Nonlinear Algebraic Identifiability and Equalizability

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Abstract: - In this contrubition, the problem of identifiability and equalizability of nonlinear channels is addressed. We use the difference algebra approach developed by Fliess, that we have presented in preceding communication, to analyse the question of perfect equalizability of multi-user channels. We propose new definitions of identifiability and equalizability for nonlinear channels. A test of those definitions in terms of ranks of Jacobian matrices is presented. The credit of this approach is, among others, that it provides an explicit analytical description of subsets of the input where equalization and identification are impossible.

Key-Words: - Equalizability, Identifiability, Nonlinear channel, Difference Algebra, Invertibility, Observability.

1 Introduction

Channel identification and equalization is a most classic subject of investigation in digital communications systems. Its purpose is to identify the channel coefficients and to recover the transmitted signal from the received one. A vast literature has been devoted to it in the linear case (see [1], [2]). Many real channels, however, possess nonlinearities that make it impossible for linear tools to provide acceptable results. Examples of such systems are encourtered in satellite communications channels, voiceband data transmission systems, high density magnetic recordings, high density optical systems and loudspeaker systems [3], [4], [5].

In this paper, the problem of identifiability and equalizability of nonlinear channels is addressed. We use the algebraic approach, that we have presented in preceding communications (see [6] and [7]). By resorting to the notions of algebraic observability and invertibility, topics which have been widely researched in systems theory, we propose new definitions of identifiability and equalizability. Now, let us annonce the main pratical interest of our concept. Despite its great generality, it is possible to test the identifiability and equalizability of a given channel. We show that the test reduces to the computation of the ranks of some jacobian matrices. This paper is organized as follows. We shall first give, in section 2, a short review on necessary background of difference algebra and on the way it permits to reformulate some parts of systems theory. The linear and nonlinear systems theory are investigated in section 3. In section 4, we give the definitions of algebraic identifiability and algebraic equalizability we introduce in this paper. We provide a test of those definitions and an illustrative example is presented. Finally, we give in section 5 some concluding remarks.

2 Mathematical background

First we give a short rewiew on the necessary background of difference algebra, introduced by Fliess in systems theory, which permit the formulation of invertibility and observability of input-output systems [8], [9], [10].

2.1 Linear algebra

Let $\mathcal{K}$ be a given ground field. Denote by $\mathcal{K}[z]$ the ring of polynomial linear difference operators, $\sum_{k=0}^{r} a_k z^k$, z is the delay operator

$$z^k u(n) = u(n-k)$$

$\mathcal{K}[z]$ is in general noncommutative (it is commutative if and only if $\mathcal{K}$ is a field of constants). Nevertheless, $\mathcal{K}[z]$ is principal ideal ring. Let M be a left $\mathcal{K}[z]$-module. An element $a \in M$ is said to be torsion if, and only if, there exists $\theta \in \mathcal{K}[z]$, $\theta \neq 0$ such that $\theta.a = 0$. A left $\mathcal{K}[z]$-module is said to be torsion if and only if all its elements are torsion. The set of all torsion elements of M constitues the tosion submodule T of M. When $T = \{0\}$, the module M is said to be free. A finitely generated $left\mathcal{K}[z]$-module spanned by $\boldsymbol{w}(n) = (w_1(n), ..., w_s(n))$ is denoted by $[\boldsymbol{w}(n)]$.

Let $\mathcal{H}$ be a given $left\mathcal{K}[z]$ module :

Definition 1 *A filtration of a $left\mathcal{K}[z]$ module $\mathcal{H}$ is an ascending chain $(\mathcal{H}_r)_{r\in\mathbb{Z}}$ of $\mathcal{K}$-vector subspaces of $\mathcal{H}$, which satisfies the following properties*

1. *Exhaustivity : $\cup_{r\in\mathbb{Z}}\mathcal{H}_r = \mathcal{H}$,*
2. *Discrete : $\mathcal{H}_r = \{0\}$ for r small enough.*

A filtration of $\mathcal{H}$ is called excellent if

1. *for all r, $\mathcal{H}_r$ is a finitely generated $\mathcal{K}$-vector subspace.*
2. *there exists $r \in \mathbb{Z}$ such that, for $s > r$,*

$$\mathcal{H}_s = \cup_{i=0}^{s-r} z^i \mathcal{H}_r.$$

Example 1 *Let $\mathcal{H}_r\{\boldsymbol{w}(n)\}$ denotes the $\mathcal{K}$-vector spaces generated by the present and the past of $\boldsymbol{w}(n)$ down to time $n-r$ as in*

$$\mathcal{H}_r\{\boldsymbol{w}(n)\} = \text{span}\{\boldsymbol{w}(n), \ldots, \boldsymbol{w}(n-r)\} \quad (1)$$

We suppose that, for $r \prec 0$,

$$\mathcal{H}_r\{\boldsymbol{w}(n)\} = 0$$

$(\mathcal{H}_r\{\boldsymbol{w}(n)\})_{r\in\mathbb{Z}}$ forms an excellent filtration of $[\boldsymbol{w}(n)]$.

