Control-Transverse and Graph Dynamics and Relations to Backstepping

EFTHIMIOS KAPPOS
Department of Applied Mathematics,
University of Sheffield, Sheffield S3 7RH
ENGLAND

Abstract: In this paper we examine the transverse geometry of control-affine systems. We point out the importance the singular set and of dynamics defined on submanifolds that are transverse to the control distribution. This setting is then used to compare the backstepping methodology for strict-feedback systems with our more general approach to nonlinear control design.

Keywords: Nonlinear control design, backstepping, singular set, Lyapunov functions

1 Introduction

Let us compare the two control-affine systems:

$$\begin{cases} \dot{x}_1 &= x_2 - f_1(x_1) \\ \dot{x}_2 &= u \end{cases} \tag{1}$$

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= u \end{cases} \tag{2}$$

The former system belongs to the class of control systems in *strict feedback form*, which allows the backstepping methodology to be applied. The latter is in some natural sense a local normal form for arbitrary control-affine systems (see section 2.) One can ask the question: When can we ascertain that the general system of equation (2) is feedback equivalent to the more restrictive form of equation (1)?

Note that we only assume that $(x_1, x_2) \in \mathbf{R}^{(n-m)} \times \mathbf{R}^m$ for some $0 < m < n$ in equation (2), while in the backstepping context x_1 and u must be one-dimensional (the more general form is given in equation (4).)

The key idea is to focus attention on the *singular set* of the control system, the set where the control directions belong to a subspace of the tangent space, rather than an *affine* subspace. We study the geometry of singular sets in section 2.

We also present a novel control design methodology that is based on graphs or, more generally, submanifolds that are transverse to the control distribution. This crucial point is somewhat obscured in the derivation of the backstepping technique, where one has to *pretend that a state is a 'virtual' control.* Our approach, like backstepping, benefits from a decomposition into dynamics transverse to the control directions and dynamics *in* the control directions. Note that, while one assumes a single control in the application of backstepping, no such assumption is necessary in the approach we take through transverse dynamics.

2 Graphs and Control Transverse Dynamics

In Nonlinear Control Theory ([19], [20]), one most commonly studies *control-affine systems* of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)\ , \quad x \in V\ , \quad n \geq m > 0, \tag{3}$$

with V some open subset of $\mathbf{R}^n$ and $f, g_1, \ldots, g_m$ smooth vector fields in $V \subset \mathbf{R}^n$. Such control-affine systems can be considered as approximations of the more general control systems $\dot{x} = f(x, u)$ near a regular section of the control bundle (see [10].) Even if the state space is a manifold M^n, we shall be working mostly in a neighbourhood of some point, and will thus assume that we have used local coordinates to write the system in an open subset V of $\mathbf{R}^n$.

Definition 1. *The* **control distribution** D *is the linear span of the vector fields* $\{g_1, \ldots, g_m\}$*. The set of* **feedback controls** $\Gamma(D)$ *is the set of smooth section of the sub-bundle* $D \subset TM$*.*

Recent *practical* control methodologies rely heavily on rather *special* forms for the control dynamics. A main example is the **backstepping** methodology, which, in its basic form, assumes the *strict feedback form*

$$\begin{cases} \dot{x}_1 &= x_2 - f_1(x_1) \\ \dot{x}_2 &= x_3 - f_2(x_1, x_2) \\ \cdots & \cdots \quad \cdots \\ \dot{x}_{n-1} &= x_n - f_{n-1}(x_1, \ldots, x_{n-1}) \\ \dot{x}_n &= u \end{cases} \tag{4}$$

Now there is a straightforward way to express the general control-affine system (3) above in a more useful form. The following is a basic definition:

Definition 2. *The* **singular set** Σ *of the control-affine system is the subset of* $\mathbf{R}^n$

$$\Sigma = \{x \in V\,;\, f(x) \in \mathrm{img}(x)\}$$

This definition can be easily modified for the case when the control is not unbounded: it is the set of all points of the state space where the 'state dynamics' $f(x)$ lies in the control set $g(x)U$, where g is the $(n \times m)$ matrix of controls and $U \subset \mathbf{R}^m$ is the control set. The singular set is a **fundamental** object in the study of nonlinear control systems. For a start, all possible equilibrium points of all possible control dynamics lie in Σ:

