ω-periodic solution of (21), and it is easy to see that all other solutions of (21) converge exponentially to it as $t \to +\infty$. This completes the proof. $\square$

4 An example

Consider the competitive-cooperative neural networks with delay

$$\begin{cases} \frac{dx_1}{dt} = -2x_1(t) + \frac{1}{2}f_1(x_1(t)) + \frac{1}{2}g_1(x_1(t-\tau)) + \sin t \\ \frac{dx_2}{dt} = -2x_2(t) + \frac{1}{2}f_2(x_2(t)) + \frac{1}{2}g_2(x_2(t-\tau)) + \cos t, \end{cases}$$

where $\tau = 1$,

$$f_i(x_i) = g_i(x_i) = \begin{cases} x_i, & \text{if } x_i \geq 0, \\ x_i^3, & \text{if } x_i < 0, \end{cases}$$

for $i = 1, 2$,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Obviously,

$$K_1 = K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Take

$$\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then $(A - B|K_1| - C|K_2|)\alpha > 0$.
Therefore, all the solutions of the system are componentwise exponential convergence and the system exists a unique exponential stability periodic solution.

Obviously, in this example the activation functions are not global Lipschitz continuous, therefore the previous results are not plausible to it. Using our criteria given in this paper, we can deduce the system exists an exponential stability periodic solution.

Let the system be supplemented with the four different constant initial values, we give the following diagram to explain our example.

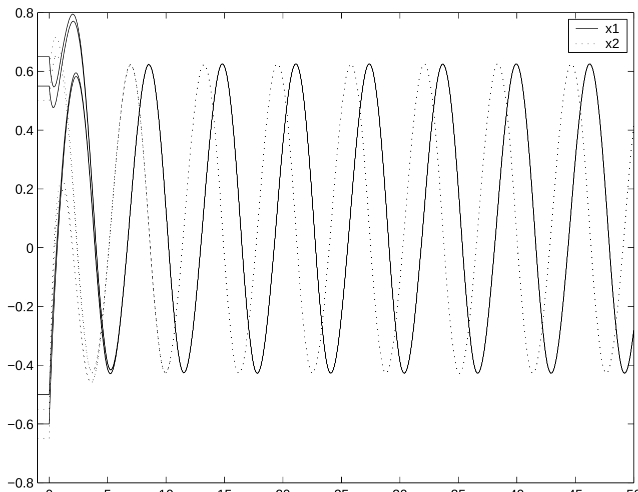

Fig.1 Transient response of state variables

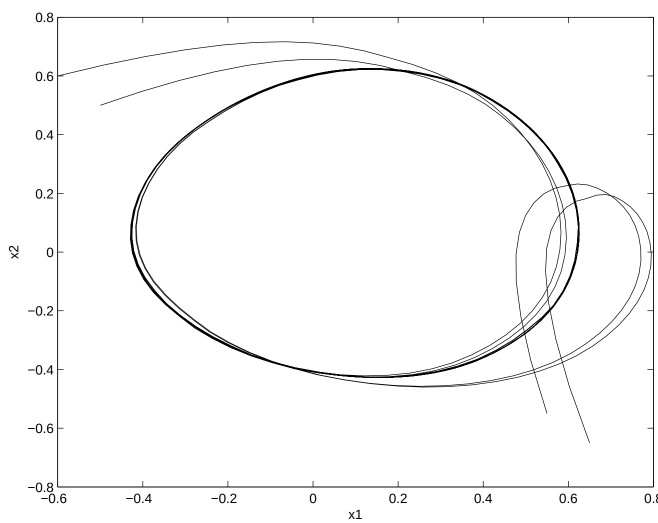

Fig.2 Phase plots of state variable $(x_1(t), x_2(t))$

5 Conclusions

We have developed a decomposition method in the more general competitive-cooperative delay neural networks. Simple necessary and sufficient conditions have been established to guarantee trapping region regions and guaranteed componentwise exponential convergence. Moreover, a set of criteria have been derived ensuring the exponential stability and the existence of periodic solution for the considered model. The criteria does not require the activation functions f_i, g_i to be differential, bounded and global Lipschitz continuous and not also require the weight-connected matrices B, C to be symmetric and the external input vector to be constants. Therefore, the results are of practical significance in designing a network with desired performance.

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