Lorenz and Rössler Systems with Piecewise-Linear Vector Fields

JIŘÍ PETRŽELA, JIŘÍ POSPÍŠIL, ZDENĚK KOLKA, STANISLAV HANUS
Department of Radio Electronics
Brno University of Technology
Purkyňova 118, 612 00 Brno
CZECH REPUBLIC

Abstract: - Dynamical systems of class *C* [1] are described by the 3rd-order autonomous differential equation with nonlinearity given as a three-segment piecewise linear (PWL) function. Argument of this function is a linear combination of state variables. These systems form an extensive group of nonlinear systems with PWL vector fields and may produce rich set of chaotic attractors. The paper shows how this group can be extended for Lorenz and Rössler systems.

Key-Words: Dynamical systems, piecewise-linear, chaos, topological conjugacy

1 Introduction

Behavior of a dynamical system is determined by its vector field and initial conditions. We will discuss the linearization procedure of dimensionless Lorenz and Rössler systems. Our goal is to find a transformation between the linearized system and the 1st canonical ODE equivalent of Chua´s equation [1] which represents a reference model.

Systems of class *C* can be expressed in compact explicit form as

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x} + \mathbf{b}\cdot h\left(\mathbf{w}^{\mathbf{T}}\mathbf{x}\right), \tag{1}$$

where $\mathbf{A} \in R^{3\times 3}$, $\mathbf{b}, \mathbf{w} \in R^3$, and PWL function

$$h\left(\mathbf{w}^{\mathbf{T}}\mathbf{x}\right) = \frac{1}{2}\left(\left|\mathbf{w}^{\mathbf{T}}\mathbf{x}+1\right| - \left|\mathbf{w}^{\mathbf{T}}\mathbf{x}-1\right|\right). \tag{2}$$

To include Lorenz and Rössler systems into class *C* it is necessary to define a starting system. It can be chosen as the 1st canonical ODE equivalent in the form [1]

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix} q_1 & -1 & 0\\ q_2 & 0 & -1\\ q_3 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}p_1-q_1\\ p_2-q_2\\ p_3-q_3\end{pmatrix}\cdot h\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right], \tag{3}$$

where p_i and q_i are coefficients of characteristic polynomials in respective segments of the state space. Any two systems having the same eigenvalues are qualitatively equivalent. The mapping of state trajectory of one element of this group onto another must be continuous but there is no requirement to be linear. We can use conditions of linear topological conjugacy (LTC) taken from [1]. The transformation matrix

$$\mathbf{T} = \widetilde{\mathbf{K}}^{-1}\cdot\mathbf{K} \tag{4}$$

converts the original system (1) to an equivalent system

$$\widetilde{\mathbf{x}} = \mathbf{T}\cdot\mathbf{x}\ ,\quad \widetilde{\mathbf{A}} = \mathbf{T}\cdot\mathbf{A}\cdot\mathbf{T}^{-1}\ ,\quad \widetilde{\mathbf{b}} = \mathbf{T}\cdot\mathbf{b}\ . \tag{5a, b, c}$$

Partial transformation matrices $\widetilde{\mathbf{K}}$ and $\mathbf{K}$ are defined by the nonsingular forms

$$\widetilde{\mathbf{K}} = \begin{pmatrix}\widetilde{\mathbf{w}}^{\mathbf{T}}\\ \widetilde{\mathbf{w}}^{\mathbf{T}}\cdot\widetilde{\mathbf{A}}\\ \widetilde{\mathbf{w}}^{\mathbf{T}}\cdot\widetilde{\mathbf{A}}^{2}\end{pmatrix},\quad \mathbf{K} = \begin{pmatrix}\mathbf{w}^{\mathbf{T}}\\ \mathbf{w}^{\mathbf{T}}\cdot\mathbf{A}\\ \mathbf{w}^{\mathbf{T}}\cdot\mathbf{A}^{2}\end{pmatrix}, \tag{6a, b}$$

fulfilling the observability condition of matrix **A** and vector **w**. The simple form of partial transformation matrix and its inversion is the reason the 1st ODE equivalent is proposed as the reference system

$$\mathbf{K_I} = \begin{pmatrix}1 & 0 & 0\\ q_1 & -1 & 0\\ q_1^{2}-q_2 & -q_1 & 1\end{pmatrix},\ \mathbf{K_I}^{-1} = \begin{pmatrix}1 & 0 & 0\\ q_1 & -1 & 0\\ q_2 & -q_1 & 1\end{pmatrix}. \tag{7a, b}$$