The following theorem, due to levin [11], plays here a key role

Theorem 1 *If $(\mathcal{H}_r)_{r\in\mathbb{Z}}$ is an excellent filtration of $\mathcal{H}$, then there exists a polynomial $P \in \mathbb{N}[r]$, of degree 1, $P(r) = \alpha r + \beta$, such that for r big enough,*

- $\dim \mathcal{H}_r = \alpha r + \beta$
- $\text{rk}(\mathcal{H}) = \dim \mathcal{H}_{r+1} - \dim \mathcal{H}_r = \alpha$

The polynomial P is called the Hilbert polynomial corresponding to the filtration $(\mathcal{H}_r)_{r\in\mathbb{Z}}$.

2.2 Nonlinear algebra

Let $\mathcal{K}$ still denotes a ground field. When $\mathcal{K}$ is equipped with the delay operator, it becomes a difference field. Let $\zeta = \{\zeta_1, \ldots, \zeta_l\}$ be a collection of elements on which acts the operator z. Define from $\mathcal{K}$ and ζ the set of all polynomials $P(\zeta_1, \ldots, \zeta_l, z\zeta_1, \ldots, z\zeta_l, z^\alpha\zeta_1, \ldots, z^\alpha\zeta_l)$ in $l(\alpha+1)$ variables with coefficients in $\mathcal{K}$ and denote this set as $\mathcal{K}[\boldsymbol{\zeta}]$. One may check that $k[\boldsymbol{\zeta}]$ is a difference ring with unity, finitely generated by $\boldsymbol{\zeta}$. The corresponding quotient field is a difference field, subsequently denoted as $\mathcal{K} < \boldsymbol{\zeta} >$. Let then $\mathcal{K} < \boldsymbol{w}(n) >$ be the difference field obtained as above

Example 2 *If $\mathcal{K}$ is the field of real numbers, then $\frac{z^2 w_1^3(n) + 7z w_1(n) w_4(n) + 3}{z w_2(n) + 2z^2 w_2^2(n) \cdot z w_3^4(n) + 5 w_1(n)}$ is an element of the difference field $\mathcal{K} < \boldsymbol{w}(n) >$.*

Definition 2 *A difference field extension L/E is given by two difference fields L and E, equipped with the same difference operator z, such that $E \subset L$.*

Definition 3 *An element $\zeta \in L$ is said to be* `transformally algebraic over` *E if and only if, it satisfies an algebraic difference equation with coefficients in E.*

It means that there exists a polynomial $P(x_0, \ldots, x_l)$ over E such that

$$P(\zeta, z\zeta, \ldots, z^l\zeta) = 0.$$

Otherwize, this element ζ is said to be `transformally` E`-transcendental`.

Definition 4 *Let I be an index set. A subset $\{\zeta_i, i \in I\}$ of L is called transformally E-algebraically independent if, and only if, the set $\{z^j\zeta_i, i \in I, j \in \mathbb{N}\}$ is E-algebraically independent.*

Such an independent set, which is maximal with respect to inclusion is called a `transformal transcendence basis` of the extension L/E. Two such bases have the same cardinality which is called the `transformal transcendence degree` of L/E and is denoted by $transf\ tr \deg L/E$.

3 Systems theory

This section is devoted to short rewiew on systems theory developed by Fliess (see [8]).

3.1 Linear systems

Take a finite set $\boldsymbol{u}(n) = (u_1(n), ..., u_m(n))$ which will play the role of input variables. We will assume that the input variables are *independent*, i.e.,$[\boldsymbol{u}(n)]$ is free.

Definition 5 *A `linear dynamics` is a finitely generated left $\mathcal{K}[z]$-module $\mathcal{D}$, which contains $\boldsymbol{u}(n)$, such that the quotient module $\mathcal{D}/[\boldsymbol{u}(n)]$ is torsion.*

The torsion of $\mathcal{D}/[\boldsymbol{u}(n)]$ means that any element in $\mathcal{D}$ can be calculate from $\boldsymbol{u}(n)$ by linear difference equations.

