Active RLC Circuit Model of the Second-order Dynamical System with Piecewise-linear Controlled Sources

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Abstract: - RLC active circuit model with voltage-controlled voltage-source (VCVS) and current-controlled current-source (CCCS) for the second-order autonomous dynamical system realization is proposed. Its circuit parameters are directly related to the state model parameters that lead to simple design formulas.

Key-Words: - Dynamical systems, State models, Active equivalent circuits, Piecewise-linear controlled sources

1 Introduction

Autonomous piecewise-linear (PWL) systems of Class *C* can be described by the general state matrix form [3], [4]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\; h\,(\mathbf{w}^{\mathbf{T}}\mathbf{x}) \;, \qquad (1)$$

where the normalized elementary PWL feedback function (Fig. 1)

$$h\,(\mathbf{w}^{\mathbf{T}}\mathbf{x}) = \tfrac{1}{2}\left(\left|\mathbf{w}^{\mathbf{T}}\mathbf{x}+1\right| - \left|\mathbf{w}^{\mathbf{T}}\mathbf{x}-1\right|\right) \qquad (2)$$

contains the regions D_0 and D_{+1} (D_{-1}). The dynamical behavior of the system is determined by two characteristic polynomials associated to these individual regions [3]. All systems of the Class *C* having the same characteristic polynomials are qualitatively equivalent and they are related by linear topological conjugacy [4]. Typical systems of this class are the Chua's model, both its canonical forms [3], and also recently derived optimized state model having the minimum sum of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters [8]. Just this low-sensitivity model is very useful as a prototype

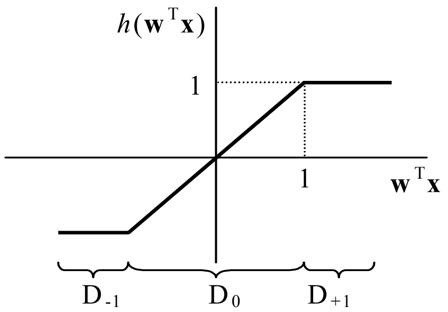

Fig. 1. Simple memoryless PWL feedback function.

for practical chaotic system realization in the form of electronic circuit. It provides the possibility to utilize the block-decomposed form of the state matrix so that the design procedure can be started from the optimized second-order system and then extended by a simple way to the optimized higher-order case [8].

State model can be used as a mathematical tool for the numerical simulation of dynamical system behavior as well as a prototype for the electronic circuit realizat-ion using available circuit technique. From the complete state equations can directly be derived either the general integra-tor-based circuit block-diagram (it is typical for both canonical forms) or the corresponding RLC active circuit (it is typical for Chua's oscillator). In both cases only a single PWL network element is used utilizing various types of active electronic blocks operating in both the voltage and current modes (op-amps, current conveyors, transimpedance amplifiers, etc.).

For the optimized low-sensitivity model first the corresponding integrator-based block diagram has been derived for both second- and third-order cases [8]. The intention of this contribution is to propose the corresponding RLC active circuits where, unlike the Chua's model, the circuit parameters have a direct relation to the model parameters.

2 Optimized Second-order State Models with Low Eigenvalue Sensitivities

The most frequently occurring autonomous dynamical systems have their complex conjugate eigenvalues in both regions of PWL function (Fig. 1), i.e. for the inner region (D_0) it is $\left(\mu_{1,2} = \mu' \pm j\mu''\right)$ and for the outer regions (D_{-1}, D_{+1}) it is $(\nu_{1,2} = \nu' \pm j\nu'')$. Then the associated characteristic polynomials are defined as follows

$$(D_0):\quad P(s) = (s-\mu_1)(s-\mu_2) = \det(s\mathbf{1}-\mathbf{A}_0) \tag{3a}$$

$$(D_{-1}, D_{+1}):\quad Q(s) = (s-\nu_1)(s-\nu_2) = \det(s\mathbf{1}-\mathbf{A}) \tag{3b}$$

where relation between the state matrices can be expressed as [3] $\quad \mathbf{A}_0 = \mathbf{A} + \mathbf{b}\mathbf{w}^{\mathrm{T}}$, (4)