Equations of Lorenz system were derived from Navier-Stokes partial differential equations in the form

$$\dot{x} = -\sigma(x-y),\ \dot{y} = rx - y - xz,\ \dot{z} = bz + xy. \tag{8a, b, c}$$

A chaotic behavior can be observer for certain set of parameters. Standard values are

$$\sigma = 10\ ,\quad r = 28\ ,\quad b = -2.667\ . \tag{9}$$

The Rössler system given as

$$\dot{x} = -y - z\ ,\ \dot{y} = x + ay\ ,\ \dot{z} = (x-c)z + b\ , \tag{10a, b, c}$$

exhibits a chaotic attractor for a wide set of parameters. The optimal values are

$$a = b = 0.3 \ , \ c = 4 \ . \tag{11}$$

2 Linearization Procedure

The linearization procedure can be divided into several steps which will be demonstrated for Lorenz and Rössler systems. First, equilibrium points of the studied system should be determined and denoted as $\bar{\mathbf{x}}_n$, where $n = 0, 1, 2, \ldots$. The second step consists in computing of the Jacobi matrix $\mathbf{J}(\bar{\mathbf{x}}_n)$ which provides a linearized flow of studied system near all equilibria. We expect a solution which produces a single scroll attractor. Thus, there are two complex conjugate eigenvalues of **J** with positive real part and a negative real eigenvalue.

It is known from linear algebra that the matrix **J** can be written as $\mathbf{J} = \mathbf{M} \cdot \mathbf{R} \cdot \mathbf{M}^{-1}$, where **R** is a block diagonal matrix. The formula represents in fact a coordinate transformation that also transforms the equilibrium points as $\tilde{\bar{\mathbf{x}}}_n = \mathbf{M}^{-1} \cdot \bar{\mathbf{x}}_n$. In the last step it is necessary to determine a plane, which separates the state space into two regions. In each region the linear system produces a single scroll. Only one requirement is needed for successful linearization – a presence of at least one coordinate of two fixed points which could be expressed as adding and subtracting the same value from a constant. We can start with the linear system

$$\dot{\mathbf{x}} = \mathbf{J}(\bar{\mathbf{x}}) \cdot [\mathbf{x} - \mathbf{F}(\mathbf{x})] \ , \tag{12a}$$

where

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \\ \bar{z}_1 \end{pmatrix} \operatorname{sgn}\left(\mathbf{q}^{\mathrm{T}} \cdot \mathbf{x}\right) , \tag{12b}$$

or in transformed Jordan form

$$\dot{\mathbf{x}} = \boldsymbol{\lambda} \cdot \left[\mathbf{x} - \tilde{\mathbf{F}}(\mathbf{x})\right] , \tag{13a}$$

where

$$\tilde{\mathbf{F}}(\mathbf{x}) = \frac{1}{\det \mathbf{M}} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} \operatorname{sgn}\left(\mathbf{q}^{\mathrm{T}} \cdot \mathbf{M} \cdot \mathbf{x}\right) , \tag{13b}$$

where $\boldsymbol{\lambda}$ is block diagonal matrix and symbols Δ_i in (13b) came from the Cramer rule and denote sub-determinants of the matrix **M**. After the third step we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & r_{33} \end{pmatrix} \cdot \begin{pmatrix} x - \tilde{\bar{x}} \cdot \operatorname{sgn}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \\ y - \tilde{\bar{y}} \cdot \operatorname{sgn}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \\ z - \tilde{\bar{z}} \cdot \operatorname{sgn}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \end{pmatrix} . \tag{14}$$

Equations (14) can be rearranged into a system of class C

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \delta & -\omega & 0 \\ \omega & \delta & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\delta \cdot \tilde{\bar{x}}_1 + \omega \cdot \tilde{\bar{y}}_1 \\ -\omega \cdot \tilde{\bar{x}} - \delta \cdot \tilde{\bar{y}}_1 \\ -\lambda_3 \cdot \tilde{\bar{z}}_0 \end{pmatrix} h\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) \tag{15}$$

where individual elements of matrix **R** are determined by eigenvalues of the single scroll $\lambda_{1,2} = \delta \pm \omega\, i$.