Definition 6 *`A linear system` with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n) = (y_1(n), ..., y_p(n))$ is a linear dynamics $\mathcal{D}$, which contains the components of $\boldsymbol{y}(n)$.*

A dynamics and an input-output system are said to be constant (resp. time-varying) if $\mathcal{K}$ is (resp. is not) a field of constants.

3.1.1 Linear invertibility

Recall that a family $\{\zeta_1, \dots, \zeta_m\}$ of elements of $[\boldsymbol{y}(n)]$ is said to be $\mathcal{K}[z]$-linearly independent if for any set $\{f_i(z)\}_{i=1}^m$ of m polynomials in $\mathcal{K}[z]$, we have

$$\sum_i f_i(z)\zeta_i = 0 \Longrightarrow f_i(z) = 0 \ \forall i.$$

The rank of $[\boldsymbol{y}(n)]$, subsequently denoted by $\mathrm{rk}[\boldsymbol{y}(n)]$, is defined as the cardinal of any maximum (w.r.t inclusion) $\mathcal{K}[z]$-linearly independent family of $[\boldsymbol{y}(n)]$.

Definition 7 *The rank ρ of the linear system with the input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is the rank of left $\mathcal{K}[z]$-module $[\boldsymbol{y}(n)]$.*

One easily verifies that :

- $\rho \preceq inf(m, p)$
- For a constant linear system, the rank ρ turns out to be equal to the rank of its transfer matrix.

This rank has been introduced by Fliess in the context of the inversion problem. The relation with the inversion is the following :

Definition 8 *The linear system with the input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is* left invertible *if, and only if, $\rho = m$.*

The intuitive meaning is clear: The left invertibility amounts to the possibility of recovring the input variables from the output variables by a finite set of difference equations.

3.1.2 Linear observability

Choose a subset $x = \{x_i / i \in I\}$ of $\mathcal{D}$ in a linear dynamics $\mathcal{D}/[\boldsymbol{u}(n)]$.

Definition 9 *An element θ in $\mathcal{D}$ is said to be `observable with respect to` x if and only if $[\theta, x]/[x]$ is Torsion.*
A subset Θ of $\mathcal{D}$ is said to be observable with respect to x if, and only if, any element of Θ is observable with respect to x.

Then θ can be calculated from x by linear difference equations.

3.2 Nonlinear systems

Let $\mathcal{K}$ be a given ground field. Denoted by $\mathcal{K} < \boldsymbol{u}(n) >$ the difference field generated by a finite set $\boldsymbol{u}(n) = (u_1(n), ..., u_m(n))$ of difference quantities. The set $\boldsymbol{u}(n)$ plays the role of input.

Definition 10 *A* `nonlinear dynamics` *is a finitely generated transformally algebraic extention* $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$.

Definition 11 *A* `nonlinear system` *with input* $\boldsymbol{u}(n)$ *and output* $\boldsymbol{y}(n)$ *is a nonlinear dynamics* $\mathcal{D}$*, which contains the components of* $\boldsymbol{y}(n)$.

As ouput variables can be viewed as sensors on the dynamics, we formelly define an output as a finite set $\boldsymbol{y}(n) = (y_1(n), ..., y_p(n))$ in $\mathcal{D}$.

3.2.1 Nonlinear invertibility

In this framework, the rank of a nonlinear system $\mathcal{S}$ admits a clear-cut definition given by

Definition 12 *The rank of the nonlinear system* $\mathcal{S}$ *with input* $\boldsymbol{u}(n)$ *and output* $\boldsymbol{y}(n)$*, denoted as* $\mathrm{rk}\{\mathcal{S}\}$*, is defined as*

$$\mathrm{rk}\{\mathcal{S}\} \triangleq \textit{transf tr}\deg \mathcal{K}(\boldsymbol{y}(n))/\mathcal{K}.$$

This rank satisfies the following properties :

- $\mathrm{rk}\{\mathcal{S}\} \leq inf(m, p)$
- $\mathrm{rk}\{\mathcal{S}\}$ extends to nonlinear system the usual transfer matrix rank of constant linear system.