and $\mathbf{1}$ is the unity matrix. The optimized low-sensitivity state model (1) have been chosen in the simplified and decomposed complex form [9], where the corresponding state matrices are

$$\mathbf{A} = \begin{bmatrix} \nu' & -\nu'' \\ \nu'' & \nu' \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} \mu' & -\mu''\mathrm{K} \\ \mu''\mathrm{K}^{-1} & \mu' \end{bmatrix}, \tag{5a,b}$$

and the optimizing coefficient K is given as the real root of the quadratic equation

$$\mathrm{K}^2 - 2\mathrm{K}(\mathrm{M}+1) + 1 = 0, \quad \text{i.e.} \quad \mathrm{K} = 1 + \mathrm{M} \pm \sqrt{\mathrm{M}(\mathrm{M}+2)},$$

where the auxiliary parameter M is

$$\mathrm{M} = \frac{(\mu'-\nu')^2 + (\mu''-\nu'')^2}{2\mu''\nu''} > 0, \qquad (\mu'', \nu'' \neq 0).$$

In the vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ one of the para-meters can be chosen, e.g. $w_1 = 1$, while the others are obtained as [8]

$$b_1 = \mu' - \nu', \quad b_2 = \frac{(\mu'-\nu')^2}{\nu'' - \mu''\mathrm{K}}, \quad w_2 = \frac{\nu'' - \mu''\mathrm{K}}{\mu' - \nu'} \tag{6a,b,c}$$

Then the complete state equations of the optimized second-order PWL autonomous system can be then written as

$$\dot{x} = \nu'[x - h(x + w_2 y)] - \nu'' y + \mu' h(x + w_2 y), \tag{7}$$

$$\dot{y} = \nu'' x + \nu' y + b_2 h(x + w_2 y), \tag{8}$$

where the parameters b_2 and w_2 are given by the formulas (6b,c). The corresponding integrator-based circuit block diagram, suitable also as the prototype for the practical realization, is shown in [8]. All the sensitivity functions are obtained in the complex form and the same functions, expressed separately for the eigenvalues real and imaginary parts, can easily be derived. Then the minimum sums of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters can be expressed for both the real and imaginary parts generally as

$$\sum S_r^2(\lambda', a_{ij}) = \sum S_r^2(\lambda'', a_{ij}) = \frac{1}{2} \tag{9}$$

where in outer regions (D_{-1}, D_{+1}) $\lambda' = \nu'$, $\lambda'' = \nu''$ and in inner region (D_0) $\lambda' = \mu'$, $\lambda'' = \mu''$ [9].

3 Equivalent active RLC circuits with piecewise-linear controlled sources

3.1 Equivalent circuit utilizing VCVS

Consider the autonomous RLC circuit introduced in Fig. 2 containing voltage-controlled voltage-source (VCVS) with PWL transfer characteristic function $u_0 = f(u_1 + R_2 i_2)$ having three segments (Fig. 3) expressed as