The main problem of linearization procedure lies on choosing the vector **q** which defines a separatrix of the state space. The separatrix of two single scrolls is nonlinear $\mathbf{w} = \begin{pmatrix} 1 & -y & 0 \end{pmatrix} \Rightarrow \mathbf{w}^{\mathrm{T}}\mathbf{x} = x - y^2$ and cannot be used for LTC. A numerical analysis shown the possibility of construction the double scroll attractor for $\mathbf{q} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. Some differences between these two variations of Rössler dynamical system are visible in Fig. 2 (yellow versus brown attractor). There is a substantial difference between (14) and (15) in the function which makes the middle region. Both PWL and signum functions can be approximated by a sigmoid function known from neural networks. This function allows us to expand or compress the middle region by varying a single parameter γ

$$h\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) = \frac{\exp\left(\gamma \cdot \mathbf{w}^{\mathrm{T}}\mathbf{x}\right) - 1}{\exp\left(\gamma \cdot \mathbf{w}^{\mathrm{T}}\mathbf{x}\right) + 1} \ . \tag{16}$$

For $\gamma \approx 2$ (16) approximates the PWL function (2) and for $\gamma > 10$ it converges towards the signum function. Any linearized dynamical system in Jordan form produces two horizontal single scrolls. Very interesting question is the existence of LTC $\mathbf{R_L} \rightarrow \boldsymbol{\lambda} \rightarrow \mathbf{R_R}$ where matrices $\mathbf{R_L}$ and $\mathbf{R_R}$ belongs to Lorenz and Rössler system, respectively. Accordingly to (4) the transformation can be decomposed into two steps represented by two transformation matrices $\mathbf{T_L} = \mathbf{K}_\lambda^{-1} \cdot \mathbf{K_L}$ and $\mathbf{T_R}^{-1} = \mathbf{K}_\lambda^{-1} \cdot \mathbf{K_R}$. The partial transformation matrix $\mathbf{K}_\lambda$ of any linearized dynamical system in Jordan form can be obtained using (6b). The result is

$$\mathbf{K}_{\lambda}=\begin{pmatrix} \sum_{i=1}^{3} q_i\, m_{i1} & \sum_{i=1}^{3} q_i\, m_{i2} & \sum_{i=1}^{3} q_i\, m_{i3} \\ \sum_{i=1}^{3} q_i\,(\delta\, m_{i1}+\omega\, m_{i2}) & \sum_{i=1}^{3} q_i\,(-\omega\, m_{i1}+\delta\, m_{i2}) & \lambda_3 \sum_{i=1}^{3} q_i\, m_{i3} \\ \sum_{i=1}^{3} q_i \left[\left(\delta^2-\omega^2\right)m_{i1}+2\delta\omega\, m_{i2}\right] & \sum_{i=1}^{3} q_i\left[-2\delta\omega\, m_{i1}+\left(\delta^2-\omega^2\right)m_{i2}\right] & \lambda_3{}^2 \sum_{i=1}^{3} q_i\, m_{i3} \end{pmatrix}. \quad (17)$$

3 Analytical Results

An analytical solution will be shown only for Lorenz and Rössler systems. The same principle can be used for other systems producing the double scroll attractor. The analytical expression for three fixed points, Jacobi matrix and vector **q** of Lorenz system is

$$\begin{pmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & r-1 \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & r-1 \end{pmatrix},$$

$$\mathbf{J}(\bar{\mathbf{x}}_1)=\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\bar{x}_1 \\ \bar{y}_1 & \bar{x}_1 & -b \end{pmatrix},\ \mathbf{q}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (18a, b, c)$$

The Lorenz system is invariant under the coordinate transformation $(x, y, z)\to(-x, -y, z)$. So there is no sigmoid function for offset in the last component of the vector $\mathbf{F}(x)$. Single scrolls generated in both regions (near fixed points $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$) have the same eigenvalues as the roots of the characteristic polynomial $\lambda^3+(\sigma+b+1)\lambda^2+b(\sigma+r)\lambda+2\sigma b(r-1)=0$. Note that equilibria exist only if $r>1$. Vector **b** and partial transformation matrix **K** have a very simple form

$$\mathbf{b}=\begin{pmatrix} 0 \\ 0 \\ -2b(r-1) \end{pmatrix},$$

$$\mathbf{K}_{\mathbf{L}}=\begin{pmatrix} 1 & 0 & 0 \\ -\sigma & \sigma & 0 \\ \sigma(\sigma+1) & -\sigma(\sigma+1) & -\sigma\sqrt{b(r-1)} \end{pmatrix}. \quad (19a, b)$$

