Hénon Mappings and Wada Lakes

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Abstract: We use an inductive limit construction to examine the existence of Wada lakes for Hénon mappings.

Key-Words: complex analytic dynamical systems, Hénon mappings, inductive limits, Wada Lakes.

1 Introduction

A continuum is a compact connected set. A continuum is called indecomposable if it is not the union of two proper subcontinua. A famous example in plane topology, attributed to a mysterious Mr. Wada by K. Yoneyama in [**5**], gives three bounded, connected, simply connected, open subsets of $\mathbb{R}^2$ with a common boundary that is an indecomposable continuum.

Imagine an island that is home to three philanthropists owning lakes of water, milk, and wine, respectively. The owner of the lake of water generously decides to build a network of canals bringing water within 100 meters of every spot of the island. It is clearly possible to do this while keeping the union of the original water lake and the water canals connected and simply connected with closures disjoint from the other lakes.

Next, the owner of lake of milk decides to bring milk to within 10 m of every spot on the island, also keeping the milk locus connected and simply connected.

Not to be outdone, the owner of the lake of wine now decides to bring wine to within 1 m of every spot on the island. Although canal building is becoming more complicated, the wine purveyor, with proper fortification, accomplishes the task.

The construction continues with each of the three philanthropists, in turn, bringing his or her product closer to the poor inhabitants of the island. Land prices soar as land becomes scarcer. The real estate market collapses.

In the limit, the construction achieves the desired result: each of the lakes, being an increasing union of connected, simply connected open sets, is a connected, simply connected set, and each point of the boundary of one is in the boundary of the other two.

We wish to show that, under appropriate circumstances, the basins of attraction of attracting cycles form Wada lakes for Hénon mappings in $\mathbb{R}^2$. As it turns out, the "strategy" of these basins is remarkably similar to that of the philanthropists above.

2 Hénon Mappings

Let us consider the Hénon family

$$H_{a,p} : \mathbb{C}^2 \to \mathbb{C}^2$$

defined by

$$H_{a,p} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}$$

where $a \in \mathbb{C} - \{0\}$, p is a polynomial of degree $d \geq 2$, and $x, y \in \mathbb{C}$. $H_{a,p}$ is invertible since

$$H_{p,a}^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ (p(y) - x)/a \end{bmatrix}.$$

For any natural number, let $f^{\circ n}$ and $f^{\circ -n}$ denote the n-fold composites of f and f^{-1}, respectively.

Given a polynomial $p(z)$, the following sets are central in the study of the dynamics of p:

$$K_p = \left\{ z \;\middle|\; \lim_{n\to\infty} p^{\circ n}(z) \neq \infty \right\}$$

and its boundary $J_p = \partial K_p$, the Julia set of p.

In [3], the sets studied for a Hénon mapping H are defined in imitation of the one-dimensional case:

$$K_\pm = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \lim_{n\to\infty} \left\| H^{\circ \pm n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \neq \infty \right\},$$

$$U_\pm = \mathbb{C}^2 - K_\pm, \qquad J_\pm = \partial K_\pm,$$

$$K = K_+ \cap K_-, \qquad J = J_+ \cap J_-.$$

K and J are compact invariant sets under H.

3 Inductive Limits

If $f : X \to X$ is a mapping from a space to itself, then the inductive limit

$$\check{X}_f = \varinjlim (X,\ f)$$

is the quotient $(X \times \mathbb{N})/\sim$, where $\sim$ is generated by setting $(x, n) \sim (f(x), n + 1)$.

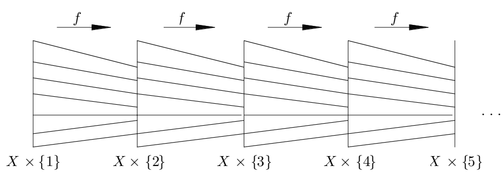

Fig. 1

Inductive limits are pathological objects in general, and will be Hausdorff only when f has some nice properties. When f is open and injective, the inductive limit is an increasing union of subsets homeomorphic to X, hence locally as nice as X.

The inductive limit comes with a map to itself: $\check{f} : \check{X}_f \to \check{X}_f$ induced by

$$\check{f} : (x, n) \mapsto (f(x), n) \sim (x, n - 1).$$

This mapping is obviously bijective, as an inverse is induced by $(x, n) \mapsto (x, n + 1)$.

Let p be a hyperbolic polynomial; then p has no critical points in its Julia set J_p. Let $D \subset \mathbb{C}$ be a disk of radius R sufficiently large so that $J_p \subset D$. Consider the mapping

$$f_{p,\alpha,R} : J_p \times D \to J_p \times \mathbb{C}$$

given by

$$f_{p,\alpha,R} \begin{bmatrix} \zeta \\ z \end{bmatrix} = \begin{bmatrix} p(\zeta) \\ \zeta + \alpha \frac{z}{p'(\zeta)} \end{bmatrix},$$

which is well defined since $p'(\zeta) \neq 0$.