$$u_0 = A_1(u_1 + R_2 i_2) + (A_0 - A_1)h(u_1 + R_2 i_2) \tag{10}$$

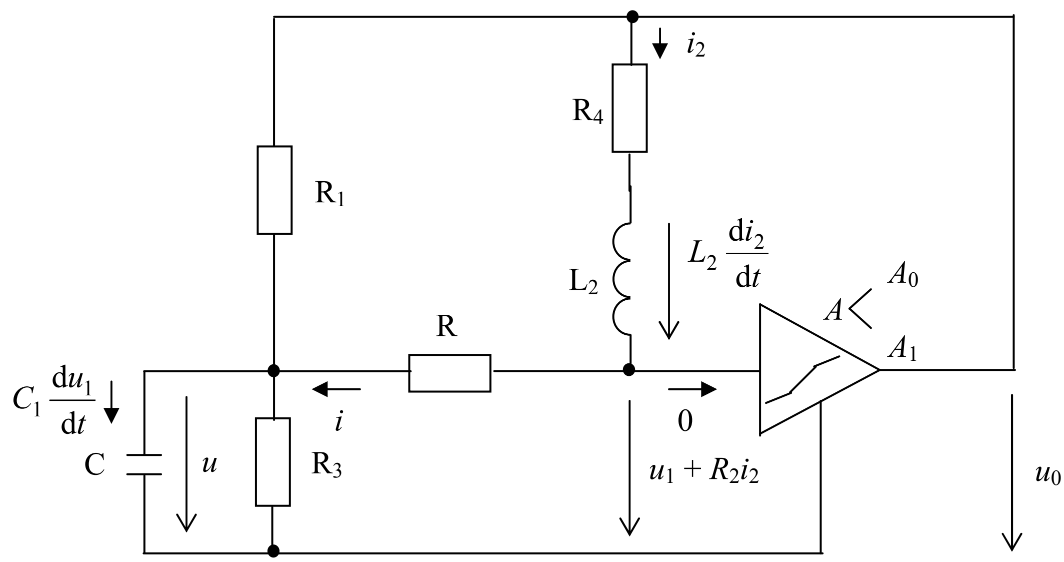

Fig. 2. Second-order autonomous circuit with PWL voltage

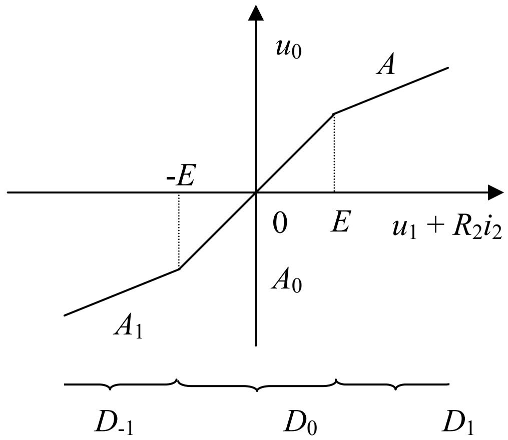

Fig. 3. Transfer PWL characteristic of VCVS.

Choosing state variables the capacitor voltage u_1 and the inductor current i_2 two Kirchhoff's equations of this circuit can be written in the basic form

$$\frac{u_0 - u_1}{R_1} + i_2 - C_1\frac{\mathrm{d}u_1}{\mathrm{d}t} - \frac{u_1}{R_3} = 0 \tag{11a}$$

$$R_4 i_2 + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_2 i_2 - (u_0 - u_1) = 0 \tag{11b}$$

and then rewritten to the complete (non-normalized) state equation form, i.e.

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} = &\frac{A_1G_1 - (G_1 + G_3)}{C_1}u_1 + \frac{A_1G_1R_2 + 1}{C_1}i_2 \\ &+ \frac{A_0 - A_1}{C_1R_1}h(u_1 + R_2 i_2)\end{aligned} \tag{12a}$$

$$\begin{aligned}\frac{\mathrm{d}i_2}{\mathrm{d}t} = &\frac{A_1 - 1}{L_2}u_1 + \frac{A_1R_2 - (R_2 + R_4)}{L_2}i_2 \\ &+ \frac{A_0 - A_1}{L_2}h(u_1 + R_2 i_2)\end{aligned} \tag{12b}$$

Utilizing reference values of voltage E (Fig. 3), resistance R_0, and capacitance C_0 the normalized state variables including the time scaling can be given as

$$x = \frac{u_1}{E}, \quad y = i_2\left(\frac{R_0}{E}\right), \quad \tau = \frac{t}{R_0C_0} \tag{13a,b,c}$$

Then the corresponding normalized capacitance, inductance and all resistances are

$$\alpha = \frac{C_1}{C_0}, \quad \beta = \frac{L_2}{R_0^{\,2}C_0}, \quad r_1 = \frac{1}{g_1} = \frac{R_1}{R_0},$$

$$r_2 = \frac{R_2}{R_0}, \quad r_3 = \frac{1}{g_3} = \frac{R_3}{R_0}, \quad r_4 = \frac{R_4}{R_0} \tag{14}$$