Although the Rössler system is topologically simpler than Lorenz system, it seems to be much harder to analyze. The Rössler system has only two equilibrium points which exists if $c^2>4ab$

$$\begin{pmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{pmatrix}=\frac{1}{2a}\begin{pmatrix} ac+a\sqrt{c^2-4ab} & -c-\sqrt{c^2-4ab} & c+\sqrt{c^2-4ab} \\ ac-a\sqrt{c^2-4ab} & -c+\sqrt{c^2-4ab} & c-\sqrt{c^2-4ab} \end{pmatrix}, \quad (20a)$$

$$\mathbf{J}(\bar{\mathbf{x}}_1)=\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \bar{z}_1 & 0 & \bar{x}_1-c \end{pmatrix},\ \mathbf{q}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{b}=\begin{pmatrix} 0 \\ 0 \\ -0.5a^{-1}c^2+2b \end{pmatrix}.$$

(20b, c, d)

Characteristic polynomials for both single scrolls are

$$\lambda^3+0.5a^{-1}\left(\mp a\sqrt{c^2-4ab}-2a^2+ac\right)\lambda^2+$$
$$+0.5a^{-1}\left[-a^2c\pm\left(a^2+1\right)\sqrt{c^2-4ab}+2a+c\right]\lambda\mp\sqrt{c^2-4ab}=0.$$

There is one unpleasant factor in linearization of Rössler system. We will meet this problem in transformation process between formulas (14) and (15). It is clear that we may write a difference between fixed points (for example coordinate *x*) as $r\pm s$ where $r=0.5c$ and $s=0.5\sqrt{c^2-4ab}$. Because of constant *r* there is another vector in (1). This is undesirable and it must be removed using temporary substitutions $\tilde{x}=x-0.5c$, $\tilde{y}=y+0.5a^{-1}c$, $\tilde{z}=z-0.5a^{-1}c$. After that a partial transformation matrix is and the resultant transformation matrix between Lorenz and the 1st canonical ODE equivalent are

$$\mathbf{T}_{\mathbf{L\text{-}I}}=\begin{pmatrix} 1 & 0 & 0 \\ \sigma+q_1 & -\sigma & 0 \\ (\sigma+q_2)(\sigma+1) & -\sigma(\sigma+q_1+1) & -\sigma\sqrt{b(r-1)} \end{pmatrix}, \quad (21)$$

$$\mathbf{K}_{\mathbf{R}}=\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2a}\left(c+\sqrt{c^2-4ab}\right) & 0 & \frac{1}{2}\left(\sqrt{c^2-4ab}-c\right) \\ -b & -\frac{1}{2a}\left(c+\sqrt{c^2-4ab}\right) & -\frac{1}{2}\left[\left(\frac{1}{a}+2c\right)\sqrt{c^2-4ab}+c\left(\frac{1}{a}-1-c\right)\right] \end{pmatrix}. \quad (22)$$

$$
\mathbf{T}_{\mathbf{R-I}} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2a}\left(c+\sqrt{c^2-4ab}\right) & 0 & q_1 - \frac{1}{2a}\left(c-2ac+\sqrt{c^2-4ab}\right) \\ -\frac{q_1 c}{2}+\frac{c^2}{4a}-b+\left(c-2ac-\frac{q_1}{2}\right)\sqrt{c^2-4ab} & -\frac{1}{2}\left(c+\sqrt{c^2-4ab}\right) & q_2-\frac{q_1 c(1+2a)}{2a}\left(\frac{c}{2a}\right)^2+c^2-\frac{b}{a}-\sqrt{c^2-4ab}\left(\frac{1}{2}+\frac{c}{a}-\frac{q_1}{2a}\right) \end{pmatrix}. \quad (23)
$$

Similarly we can get the transformation matrix (23) for Rössler system

Figures 1 to 5 show some examples of phase portraits of the original and linearized systems.

4 Conclusion

Lorenz and Rössler system can be used as prototypes of dissipative nonlinear dynamical systems for education purpose. These systems can be linearized near equilibrium points and considered as systems of class *C*. After that a LTC can be derived (in analytical form) and displayed in relation to the 1st or 2nd ODE equivalent of Chua´s equation. The linearization procedure is applicable for such famous nonlinear systems as Duffing or Van der Pol oscillator. Realization as electronic circuit may be likely the biggest advantage of the linearization procedure.