Proposition 1. *For sufficiently small $|\alpha| \neq 0$, the mapping $f_{p,\alpha,R}$ is open and injective with*

$$f_{p,\alpha,R}(J_p \times D) \subset J_p \times D.$$

Proof. Clearly if $|\alpha|$ is sufficiently small, the image lies in $J_p \times D$. Moreover, if there are no critical points in J_p, then there exists $\varepsilon > 0$ such that when $\zeta_1, \zeta_2 \in J_p$ with $\zeta_1 \neq \zeta_2$ and $p(\zeta_1) = p(\zeta_2)$, then $|\zeta_1 - \zeta_2| > \varepsilon$. If α is chosen such that

$$0 < |\alpha| < \frac{\varepsilon R}{\inf_{\zeta \in J_p} |p'(\zeta)|},$$

then $f_{p,\alpha,R}$ is clearly injective. The mapping is open because it is a local homeomorphism. $\square$

In general, if $\psi : X \to Y$ is a homeomorphism conjugating $f : X \to X$ and $g : Y \to Y$, then ψ induces a homeomorphism $\check{\psi} : \check{X}_f \to \check{Y}_g$ conjugating $\check{f} : \check{X}_f \to \check{X}_f$ to $\check{g} : \check{Y}_g \to \check{Y}_g$.

Proposition 2. *For all α_1, α_2 sufficiently small and all R_1 and R_2 sufficiently large, there is a homeomorphism*

$$\psi : J_p \times D_{R_1} \to J_p \times D_{R_2}$$

conjugating f_{p,α_1,R_1} to f_{p,α_2,R_2}.

This proposition shows that the indices α and R may be dropped. Thus when p is hyperbolic and $|\alpha|$ is sufficiently small and R is sufficiently large, we may set

$$\check{\mathbb{C}}_p = \check{\mathbb{C}}_{p,\alpha,R} = \varinjlim (J_p \times D,\ f_{p,\alpha,R}),$$

and

$$\check{p} = \check{f}_{p,\alpha,R} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p.$$

The space $\check{\mathbb{C}}_p$ is quite difficult to understand. The only case where it is anything familiar is when J_p is a Jordan curve; in that case $\check{\mathbb{C}}_p$ is homeomorphic to the complement of a solenoid in a 3-sphere.

When p is a real hyperbolic polynomial, the real part $\check{\mathbb{R}}_p$ is often the common separator of Wada lakes. This illustrates some of the unavoidable complexity.

4 Dense Polynomials

A dense polynomial, p, satisfies the following properties:

(1) p is a real hyperbolic polynomial,

(2) the Julia set of p is connected,
(3) all the attracting cycles of p are real, and
(4) for each such fixed point x, its real domain of attraction $\Omega_x \cap \mathbb{R}$ is dense in $J_p \cap \mathbb{R}$.

There are lots of dense polynomials. The following lemma describes some of them in degree 2.

Lemma 1. *Let p be a real quadratic polynomial with an attracting cycle of period k, with k an odd prime. Then the k basins $U_1 = 0, \dots, U_{k-1}$ of the attracting fixed points of $p^{\circ k}$ in $\mathbb{R}$ are all dense in $J_p \cap \mathbb{R}$.*

Proof. Denote by I_0 the largest bounded interval invariant under the polynomial; it is bounded by the "external" fixed point and its inverse image. Without loss of generality we may assume that the critical point is periodic of period k; let

$$c_0, c_1, \dots, c_{k-1}, c_k = c_0$$

be the critical orbit; all the interesting dynamics occurs in the interval $I = [c_1, c_2] \subset I_0$.

The polynomial p also has an "internal" fixed point $\alpha \in [c_0, c_1]$. If $J \subset I$ is any interval containing α, then $\cup p^{\circ n}(J) = I$. The alternative is that $\cup p^{\circ n}(J) = J_0$ is an interval in $[c_0, c_1]$ bounded by a cycle of period 2, and there are no such cycles in $[c_0, c_1]$ (here we are using that p is a polynomial, not just a unimodal map). It follows from this that each of the basins U_i accumulates at α.

Thus to prove the lemma, it is enough to show that the real inverse images of α are dense in the real Julia set $J_p \cap \mathbb{R}$. Let us denote by $V_0, \dots, V_{k-1}$ the immediate domains of attraction in $\mathbb{R}$. It is known that if k is an odd prime (or more generally simply odd) the V_i have disjoint closures; let

$$\mathfrak{T} = \{T_1, \dots, T_{k-1}\}$$

be the bounded components of $I - \cup V_i$.