Lemma 1. *The singular set contains all equilibrium points of any control dynamics and is generically a manifold of dimension equal to the dimension of the control, m. Generic here means for a residual set in $\mathcal{X}(V)^{n+1}$ the set of $(n+1)$-tuples of vector fields on V.*

From now on, we shall make it an assumption that Σ is indeed a manifold of the appropriate dimension:

Assumption 1. *We assume that the singular set is an m-dimensional submanifold: $\Sigma^m \subset V \subset \mathbf{R}^n$.*

But the role of the singular set is revealed best when considering the **stabilization problem** in control—see Proposition 4 below. This is the problem of the existence of a control strategy that yields control dynamics with, at least locally, an asymptotically stable equilibrium.

The subject of stabilization has witnessed considerable development (see for example [1]), yet still lacks a good constructive methodology. The **backstepping** technique, which appeared in the mid-nineties, **is explicitly** a stabilization method. However, it had to rely on the control system taking a rather special form.

We shall examine the singular set for control-affine systems, and see how they take a special form in the case of strict feedback systems in the next section. The main interest is when the set V is a neighbourhood of a point $p \in \Sigma$, a point that we wish to stabilize, for example. For now, we proceed with the normal form for control-affine systems.

Assumption 2. *We assume that the control distribution, defined point-wise by*

$$D_x = \mathrm{span}\{g_1(x), \ldots, g_m(x)\},$$

has constant-rank m in an open, star-shaped domain of $\mathbf{R}^n$ containing V.

It is a simple fact from Algebraic Topology that the control distribution $D \subset T\mathbf{R}^n|_V$ forms a **trivial** vector bundle, in other words there is a diffeomorphism

$$D \simeq V \times \mathbf{R}^n.$$

Proposition 1. *In a relatively compact subset of V, the control affine system of equation (3) is feedback equivalent to the system*

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= g_2(x_1, x_2)u, \end{cases} \tag{5}$$

where the square $m \times m$ matrix is nonsingular, so that we can further simplify the above system by setting $g_2 = I_m$ (thus obtaining the system of equation (2).

This result says that, when considering control dynamics, one can decompose the state into the 'control directions' and the directions transverse to the control. In particular, one can consider submanifolds and foliations transverse to the control distribution D and define on these geometric objects dynamics, which will be called **control-transverse dynamics**. By varying these transverse manifolds or foliations, we vary the control-transverse dynamics. Let us give some more detail (also see references [14], [12], [10], [13].)

We shall consider (local) coordinate systems such that the control distribution is *constant*. We canonically identify $\mathbf{R}^n$ with each of its tangent spaces $T_x\mathbf{R}^n$. We have the direct sum

$$\mathbf{R}^n = E^{n-m} \oplus D^m$$

and we write $x = (x_1, x_2)$ $x_1 \in E, x_2 \in D$ and, by the above identification also $\dot{x} = (\dot{x}_1, \dot{x}_2)$.

The main geometrical objects on which we shall define dynamics are **submanifolds** N^{n-m} and **foliations** $\mathcal{F}$ transverse to the sub-bundle D and of complementary dimension. This means, for a submanifold N, that

$$T_xN \oplus D_x = T_x\mathbf{R}^n, \quad \forall x \in N$$

and similarly for each leaf of $\mathcal{F}$.

Locally, control-transverse manifolds are in a one-to-one correspondence with graphs of functions, as follows:

Proposition 2. *Let $p \in M^n$ and assume the control distribution is trivial in a neighbourhood of the point p. We can assume moreover that we decompose $TM|_W$ into $D \oplus E$, D, E fixed subspaces. Then for every submanifold N^{n-m} transverse to D at p (and hence locally), we can find a neighbourhood $W \supset N \ni p$ and a function*

$$\psi : E \to D$$

such that $\mathrm{graph}\psi = N|_W$.