5 Acknowledgement

This research is partially supported by the Grant Agency of the Czech Republic, Grant projects No. 102/02/1312/A and No. 102/01/0229. It represents the part of the Research Programs of the Czech Ministry of Education – CEZ: J22/98: 262200011 and CEZ: J22/98: 262200022.

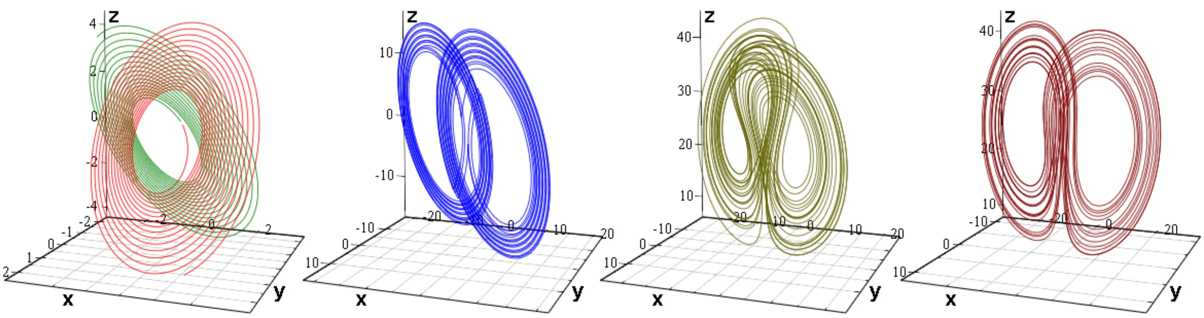

Fig. 1 Position of two single scrolls (linearized, classical and canonical Lorenz system).

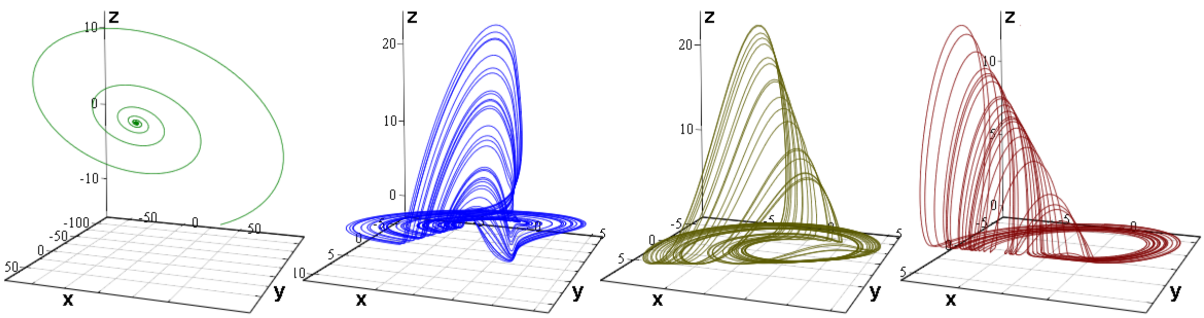

Fig. 2 Single scroll (linearized, classical and modified Rössler system).

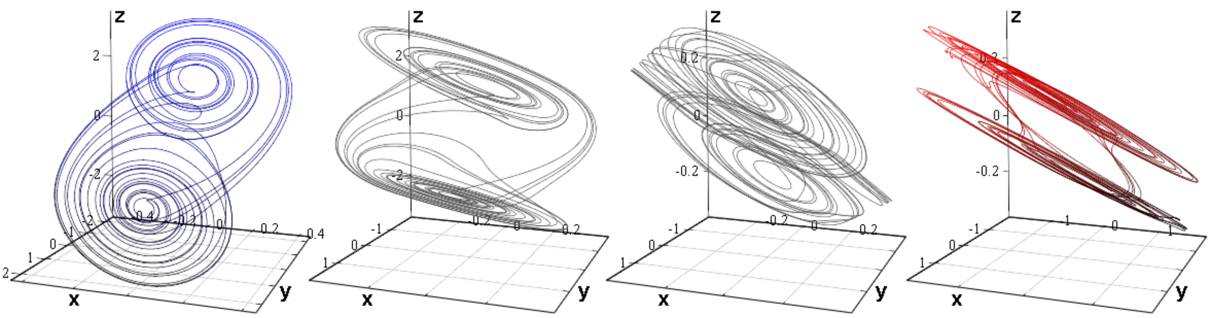

Fig. 3 LTC between 1st canonical ODE equivalent and linearized Lorenz system.