Now that we have a clear definition for the rank, the invertibility of the nonlinear system $\mathcal{S}$ mimiks the linear case *i.e.*

Definition 13 *The nonlinear system* $\mathcal{S}$ *with the input* $\boldsymbol{u}(n)$ *and output* $\boldsymbol{y}(n)$ *is* left invertible *if, and only if,* $\mathrm{rk}\{\mathcal{S}\} = m$

The left invertibility means that the extension $\mathcal{K} < \boldsymbol{u}(n), \boldsymbol{y}(n) > /\mathcal{K} < \boldsymbol{y}(n) >$ is transformally algebraic, *i.e.*, the input variables may be recovred from the output variables by a finite set of difference equations.

3.2.2 Nonlinear observability

Choose a subset $x = \{x_i/i \in I\}$ of $\mathcal{D}$ in a nonlinear dynamics $\mathcal{D}/k < \boldsymbol{u}(n) >$.

Definition 14 *An element* θ *in* $\mathcal{D}$ *is said to be observable with respect to* x *if and only if it is algebraic over* $\mathcal{K} < x >$.
A subset Θ *of* $\mathcal{D}$ *is said to be observable with respect to* x *if, and only if, any element of* Θ *is obsevable with respect to* x.

The intuitive meaning is the following : θ can be expressed as an algebraic function of x and a finite number of their delays.

4 Nonlinear algebraic identifiability and equalizability

This section is devoted to a new setting for the problem of channel identification and equalization. We propose new definitions of identifiability and equalizability for nonlinear channels. Those definitions clarify the fundamental link between the notions of observability and invertibility of input-output system with the channel parameters identification and equalization. We provide a test of those definitions in terms of ranks of jacobian matrices.
Consider a nonlinear channel $\mathcal{C}$ described by a nonlinear dynamics $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$ with the output $\boldsymbol{y}(n)$. Let $\Theta = (\theta_1, ..., \theta_N)$ a set of unknown parameters in $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$.

Definition 15 *A parameters* $\Theta = (\theta_1, ..., \theta_N)$ *are* `algebraically identifiable`*, if and only if, the subset* $\Theta = (\theta_1, ..., \theta_N)$ *are observable with respect to* $(\boldsymbol{u}(n), \boldsymbol{y}(n))$*, i.e. if they are algebraic over* $\mathcal{K} < \boldsymbol{u}(n), \boldsymbol{y}(n) >$.

Which means that sevral values of the parametres are possible as algebraic equations must be solved.

Definition 16 *The nonlinear channel* $\mathcal{C}$ *is* `algebraically equalizable` *if and only if*

$$\mathrm{rk}\{\mathcal{C}\} \triangleq \textit{transf tr}\deg \mathcal{K}(\boldsymbol{y}(n))/\mathcal{K} \triangleq m.$$

The intuitive meaning is clear : algebraic equalizability is equivalent to stating that the input signal can be recovred from the output signal by a finite set of difference equations.
The previous definitions of identifiability and

equalizability lead to test which may be formally be performed in terms of ranks of Jacobian matrices. The part of mathematics underlying this results is the theory of `Kähler differentials` (see [12] and [11]). `Kähler differentials` can be seen as the algebraic version of the usual infinitesimal differential calculus. Their role is to translate properties of fields extensions into language of linear algebra. Then, to the difference field extension $\mathcal{K}(\boldsymbol{y}(n))/\mathcal{K}$ attach the left $\mathcal{K}(\boldsymbol{y}(n))[z]$-module $[d\boldsymbol{y}(n)]$ spanned by the so-called `Kähler differentials` dx; for $x \in \mathcal{K}(\boldsymbol{y}(n))$. The mapping

$$d : \mathcal{K}(\boldsymbol{y}(n)) \longrightarrow [d\boldsymbol{y}(n)]$$

satisfies the following rules

$$\begin{aligned} z(d\alpha) &= d(z\alpha) \quad \forall\ \alpha \in \mathcal{K}(\boldsymbol{y}(n)) \\ d(\alpha\beta) &= d(\alpha)\beta + \alpha d(\beta) \quad \forall\ \alpha, \beta \in \mathcal{K}(\boldsymbol{y}(n)) \\ d(c) &= 0 \quad \forall\ c \in \mathcal{K} \end{aligned}$$