Now, given the normal form of the Lemma, choosing a control-transverse submanifold or equivalently, choosing, locally, a function

$$\psi : E \to D \simeq \mathbf{R}^m$$

($E \subset \mathbf{R}^{n-m}$) we can define the dynamics of the system

$$\dot{x}_1 = f_1(x_1, \psi(x_1)). \tag{6}$$

Proposition 3. *There is a choice of a smooth feedback control in a tubular neighbourhood of a control-transverse manifold N that makes the manifold invariant under the control flow. The dynamics thus obtained on N are topologically orbitally equivalent to the dynamics on the graph of a suitable ψ as above in equation (6).*

Proof. Since we have that the manifold is locally the graph of a function $\psi : E \to D$, we can make it invariant by choosing

$$u = \dot{x}_2 = \frac{\partial \psi}{\partial x}\dot{x}_1 = \frac{\partial \psi}{\partial x} f_1(x_1, \psi(x_1)).$$

Since we can control the D-directions, we can then extend the control to a tubular neighbourhood of N such that N is a hyperbolic invariant set (we can make N asymptotically stable, for example.) The last part just says that the dynamics on N are given by

$$\dot{x}_1 = f_1(x_1, \psi(x_1)),$$

which is clear. □

Here is an important result that uses the control-transverse dynamics notion.

Proposition 4. *The control-affine system is smoothly stabilizable to the point $p \in \Sigma$ if and only if there is an $(n-m)$-dimensional submanifold transverse to D and containing p such that the invariant dynamics are locally asymptotically stable*

Compare this with the backstepping approach for the system (1): this considers the state component x_2 as a virtual control and takes

$$x_2 = f_1(x_1) - \alpha x_1,$$

for example, which yields a stable subsystem. A Lyapunov function for the subsystem is then extended to the whole state by defining

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - f_1(x_1) - \alpha x_1)^2.$$

This step-by-step approach can easily extend to the full system of the form of equation (4.)

3 Singular set geometry

Given a control-affine system in the 'normal form' of equation (5), the singular set is obtained by setting f_1 to zero:

$$\Sigma = \{x \in V \ ; \ f_1(x_1, x_2) = 0\}.$$

(so we have $(n-m)$ equations in n variables.) Here is the relevant result from Differential Geometry:

Proposition 5. *Suppose that zero is a regular value of f_1. Then Σ is an m-dimensional smooth submanifold. In fact, the distribution* $\ker df_1$ *is then regular nearby, and hence, since it also integrable, we can find a foliation by the level sets of f_1 near the value* 0.

The proof is standard and ultimately rests on the *implicit function theorem* (see, for example, Conlon [3].)

Singular sets and control distribution We saw that submanifolds that are transverse to the singular set play a special role in control theory. Proposition 4, for example, gave a necessary and sufficient condition for smooth stabilization in terms of a control-transverse manifold with stable dynamics. But there are stronger connections that bring us closer to the linear system idea of **controllability**. More specifically, we assert that what is of more interest, from the dynamical viewpoint, in the strict feedback form is **not** that we can derive a stabilizing feedback, but that *we can transform the system to a* **linear** *controllable form!*

Now the singular set for the *strict feedback form* of the backstepping method is found by setting $\dot{x}_1 = \dot{x}_2 = \ldots = \dot{x}_{n-1} = 0$ in the system (4), in other words, setting

$$\left\{ \begin{array}{lll} x_2 & = & f_1(x_1) \\ x_3 & = & f_2(x_1, x_2) \\ \cdots & \cdots & \cdots \\ x_n & = & f_{n-1}(x_1, \dots, x_{n-1}) \end{array} \right. \tag{7}$$

The last equation certainly gives a graph transverse to $D =< e_n >$, in agreement with the Proposition. Successive equations give graphs—and hence transverse manifolds— on the previous graph until we get to the two-dimensional one given by $x_2 = f_1(x_1)$, so that the idea of 'virtual controls' seems a natural one. In the design process, we take these graphs in the reverse order, i.e. in the order they are listed in the equation above, and use graph dynamics in the manner we described to get a stable system overall. What is as important, though, is that if we forget the stabilization issue, we can obtain the transformed system (a 'feedback-equivalent system)

$$\left\{ \begin{array}{lll} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \cdots & \cdots & \cdots \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & u \end{array} \right. \tag{8}$$