Denoting $k = \mathrm{sgn}(R_0C_0)$ the state equations (12) can be rewritten into the normalized form

$$\begin{aligned}\dot{x} = &\frac{k}{\alpha}[(A_1 - 1)g_1 - g_3]x + \frac{k}{\alpha}(1 + A_1g_1r_2)y \\ &+ \frac{k}{\alpha}(A_0 - A_1)g_1h(x + r_2y)\end{aligned}, \tag{15a}$$

$$\begin{aligned}\dot{y} = &\frac{k}{\beta}(A_1 - 1)x + \frac{k}{\beta}[(A_1 - 1)r_2 - r_4]y \\ &+ \frac{k}{\beta}(A_0 - A_1)h(x + r_2y)\end{aligned} \tag{15b}$$

Comparing them with general matrix form (1) for the second-order system

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{16}$$

the following equations are obtained

$$a_{11} = \frac{k}{\alpha}[(A_1 - 1)g_1 - g_3], \quad a_{12} = \frac{k}{\alpha}(1 + A_1g_1r_2),$$

$$b_1 = \frac{k}{\alpha}(A_0 - A_1)g_1, \qquad w_1 = 1, \tag{17a,b,c,d}$$

$$a_{22} = \frac{k}{\beta}[(A_1 - 1)r_2 - r_4], \quad a_{21} = \frac{k}{\beta}(A_1 - 1),$$

$$b_2 = \frac{k}{\beta}(A_0 - A_1), \qquad w_2 = r_2 \tag{18a,b,c,d}$$

and then utilized as independent formulas for design of the individual circuit parameters. For the case when α, β and k are chosen as free parameters it is summarized in the next design formulas where both the general and optimized state model are considered.

$$g_1 = \frac{1}{r_1} = \frac{R_0}{R_1} = \frac{\alpha}{\beta}\frac{b_1}{b_2} = \frac{\alpha}{\beta}\frac{\nu'' - \mu''K}{\mu' - \nu'}$$

$$r_2 = \frac{R_2}{R_0} = w_2 = \frac{\nu'' - \mu''K}{\mu' - \nu'}$$

$$g_3 = \frac{1}{r_3} = \frac{R_0}{R_3} = \frac{\alpha}{k}(a_{21}w_2 - a_{11}) = \frac{\alpha}{k}\left(\nu''\frac{\nu'' - \mu''K}{\mu' - \nu'} - \nu'\right)$$

$$r_4 = \frac{R_4}{R_0} = \frac{\beta}{k}(a_{21}w_2 - a_{22}) = \frac{\beta}{k}\left(\nu''\frac{\nu'' - \mu''K}{\mu' - \nu'} - \nu'\right)$$

$$A_1 = 1 + \frac{\beta}{k}a_{21} = 1 + \frac{\beta}{k}\nu''$$

$$A_0 = 1 + \frac{\beta}{k}(a_{21} + b_2) = 1 + \frac{\beta}{k}\left[\nu'' + \frac{(\mu' - \nu')^2}{\nu'' - \mu''K}\right]$$

Any other details about realization conditions of the individual circuit elements are introduced in [9].

3.2 Equivalent circuit utilizing CCCS

Consider the autonomous RLC circuit introduced in Fig. 4 containing current-controlled current-source (CCCS) with PWL transfer characteristic function $i_0 = f(i_1 + G_2 u_2)$ having three segments (Fig. 5) expressed as

$$i_0 = B_1(i_1 + G_2 u_2) + (B_0 - B_1)h(i_1 + G_2 u_2) \quad (19)$$

Choosing state variables the inductor current i_1 and the capacitor voltage u_2 two Kirchhoff's equations of this circuit can be written in the basic form

$$R_3 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - u_2 - R_1(i_0 - i_1) = 0 \quad (20a)$$

$$i_0 - i_1 - C_2 \frac{\mathrm{d}u_2}{\mathrm{d}t} - \frac{u_2}{R_4} - \frac{u_2}{R_2} = 0 \quad (20b)$$

and then rewritten to the complete (non-normalized) state equation form, i.e.