Example 3 *Consider $s(n)$, an element of $\mathcal{K}(\boldsymbol{y}(n))$, where $\mathcal{K} = \mathbb{R}$, given by*

$$s(n) = 2y(n-1)^2 + y(n-2).$$

Then we have

$$ds(n) = 4y(n-1)dy(n-1) + dy(n-2).$$

4.1 Identifiability test

One can readily check that the image of the nonlinear dynamic $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$ by d is a linear left $\mathcal{D}[z]$-module $\Omega_{\mathcal{D}/\mathcal{K}}$. This module can be viewed as a linear dynamics with input $d\boldsymbol{u}(n) = (du_1(n), ..., du_m(n))$. It is called the tanjent dynamics of $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$. When there is an output $\boldsymbol{y}(n) = (y_1(n), ..., y_p(n))$ in $\mathcal{D}$, it gives rise to an output $d\boldsymbol{y}(n) = (dy_1(n), ..., dy_p(n))$ in $\Omega_{\mathcal{D}/\mathcal{K}}$. We have, from [13], the following theorem :

Theorem 2 *The dynamics $\mathcal{D}/\mathcal{K} < \boldsymbol{u}(n) >$ with output $\boldsymbol{y}(n)$ is obsevable if, and only, if the linear dynamics $\Omega_{D/\mathcal{K}}$ is observable.*

It yields the

Proposition 1 *The parameters $\Theta = (\theta_1, ..., \theta_N)$ are algebraically identifiable, if and only if, the set $d\Theta = (d\theta_1, ..., d\theta_N)$ are observable with respect to $(d\boldsymbol{u}(n), d\boldsymbol{y}(n))$.*

Then, if $\boldsymbol{J}_N^\theta$ is the jacobian matrix of $(\boldsymbol{y}(n), \ldots, \boldsymbol{y}(n-N))$ with respect to $(\theta_1, \theta_2, ..., \theta_N)$

$$\boldsymbol{J}_N^\theta = \frac{\partial(\boldsymbol{y}(n), \ldots, \boldsymbol{y}(n-N))}{\partial(\theta_1, \ldots, \theta_N)}$$

we have the following rank condition :

Proposition 2 *The parameters $\Theta = (\theta_1, ..., \theta_N)$ are algebraically identifiable if and only if,*

$$\mathrm{rk}(\boldsymbol{J}_N^\theta) = N$$

Example 4 *Let us discuss the following nonlinear example :*

$$y(n) = \theta_1 u(n) + \theta_1\theta_2 u(n-1)^2$$

the Jacobian matrix is given by,

$$J_2^\theta = \begin{bmatrix} u(n) + \theta_2 u^2(n-1) & \theta_1 u^2(n-1) \\ u(n-1) + \theta_2 u^2(n-2) & \theta_1 u^2(n-2) \end{bmatrix}$$

We may check that if

$$\theta_1(u(n)u^2(n-2) - u^3(n-1) \neq 0$$

,

$$\mathrm{rk} J_2^\theta = 2$$

Since $\theta_1 \neq 0$, we thus deduce the condition required to identify the nonlinear channel :

$$u(n)u^2(n-2) - u^3(n-1) \neq 0$$

which means that we can identify the nonlinear channel, unless the input lies on the hypersurface defined by :

$$u(n)u^2(n-2) - u^3(n-1) = 0, \forall\ n \in \mathbb{N}$$

4.2 Equalizability test

One can readily check that the image of $\mathcal{K}(\boldsymbol{y}(n))$ by d is the $\mathcal{K}(\boldsymbol{y}(n))[z]$-left module generated by $d\boldsymbol{y}(n)$ that we denote by $[d\boldsymbol{y}(n)]$.

This module corresponds to a **linear** time-varying system with input $d\boldsymbol{u}(n)$ and output $d\boldsymbol{y}(n)$, representing the tangent system of $\mathcal{S}$.
A set $\boldsymbol{w}(n) = (w_1(n), \dots, w_s(n))$ is a transformal transcendence basis of $\mathcal{K}(\boldsymbol{y}(n))/\mathcal{K}$ if, and only if, $d\boldsymbol{w} = (dw_1, \dots, dw_s)$ is a maximal set of $\mathcal{K}(\boldsymbol{y}(n))[z]$-lineary independent elements in $[d\boldsymbol{y}(n)]$. Thus, we have the fundamental result [12], [11]:

$$\mathsf{transf\ tr}\deg \mathcal{K}(\boldsymbol{y}(n))/\mathcal{K} = \mathrm{rk}[d\boldsymbol{y}(n)]. \qquad (2)$$

from which we deduce

Proposition 3 *The nonlinear channel $\mathcal{C}$ with input $\boldsymbol{u}(n)$ and output $\boldsymbol{y}(n)$ is algebraically equalizable if, and only if*

$$\mathrm{rk}[d\boldsymbol{y}(n)] = m.$$

Our next concern will be how to compute this rank in terms of the rank of a given transfer matrix. Indeed this transfer matrix appears to be that of the tangent system, obtained thanks to the `Kähler differentials`.
Let $\mathcal{H}_r\{d\boldsymbol{y}(n)\}$ denotes the $\mathcal{K}(\boldsymbol{y}(n))$-vector spaces generated by the present and the past of $d\boldsymbol{y}(n)$ down to time $n-r$ as in

$$\mathcal{H}_r\{d\boldsymbol{y}(n)\} = \mathrm{span}\{d\boldsymbol{y}(n), \dots, d\boldsymbol{y}(n-r)\} \qquad (3)$$