Let us turn now to the singular set for the system (2). In order to be able to compare with the strict feedback case, we shall work in the case when D is one-dimensional. Fix a point $p \in \Sigma$. By our assumption, the one forms $df_1, \dots, df_{n-1}$ are linearly independent at, and near, p. In general, D and Σ will not be tangent at p. The derivative matrix for the strict feedback system is

$$df = \begin{pmatrix} \frac{df_1}{dx_1} & 1 & 0 & \cdots & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} & 1 \end{pmatrix} \tag{9}$$

and we note that this linear map preserves the *standard flag* in $\mathbf{R}^n$

$$0 = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^n = \mathbf{R}^n,$$

where $V^1 =< e_n >, V^2 =< e_{n-1}, e_n >$ etc. A general position argument will easily show that we can at least get a transverse graph

$$x_{n-1} = \tilde{f}_{n-1}(x_1, \dots, x_{n-1}),$$

by assuming, without loss of generality, that $df_{n-1}(D) \neq 0$ and appealing to the implicit function theorem. It is not clear how this can be continued.

4 Singular sets in the plane

In Figure 1, we give some typical positions for the singular set in the plane of a system with $m = 1$ in relation to the control distribution D, which is taken to be vertical, $D =< e_2 >$ in the figure (where, as usual, we have identified $\mathbf{R}^2 \simeq T_0\mathbf{R}^2$.

First, we fix a point $p \in \Sigma$. We have the following cases:

1. *The singular set is a complement of D at p (Fig.1(A)):*

$$T_p\Sigma + D_p = T_p\mathbf{R}^2$$

and 0 *is a regular value of the function* f_1. This is the generic case and also the one assumed in the backstepping approach. However, let us note that it is possible to choose graphs with *arbitrary* dynamics: stable, unstable, semi-stable. As a result, we can make p not only into a locally asymptotically stable equilibrium, but also a repeller or a saddle point in the plane.

2. *We have that $T_p\Sigma = D$, but* 0 *is still a regular value of f_1 (Fig.1(B) and (C)).* If $f_1(x_1, \epsilon) \cdot x_1$ is negative,then it is **still** possible to stabilize the system, by choosing a graph as shown. If the product is positive, then we **cannot** achieve stable local dynamics using smooth feedback. However, it may still be possible to achieve stable local dynamics by *non-smooth* feedback–by picking a graph with infinite slope in Figure 2(C). This is the case highlighted by Kawski in [15].

3. *We have that $T_p\Sigma = D$, but now* 0 *is not a regular value of f_1 (Fig.1(D)).* The range of dynamics we can achieve is now limited and stabilization is not possible (the semi-stable one-dimensional dynamics shown can be extended to saddle-node dynamics in the plane, for example.)

5 Conclusions

This paper promulgates an approach to nonlinear control dynamics through the dynamics defined on submanifolds and foliations transverse to (and of complementary dimension) the control distribution $D =< g >$. The backstepping method certainly makes use of this idea, indirectly, although it does not give prominence to the transverse geometry, as we do.

References

[1] A. Bacciotti: *Local Stabilizability of Nonlinear Control Systems*, World Scientific, 1992.

[2] R. Brockett, "Asymptotic Stability and Feedback Stabilization," in R. Brockett, R. Millmann, H. Sussmann eds: Differential Geometric Control Theory, Birkhäuser, 1983, pp.181–191.

[3] L. Conlon: *Differentiable Manifolds*, Birkhäuser, Boston, 2001.

[4] J.M. Coron, "A Necessary Condition for Feedback Stabilization," System and Control Letters **14**, 1990, pp.227–232.

[5] J. Dugundji: *Topology*, Allyn and Bacon, 1966.

[6] A.T. Fomenko: *Variational Problems in Topology*, Gordon and Breach, 1990.

[7] S.-T. Hu: *Homotopy Theory*, Academic Press, 1959.

[8] E. Kappos, "The Conley index and Global Bifurcations, Part I: Concepts and Theory," Int.J. Bif. and Chaos, **5**–4, 1995, pp.937–953.

[9] E. Kappos, "The Conley index and Global Bifurcations, Part II: Illustrative Applications" Int.J. Bif. and Chaos, **6**–12B, 1996, pp.2491–2505.

[10] E. Kappos: *Global Controlled Dynamics*, book manuscript, 2003.