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{B_1 R_1 - (R_1 + R_3)}{L_1} i_1 + \frac{B_1 R_1 G_2 + 1}{L_1} u_2 + \frac{B_0 - B_1}{L_1 G_1} h(i_1 + G_2 u_2) \quad (21a)$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = \frac{B_1 - 1}{C_2} i_1 + \frac{B_1 G_2 - (G_2 + G_4)}{C_2} u_2 + \frac{B_0 - B_1}{C_2} h(i_1 + G_2 u_2) \quad (21b)$$

Utilizing reference values of voltage E (in Fig. 5 $I_0 = E/R_0$), resistance R_0, and capacitance C_0 the normalized state variables including the time scaling can be established as

$$x = i_1\left(\frac{R_0}{E}\right), \quad y = \frac{u_2}{E}, \quad \tau = \frac{t}{R_0 C_0} \quad (22a,b,c)$$

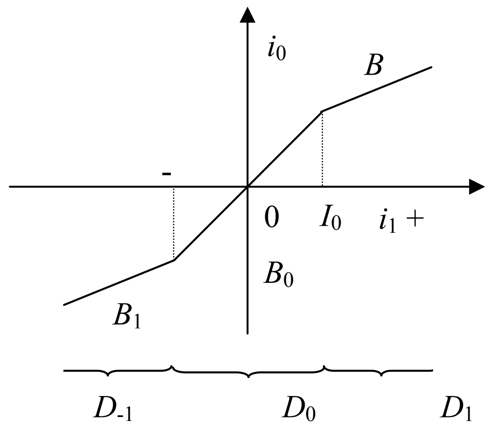

Fig. 5. Transfer PWL characteristic of CCCS.

Then the corresponding normalized inductance, capacitance, and all resistances are

$$\alpha = \frac{L_1}{R_0{}^2 C_0}, \quad \beta = \frac{C_2}{C_0}, \quad r_1 = \frac{R_1}{R_0}, \quad r_2 = \frac{1}{g_2} = \frac{R_2}{R_0}, \quad r_3 = \frac{R_3}{R_0}, \quad r_4 = \frac{1}{g_4} = \frac{R_4}{R_0} \quad (23)$$

Denoting $k = \mathrm{sgn}(R_0 C_0)$ the state equations (21) can be rewritten into the normalized form

$$\dot{x} = \frac{k}{\alpha}[(B_1 - 1)r_1 - r_3]x + \frac{k}{\alpha}(1 + B_1 r_1 g_2)y + \frac{k}{\alpha}(B_0 - B_1)r_1 h(x + g_2 y), \quad (24a)$$

$$\dot{y} = \frac{k}{\beta}(B_1 - 1)x + \frac{k}{\beta}[(B_1 - 1)g_2 - g_4]y + \frac{k}{\beta}(B_0 - B_1)h(x + g_2 y) \quad (24b)$$