We suppose that, for $r \prec 0$, $\mathcal{H}_r\{\boldsymbol{y}(n)\} = 0$

Proposition 4 *The channel $\mathcal{C}$ is algebraically equalizable, if and only if, for r big enough*

- $\dim \mathcal{H}_r\{d\boldsymbol{y}(n)\} = mr + \beta$
- $\dim \mathcal{H}_{r+1}\{d\boldsymbol{y}(n)\} - \dim \mathcal{H}_r\{d\boldsymbol{y}(n)\} = m$

Proof: The module $[d\boldsymbol{y}(n)]$ being generated by $d\boldsymbol{y}(n)$. It is easly to show that $(\mathcal{H}_r\{d\boldsymbol{y}(n)\})_{r\in\mathbb{Z}}$ is an excellent filtration of $[d\boldsymbol{y}(n)]$. The conclusion immediatly follows from theorem1.
The computation of this rank may be achieved using a matrix formulation. To see this, let $\boldsymbol{y}(n)$ be the ouput of a nonlinear channel $h(\cdot)$ with input $\boldsymbol{u}(n)$ as in:

$$\boldsymbol{y}(n) = h(\boldsymbol{u}(n),\ \boldsymbol{u}(n-1), \dots, \boldsymbol{u}(n-N))$$

Then, the Kähler differential of $\boldsymbol{y}(n)$ is given by:

$$d\boldsymbol{y}(n) = \sum_{j=0}^{N} \frac{\partial h}{\partial \boldsymbol{u}(n-j)} d\boldsymbol{u}(n-j),$$

so that we have

$$\begin{bmatrix} d\boldsymbol{y}(n) \\ d\boldsymbol{y}(n-1) \\ \vdots \\ d\boldsymbol{y}(n-r) \end{bmatrix} = \boldsymbol{J}_r^u \begin{bmatrix} d\boldsymbol{u}(n) \\ d\boldsymbol{u}(n-1) \\ \vdots \\ d\boldsymbol{u}(n-N-r) \end{bmatrix}$$

where $\boldsymbol{J}_r^u$ denotes the jacobian matrix of $\{\boldsymbol{y}(n),\ \boldsymbol{y}(n-1), \dots, \boldsymbol{y}(n-r)\}$ with respect to $\{\boldsymbol{u}(n),\ \boldsymbol{u}(n-1), \dots, \boldsymbol{u}(n-N-r)\}$. Then

Proposition 5 *The nonlinear channel $h(\cdot)$ is algebraically equalizable if and only if, for r big enough*

- $\mathrm{rk}\boldsymbol{J}_r^u = mr + \beta$
- $m = \mathrm{rk}\boldsymbol{J}_{r+1}^u - \mathrm{rk}\boldsymbol{J}_r^u$

Example 5 *We consider the nonlinear preceding example :*

$$y(n) = \theta_1 u(n) + \theta_1\theta_2 u(n-1)^2$$

The jacobian matrices are given by,

$$J_2^u = \begin{bmatrix} \theta_1 & 2\theta_1\theta_2 u(n-1) & 0 \\ 0 & \theta_1 & 2\theta_1\theta_2 u(n-2) \end{bmatrix}$$

Since $\theta_1 \neq 0$, we have

$$\mathrm{rk}J_2^u = 2$$

and

$$\mathrm{rk}J_3^u = 3$$

we may chech that for $r \geq 0$

$$1 = \mathrm{rk}\boldsymbol{J}_{r+1}^u - \mathrm{rk}\boldsymbol{J}_r^u$$

which means that the nonlinear channel is equalizable.

5 Conclusion

In this paper, we just provided some insight in the theory of algebraic identifiability and equalizability, by giving a simple and clear definition, which permit to present rank condition. Of course, many things regarding algebraic identifiability and equalizablity theory remain to be clarified in our setting. How to calculate the unkown parameters.The questions pertaining to stability and causality have not been considered in this paper neither those concerning the computation of the inverse.

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