[11] E. Kappos, "A Global, Geometrical, Input-Output Linearization Theory," IMA J. Math. Con. and Info., **9**–1, 1992, pp.1–21.

[12] E. Kappos, "Local Controlled Dynamics," in D. Owens, A. Zinober eds.: Nonlinear and Adaptive Control, Springer, 2003.

[13] E. Kappos, "A Condition for Smooth Stabilization," Proc. of the MCSS Conference, Pérpignan, 2000.

[14] E. Kappos, "The Role of Morse-Lyapunov Functions in the Design of Nonlinear Global Feedback Dynamics," in A. Zinober ed.: Variable Structure and Lyapunov Control, Lect. Notes in Control and Infor. Sciences **193**, Springer, 1994, pp.249–267.

[15] M. Kawski, "Stabilization of nonlinear systems in the plane," Systems Control Lett. **12**-2, 1989, pp.169–175.

[16] H. Khalil: *Nonlinear Systems*, 2nd Ed. Prentice Hall, 1996.

[17] M. Krstić, P. Kokotović and I. Kanellakopoulos: *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

[18] J. Milnor: *Topology from the Differentiable Viewpoint*, The University Press of Virginia, 1965.

[19] S. Sastry: *Nonlinear Systems: Analysis, Stability and Control*, Springer, 1999.

[20] M. Vidyasagar: *Nonlinear System Theory*, Prentice–Hall, 1978.

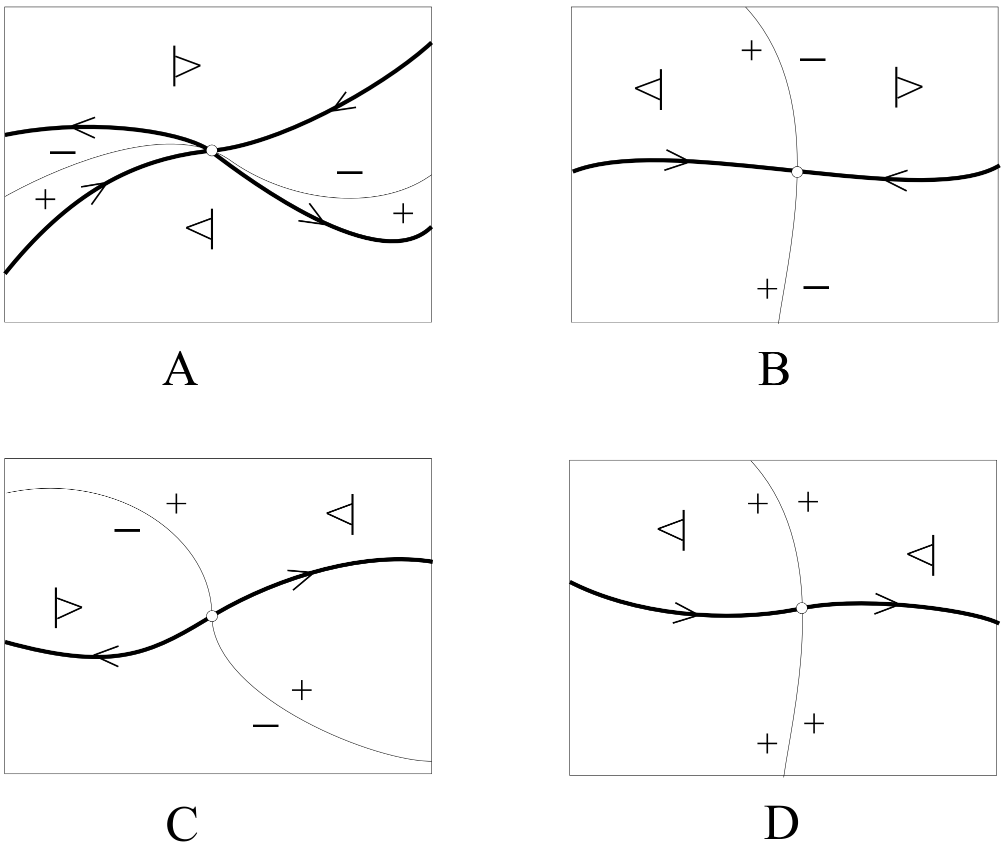

Figure 1: Singular sets in the plane and some transverse dynamics.