Abstract—An algorithm basing on numerical differentiation and central Lagrange interpolation with multi-points is presented for the fundamental frequency estimation of non-sinusoidal signals in this paper. The signal is sampled at a fixed sample frequency of 25600Hz with the unknown parameters, the frequency is estimated with 7-point consequences using the high-order differentiation at a high accuracy of 0.001% over a very wide range varying from 2Hz to 1MHz in at most 1 cycle. Comparing with other algorithms, this algorithm spends little time and computation for frequency of the signal. The proposed algorithm is simulated in Matlab software using a testing study example with satisfactory results.

Index Terms—frequency estimation, fundamental harmonic, nonsinusoidal signal, numerical differentiation, Lagrange interpolation

1 Introduction

Frequency estimation has been one of important task in intelligent instrumentation and metering. Many of well-proven techniques such as zero crossing technique [1]-[2], level crossing technique [3], least squares error technique [4]-[6], Newton method [7], Kalman filter [8]-[12], Fourier transform [13]-[19], wavelet transform [18] have been used for this purpose, and some estimation results in accuracy give us a helpful guide. However, larger errors in frequency measurement of signal are often brought in, and much more time and computation must need so as not to be applied in real-time measurement and control.

The algorithm proposed in this paper is developed to estimate the fundamental frequency of non-sinusoidal signals with a frequency varying from 6Hz to 1MHz. This algorithm is based on numerical differentiation and central Lagrange interpolation with multi-points. Comparing with other algorithms, this algorithm spends little time and computation over a wide range at a high accuracy.

In section 1, the pioneering works in frequency estimation are described. In section 2, the proposed algorithm is presented. In section 3, the steps for the algorithm implementation are discussed. In section 4, a study case is simulated with Matlab software to illustrate the results of the proposed algorithm.

2 The Proposed Algorithm

2.1 Numerical Differentiation

Given a function of voltage signal:

\[ v(t) = 0 \]  

At discrete points such as \((t_i, v_i)\) and \((t_j, v_j)\), \(i = 0,1,2,\ldots, M-1, j = 0,1,2,\ldots, M-1\), the Taylor series expansion is expressed as:

\[
v(t_j) = v(t_j) + \Delta t \frac{dv}{dt} \bigg|_{t_j} + \frac{(\Delta t)^2}{2!} \frac{d^2v}{dt^2} \bigg|_{t_j} + \frac{(\Delta t)^3}{3!} \frac{d^3v}{dt^3} \bigg|_{t_j} + \ldots

\]  

\[
+ \frac{(\Delta t)^M}{M!} \frac{d^Mv}{dt^M} \bigg|_{t_j} + \ldots
\]  

where \(\Delta t = t_i - t_j\).

Without consideration of \(M\) order and higher order derivatives, we have central difference formulas such that:

\[
v(t_{i+k}) = v(t_i) + (t_{i+k} - t_i) v'(t_i) + \frac{(t_{i+k} - t_i)^2}{2!} v''(t_i)
\]  

\[
+ \frac{(t_{i+k} - t_i)^3}{3!} v^{(3)}(t_i) + \ldots + \frac{(t_{i+k} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i)
\]  

\[
\ldots
\]  

\[
v(t_{i+k}) = v(t_i) + (t_{i+k} - t_i) v'(t_i) + \frac{(t_{i+k} - t_i)^2}{2!} v''(t_i)
\]  

\[
+ \frac{(t_{i+k} - t_i)^3}{3!} v^{(3)}(t_i) + \ldots + \frac{(t_{i+k} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i)
\]  

\[
v(t_{i-1}) = v(t_i) + (t_{i-1} - t_i) v'(t_i) + \frac{(t_{i-1} - t_i)^2}{2!} v''(t_i)
\]  

\[
+ \frac{(t_{i-1} - t_i)^3}{3!} v^{(3)}(t_i) + \ldots + \frac{(t_{i-1} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i)
\]  