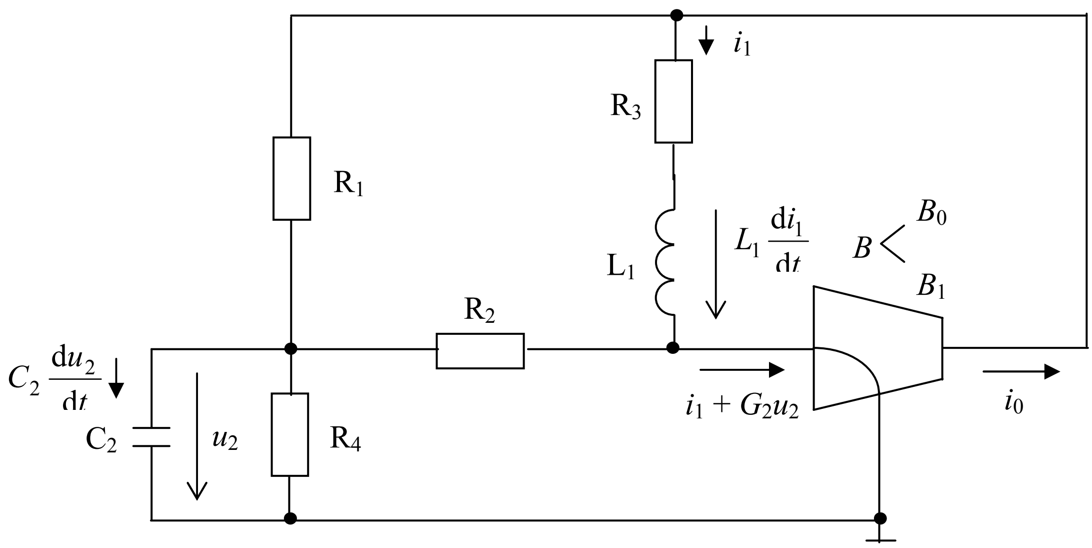

Fig. 4. Autonomous 2nd order circuit with PWL current source

Comparing them with general matrix form (1) for the second-order system

$$\mathbf{x}=\begin{bmatrix} x \\ y \end{bmatrix},\ \mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ \mathbf{w}=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\quad (25)$$

the following equations are obtained

$$a_{11}=\frac{k}{\alpha}[(B_1-1)r_1-r_3],\quad a_{12}=\frac{k}{\alpha}(1+B_1r_1g_2),$$
$$b_1=\frac{k}{\alpha}(B_0-B_1)r_1,\qquad w_1=1,\qquad (26a,b,c,d)$$

$$a_{22}=\frac{k}{\beta}[(B_1-1)g_2-g_4],\quad a_{21}=\frac{k}{\beta}(B_1-1),$$
$$b_2=\frac{k}{\beta}(B_0-B_1),\qquad w_2=g_2\qquad (27a,b,c,d)$$

and then utilized as independent formulas for design of the individual circuit parameters. For the case when α, β and k are chosen as free parameters it is summarized in the resultant design formulas where both the general and optimized state model are considered.

$$r_1=\frac{R_1}{R_0}=\frac{\alpha}{\beta}\frac{b_1}{b_2}=\frac{\alpha}{\beta}\frac{\nu''-\mu''K}{\mu'-\nu'}$$
$$g_2=\frac{1}{r_2}=\frac{R_0}{R_2}=w_2=\frac{\nu''-\mu''K}{\mu'-\nu'}$$

$$r_3=\frac{R_3}{R_0}=\frac{\alpha}{k}(a_{21}w_2-a_{11})=\frac{\alpha}{k}\left(\nu''\frac{\nu''-\mu''K}{\mu'-\nu'}-\nu'\right)$$
$$g_4=\frac{1}{r_4}=\frac{R_0}{R_4}=\frac{\beta}{k}(a_{21}w_2-a_{22})=\frac{\beta}{k}\left(\nu''\frac{\nu''-\mu''K}{\mu'-\nu'}-\nu'\right)$$

$$B_1=1+\frac{\beta}{k}a_{21}=1+\frac{\beta}{k}\nu''$$
$$B_0=1+\frac{\beta}{k}(a_{21}+b_2)=1+\frac{\beta}{k}\left[\nu''+\frac{(\mu'-\nu')^2}{\nu''-\mu''K}\right]$$

Any other details about realization conditions of the individual circuit elements are introduced in [9].

4 Conclusion

This contribution deals with the second-order nonlinear dynamical systems and their realization using active RLC circuit where the active elements the VCVS and CCCS with three-segment PWL symmetric transfer characteristic is considered, i.e. suitable especially for voltage- and current mode realization. Dynamical behavior of such a system is determined by two sets of complex conjugate state matrix eigenvalues associated with the corresponding regions.

The complete and normalized state equations are introduced where simple relation between model and circuit parameters entails also very simple design formulas in the synthesis procedure either in general or optimized (low eigenvalue sensitivities) forms. The circuits proposed represent one possibility of the second-order system realization and can be easily extended also for the third-order system utilizing the block decomposition of the state matrix. Such higher-order equivalent circuit can model also chaotic behavior of the system.

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