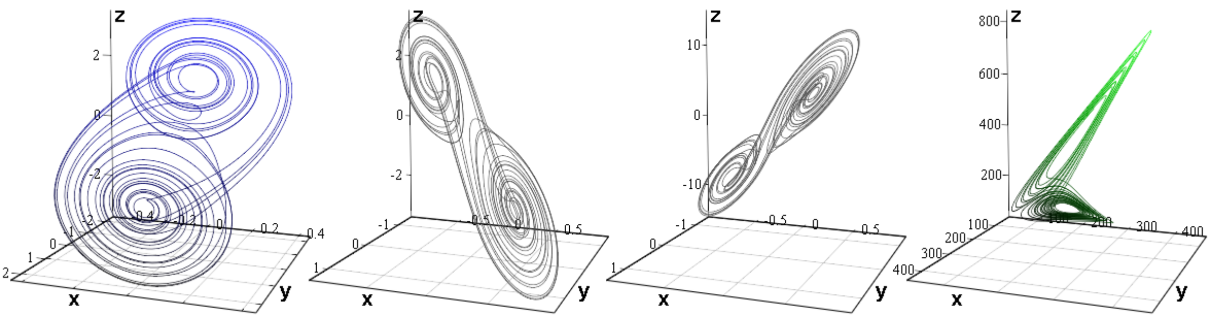

Fig. 4 LTC between 1st canonical ODE equivalent and linearized Rössler system.

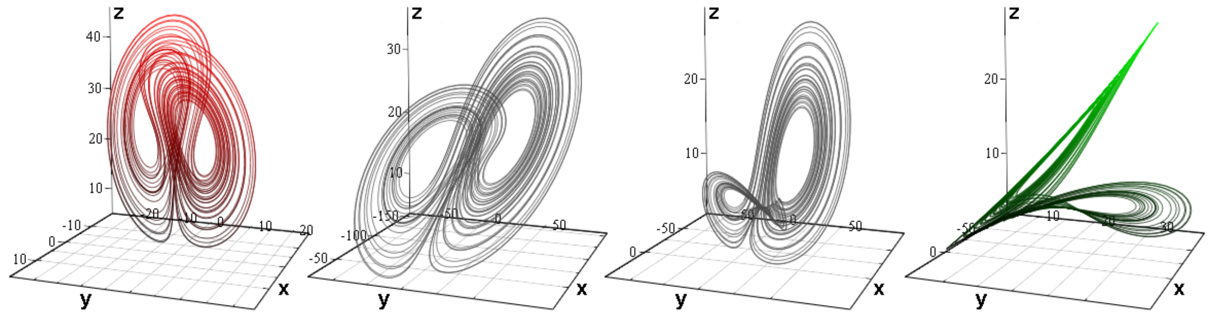

Fig. 5 Transformation through a Jordan form: from Lorenz to Rössler system.

References:

[1] POSPISIL J., BRZOBOHATY J., KOLKA Z., HORSKA J.: Simplest ODE Equivalents of Chua's Equations. *Int. Journ.of Bifurc.&Chaos,* vol.10, no.1, p.1-23, 2000.

[2] BROWN, R.: Generalizations of the Chua equations, *IEEE Transactions on Circuit and Systems-I*, vol. 40, no. 11, pp 878-884, 1993.

[3] PIVKA, L., WU, Ch. W., HUANG, A.: Lorenz equation and Chua´s equation, *Int. Journ.of Bifurc.&Chaos,* vol. 6, no. 12B, pp 2443-2489, 1996.

[4] SPROTT, J.C., LINZ, S.J.: Algebraically simple chaotic flows, *Intern. Journ. of Chaos th. and applic.*, vol.5, no.2, pp 1-20, 2000.

[5] POSPISIL J., BRZOBOHATY J., KOLKA Z., HORSKA J.: New reference state model of the third order piecewise linear dynamical system, *Radioengineering*, vol. 9, no. 3, pp 1-4.

[5] ZHONG, G., MAN, K., CHEN, G.: A systematic approach to generating *n*-scroll attractors, *Int. Journ.of Bifurc.&Chaos*, vol. 12, no. 12, pp 2907-2915, 2002.