\[
v(t_{i-2}) = v(t_i) + (t_{i-2} - t_i) v'(t_i) + \frac{(t_{i-2} - t_i)^2}{2!} v''(t_i)
\]  

\[
+ \frac{(t_{i-2} - t_i)^3}{3!} v^{(3)}(t_i) + \ldots + \frac{(t_{i-2} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i)
\]
\[
\begin{align*}
&\frac{(t_{i+2} - t_i)^3}{3!} v^{(3)}(t_i) + \cdots + \frac{(t_{i+k} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
&\quad + \frac{(t_{i+1} - t_i)^3}{3!} v^{(3)}(t_i) + \cdots + \frac{(t_{i+k} - t_i)^{M-1}}{(M-1)!} v^{(M-1)}(t_i) = \sum_{j=0}^{M-1} \prod_{\ell=0}^{M-1} (t_{i+\ell} - t_j) \\
&\quad + \sum_{j=0}^{M-1} \prod_{\ell=0}^{M-1} (t_{i+\ell} - t_i) v(t_i) = 0 \\
\end{align*}
\]

where \( M \) is interpolation number, \( k = \lfloor M/2 \rfloor \). floor(\( A \)) rounds the elements of A to the nearest integers less than or equal to A.

Given \( (t_0, v_0), (t_1, v_1), \ldots, (t_M, v_M) \) with regular spaced \( h \), we have the following relationship:

\[
\begin{align*}
t_0 &\quad, t_1 = t_0 + h \quad, t_2 = t_0 + 2h \quad, t_3 = t_0 + 3h \quad, \ldots \\
t_M = t_0 + Mh
\end{align*}
\]

So that equation (3)-(8) become:

\[
\begin{align*}
v(t_{i+1}) &= v(t_i) + kh'v(t_i) + \frac{(kh^2)}{2!} v'(t_i) \\
&\quad + \frac{(kh)^3}{3!} v^{(3)}(t_i) + \cdots + \frac{(kh)^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
v(t_{i+2}) &= v(t_i) + 2hv(t_i) + \frac{(2h)^2}{2!} v'(t_i) \\
&\quad + \frac{(2h)^3}{3!} v^{(3)}(t_i) + \cdots + \frac{(2h)^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
v(t_{i+3}) &= v(t_i) + hv(t_i) + \frac{h^2}{2!} v'(t_i) \\
&\quad + \frac{h^3}{3!} v^{(3)}(t_i) + \cdots + \frac{h^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
v(t_{i+4}) &= v(t_i) - hv(t_i) + \frac{h^2}{2!} v'(t_i) \\
&\quad + \frac{h^3}{3!} v^{(3)}(t_i) + \cdots + \frac{h^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
v(t_{i+5}) &= v(t_i) - 2hv(t_i) + \frac{(2h)^2}{2!} v'(t_i) \\
&\quad + \frac{(2h)^3}{3!} v^{(3)}(t_i) + \cdots + \frac{(2h)^{M-1}}{(M-1)!} v^{(M-1)}(t_i) \\
v(t_{i+6}) &= v(t_i) - hv(t_i) + \frac{h^2}{2!} v'(t_i) \\
&\quad + \frac{h^3}{3!} v^{(3)}(t_i) + \cdots + \frac{h^{M-1}}{(M-1)!} v^{(M-1)}(t_i)
\end{align*}
\]

From equation (9)-(14), the \( s \) order differentiation of \( v(t) \) can mathematically be concluded as:

\[
v^{(s)}(t)
\]
The value of \( v' (p) \) at point \( p \) is expressed as:

\[
v' (p) = (-1)^0 \sum_{j=0}^{(M-1)/2} \frac{2 \prod_{j=0}^{M-1} (t_p - t_j)}{j! h^{M-1}} [v_{M-1} + v_0]
\]

\[
+ (-1)^1 \sum_{j=0}^{(M-1)/2} \frac{2 \prod_{j=0}^{M-1} (t_p - t_j)}{j! h^{M-1}} [v_{M-2} + v_1] + \ldots
\]

\[
- (-1)^k \frac{2 \prod_{j=0}^{M-1} (t_p - t_j)}{k! h^{M-1}} [v_{(M-(k-1))} + V_{k-1}]
\]

\[
+ \frac{2 \prod_{j=0}^{M-1} (t_p - t_j)}{k! h^{M-1}} [v_{k-1}]
\]