Sublemma. *If there is an inverse image of α in each T_j, then p is a dense polynomial.*

Proof of Sublemma. The Julia set is

$$J_p \cap \mathbb{R} = I_0 - \bigcup_{i=0}^{k-1} \bigcup_{n=0}^{\infty} p^{-n}(V_i).$$

If each component of

$$X_M = I_0 - \bigcup_{i=0}^{k-1} \bigcup_{n=0}^{M} p^{-n}(V_i)$$

contains an inverse image of α, then these inverse images will accumulate on all of $J_p \cap \mathbb{R}$. But if each component of X_M contains an inverse image of α, then this is also true of each component of X_{M+1}, since p maps each component of X_{M+1} to a component of X_M. Thus it is enough to start the induction, which is the hypothesis of the sublemma.

$\square$ Sublemma

There is a repelling cycle Z of length k such that all endpoints of intervals $T \in \mathfrak{T}$ are either in Z or in its inverse images. Let us denote $\mathfrak{T}'$ those intervals for which at least one end-point is periodic, and $\mathfrak{T}''$ the others. Moreover set

$$A = \bigcup_{T \in \mathfrak{T}'} \bigcup_{n=0}^{\infty} p^n(T).$$

Now there are two possibilities:

(a) If $\alpha \in A$, there is an inverse image of α in some $T' \in \mathfrak{T}'$. But then there must be an inverse image of α in every $T \in \mathfrak{T}$, since each endpoint of T will eventually land on every point of Z, in particular on an end-point of T'; that iterate of T will cover T'. Then by the Sublemma, p is dense.

(b) If $\alpha \notin A$, then A is disconnected, and p permutes the components of A circularly, with period k' with $1 < k' < k$. This is because some interval $T \in \mathfrak{T}'$ must have both endpoints in Z, as there is one more point in Z than there are intervals in $\mathfrak{T}$. That interval must return to itself in fewer that k moves. Moreover k' divides k, since the map $Z \to \pi_0(A)$ is equivariant, *i.e.*, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[d, "p"'] & \pi_0(A) \arrow[d, "\pi_0(p)"] \\ Z \arrow[r, "\psi"'] & \pi_0(A) \end{tikzcd}$$

This cannot happen if k is prime. $\square$ (Lemma 1)

Fig. 2, for the polynomial

$$z^2 - 1.785866\dots,$$

with a superattractive cycle of length 9, should illustrate what is going on.

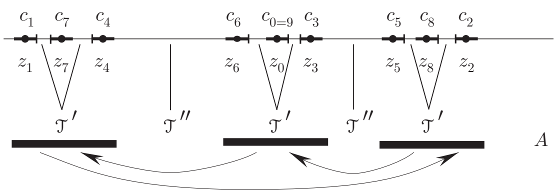

Fig. 2

For this polynomial, the critical point is periodic of period 9. We have used heavy lines to indicate the immediate basin, and the line segments pointing down form the repelling cycle $Z = \{z_0, \dots, z_8\}$. The 8 intervals forming $\mathcal{T}$ break up into 6 in $\mathcal{T}'$, and two in $\mathcal{T}''$. The forward images of the intervals in $\mathcal{T}'$ form the set A which consists of 3 intervals which are permuted circularly. The point α is not in A, and this polynomial is not dense.

Remark. The proof above shows that if a hyperbolic polynomial is not dense, then it is renormalizable in an appropriate sense.

5 Constructing Wada Lakes

We will now construct Wada lakes for Hénon mappings that are "small perturbations" of dense polynomials.

Theorem 1. *If p is a dense polynomial and if $|a|$ is sufficiently small, then the Hénon mapping $H_{a,p}$ has attractive cycles close to those of p, and the boundaries of all the components of the basins coincide.*

Remark. The proof of Theorem 1 depends on Lemma 6.3, Proposition 6.1, and Theorem 7.7 of [**4**]. Let us list these for reference, renaming them Lemma A, Proposition B, and Theorem C.

W^S is the stable manifold of J, the Julia set of $H_{a,p}$.

Lemma A. *There is a unique projection*

$$\pi : W^S \to J_p$$

such that the diagram

$$\begin{tikzcd} W^S \arrow[r, "H_{a,p}"] \arrow[d, "\pi"'] & W^S \arrow[d, "\pi"'] \\ J_p \arrow[r, "p"] & J_p \end{tikzcd}$$

commutes, and the fibers of π are stable disks of the crossed mappings.

For each $\zeta \in J_p$, let L_ζ be the inductive limit of

$$\{\zeta\} \times D \overset{f_p}{\hookrightarrow} \{p(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \{p^{\circ 2}(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \dots,$$

an increasing union of discs.