In the same form, the 3rd, 4th, 5th and 6th order derivative of \( v(t) \) at point \( p \) is written respectively:

\[
v^{(3)} (p) = \left[ v (p + 3) - 3v (p - 3) + 3v (p - 3) - v (p - 7) \right] / 12 h^3
\]

\[
v^{(4)} (p) = \left[ v (p + 4) - 4v (p - 4) + 6v (p - 4) - v (p - 8) \right] / 12 h^4
\]

\[
v^{(5)} (p) = \left[ v (p + 5) - 5v (p - 5) + 10v (p - 5) - v (p - 9) \right] / 12 h^5
\]

\[
v^{(6)} (p) = \left[ v (p + 6) - 6v (p - 6) + 15v (p - 6) - v (p - 10) \right] / 12 h^6
\]

With to \( M = 7 \), the 1st order derivative of \( v(t) \) at point \( p \) is expressed as follow:

\[
v' (p) = [ v(p + 3) - v(p - 3)] / 12 h^3
\]

The 2nd order derivative of \( v(t) \) at point \( p \) is expressed as follow:

\[
v'' (p) = [2[ v(p + 3) + v(p - 3)] - 27[ v(p + 2) + v(p - 2)] + 39[ v(p + 1) + v(p - 1)] / 180 h^2
\]

In the same form, the 3rd, 4th, 5th and 6th order derivative of \( v(t) \) at point \( p \) is respectively:

\[
v^{(3)} (p) = [ v (p + 3) - 3v (p - 3) + 3v (p - 3) - v (p - 7) ] / 12 h^3
\]

\[
v^{(4)} (p) = [ v (p + 4) - 4v (p - 4) + 6v (p - 4) - v (p - 8) ] / 12 h^4
\]

\[
v^{(5)} (p) = [ v (p + 5) - 5v (p - 5) + 10v (p - 5) - v (p - 9) ] / 12 h^5
\]

\[
v^{(6)} (p) = [ v (p + 6) - 6v (p - 6) + 15v (p - 6) - v (p - 10) ] / 12 h^6
\]

With to \( M = 15 \), the 2nd order differentiation of \( v(t) \) at point \( p \) is expressed as follow:

\[
v'' (p) = \frac{10368}{871782912 h^2} [ v(p + 7) + v(p - 7) ]
\]

\[
- \frac{14112}{62270208 h^2} [ v(p + 6) + v(p - 6) ]
\]

\[
+ \frac{2032128}{958003200 h^2} [ v(p + 5) + v(p - 5) ]
\]

\[
- \frac{3175200}{239508000 h^2} [ v(p + 4) + v(p - 4) ]
\]

\[
+ \frac{56448}{870912 h^2} [ v(p + 3) + v(p - 3) ]
\]

\[
- \frac{435456}{1270080 h^2} [ v(p + 2) + v(p - 2) ]
\]

\[
+ \frac{508032}{2903040 h^2} [ v(p + 1) + v(p - 1) ]
\]

\[
- \frac{435456}{1270080 h^2}
\]

2.2 Frequency Estimation

Without loss of generality, a non-sinusoidal signal with 3rd -order harmonics is taking into consideration:

\[
v(n) = V_1 \sin(2 \pi f t + \phi_1) + V_2 \sin(2 \pi f t + \phi_2)
\]

\[
+ V_3 \sin(3 \pi f t + \phi_3)
\]
The 1st-order differentiation of \( v(n) \) is formulated:
\[
v'(n) = V_1 \left( 2/\pi \right) \sin(2\pi f t_s + \phi_1) + V_2 \left( 2/\pi \right) \cos(2\pi f t_s + \phi_2) + V_3 \left( 2/\pi \right) \cos(3\pi f t_s + \phi_3)
\]
(27)

The 2nd-order differentiation of \( v(n) \) is formulated:
\[
v''(n) = V_1 \left( 4/\pi^2 \right) \sin(2\pi f t_s + \phi_1) - V_2 \left( 4/\pi^2 \right) \sin(2\pi f t_s + \phi_2) - V_3 \left( 4/\pi^2 \right) \sin(3\pi f t_s + \phi_3)
\]
(28)

From equation (27) and (28), we get the odd-order and even-order differentiation of \( v(n) \) is expressed respectively:
\[
v^{(o)}(n) = (-1)^{\frac{o-1}{2}} V_1 \left( 2/\pi \right)^o \cos(2\pi f t_s + \phi_1)
+ \left(\frac{-1}{2}\right)^{\frac{o-1}{2}} V_2 \left( 2/\pi \right)^o \cos(2\pi f t_s + \phi_2)
+ \left(\frac{-1}{2}\right)^{\frac{o-1}{2}} V_3 \left( 2/\pi \right)^o \cos(3\pi f t_s + \phi_3)
\]
(29)

\[
v^{(e)}(n) = (-1)^{\frac{o-2}{2}} V_1 \left( 2/\pi \right)^o \sin(2\pi f t_s + \phi_1)
+ \left(\frac{-1}{2}\right)^{\frac{o-2}{2}} V_2 \left( 2/\pi \right)^o \sin(2\pi f t_s + \phi_2)
+ \left(\frac{-1}{2}\right)^{\frac{o-2}{2}} V_3 \left( 2/\pi \right)^o \sin(3\pi f t_s + \phi_3)
\]
(30)

Step 2: Basing on central numerical differentiation of 7 points, compute the \( 2^{nd} \) to \( 5^{th} \) differentiation of the signal basing equation (16) - (25).

\[ v^{(o)}(n) = \sum_{k=1}^{N} V_k \sin(2\pi f k T / N + \phi_k) \]
(39)

The tests of numerical simulation for a non-sinusoidal signal with 3 order harmonics are carried out in Matlab codes. The tested signal is shown as the following equation.

\[
v(t) = 20 \sqrt{2} \sin(2\pi f t + \phi_1) + 0.8 \sqrt{2} \sin(2\pi f t + 6\phi) + 1.6 \sqrt{2} \sin(2\pi f t + 12\phi)
\]
(40)
Basing on the implementation steps, a fixed sample frequency: 512×50Hz is used for frequency estimation. The results of estimation for fundamental frequency are shown in Table I. From the table, it is seen that the fundamental frequency is estimated at an accuracy of 0.001% over a range varying from 2Hz to 1MHz. Over wide frequency range, the relative errors are retained at 0.001% or smaller. However, the relative errors and he absolute are all very small when the frequency varies from 2Hz to 40kHz while the relative errors are small and the absolute errors are large when the frequency varies from 40kHz to 1MHz.

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5 Conclusion

Basing on numerical differentiation and central Lagrange interpolation with multi-points, an algorithm is presented for the fundamental frequency estimation in this paper. One advantage of the proposed algorithm is that the fundamental frequency of non-sinusoidal signals with multi-components is estimated at a high accuracy of 0.001% over a very wide range varying from 2Hz to 1MHz in at most 1 cycle. Basing on the proposed algorithm, the parameters of the sampled signals such as amplitudes and the phase angles of the fundamental harmonic and other harmonic need not to be known. At a great degree, the frequency range with a higher accuracy is dependent on the sample frequency and the coefficients for the computation, which comes from the experiences.

Over wide frequency range, the relative errors are retained at 0.001% or smaller. However, the relative errors and he absolute are all very small when the frequency varies from 2Hz to 40kHz while the relative errors are small and the absolute errors are large when the frequency varies from 40kHz to 1MHz.

Comparing with other algorithms, this algorithm spends little time and computation over a wide range at a high accuracy due to the use of numerical differentiation and central Lagrange interpolation with multi-points.

References