Proposition B. *Each L_ζ is a Riemann surface isomorphic to $\mathbb{C}$, and is dense in $\check{\mathbb{C}}_p$. The foliation is compatible with the dynamics in the sense that*

$$\check{p}(L_\zeta) = L_{p(\zeta)}.$$

Theorem C. *If $|a|$ is sufficiently small, then $H_{a,p}$ has an attractive fixed point $z(a)$ corresponding to z, and the accessible boundary of its basin is $\Phi_+((\partial\Omega)\check{})$.*

Remark. General theorems of Bedford and Smillie [**BS3**], and independently by Sibony and Fornæss [**FS**], assert that for any saddle point of a Hénon mapping (and many other mappings besides), the stable manifold is dense in J_+. We will use an analogous statement, in the much more restricted class of mappings to which Lemma A applies. But Theorem 1 does not immediately follow from this density argument. For instance, the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 - 1.05 - .38y \\ x \end{bmatrix}$$

has an attractive cycle of period 3 (as well as an attractive fixed point), and the basin of this cycle is bounded by the stable manifold of a cycle of period 3 which is a saddle. Of course, in $\mathbb{C}^2$, each path component of this stable manifold is dense in J_+, and in particular each path component accumulates onto the others. But not in $\mathbb{R}^2$: in the real, each of these path components accumulates exactly on the stable manifold of the saddle fixed point.

Proof of Theorem 1. The proof of Lemma A is valid over the reals. Thus for $|a|$ sufficiently small, $\Phi_+ : \check{\mathbb{R}}_p \to J_+ \cap \mathbb{R}^2$ is a homeomorphism, where the space

$$\check{\mathbb{R}}_p = \varinjlim \left((J_p \cap \mathbb{R}) \times I,\ f_p |_{(J_p \cap \mathbb{R}) \times I} \right)$$

is obtained by the same inductive limit construction as in the complex. Fig. 3, Fig. 4, Fig. 5, and Fig. 6 illustrate this construction.

Moreover, Theorem C is also valid over the reals: if x is a fixed point of $p^{\circ k}$ with immediate basin Ω, the accessible boundary of each basin is

$$(\partial(\Omega \cap \mathbb{R}))\check{} = \varinjlim(\partial\Omega \times I,\ f_p^{\circ k}).$$

But $\Omega \cap \mathbb{R}$ is an interval, bounded by a repelling fixed point ξ of $p^{\circ k}$ and one of its inverse images ξ'. As such, the inductive limit above is a real line, which maps by Φ_+ to the the stable manifold of the fixed point $\xi(a)$ of $H_{a,p^{\circ k}}$. Thus we understand exactly what the accessible boundary of each basin is, and what its inverse image by Φ_+ is. So far, none of this required that p be dense.

If p is dense, then every point of $J_p \cap \mathbb{R}$ can be approximated by inverse images $\xi_n \in p^{-nk}(\xi)$; the curves $\pi_{U'}^{-1}(\xi_n)$ are then part of $(\partial(\Omega \cap \mathbb{R}))\check{}$, by the argument of Proposition B. Thus $(\partial(\Omega \cap \mathbb{R}))\check{}$ is dense in $(J_p \times I) \times \{0\}$, the first term in the inductive limit defining $\check{\mathbb{R}}_p$, and by the argument of Proposition B, this shows it is dense in all of $\check{\mathbb{R}}_p$. Thus the accessible boundary of each basin is dense in $J_+ \cap \mathbb{R}^2$, so they do have common boundary. □

The following pictures carry out the construction of $\check{\mathbb{R}}_p$ for p a real quadratic polynomial with an attractive cycle of period 3. It is of course easy to imagine the first step of the construction $(J_p \cap \mathbb{R}) \times I$, which is a product of a Cantor set by an interval.

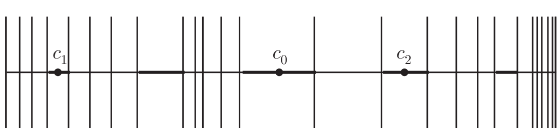

Fig. 3

Fig. 3 shows the set $(J_p \cap \mathbb{R}) \times I$, the first step in the construction.

How should we imagine the inclusion

$$\big((J_p \cap \mathbb{R}) \times I\big) \times \{0\} \hookrightarrow \big((J_p \cap \mathbb{R}) \times I\big) \times \{1\}?$$

Note f_p maps the two intervals through the endpoints of the immediate basin of c_0 to two disjoint subintervals in the interval through the right endpoint of the immediate basin of c_1. Note also that the $p'(\zeta)$ in the denominator in the definition of f_p is essential for the orientations to be as indicated by the arrows in Fig. 4.

Thus in $\big((J_p \cap \mathbb{R}) \times I\big) \times \{1\}$ there must be an arc joining the two intervals above, so that these intervals and the arc will map to the interval where the arrows end. Similarly one sees that there must be an arc joining every pair of symmetric intervals.

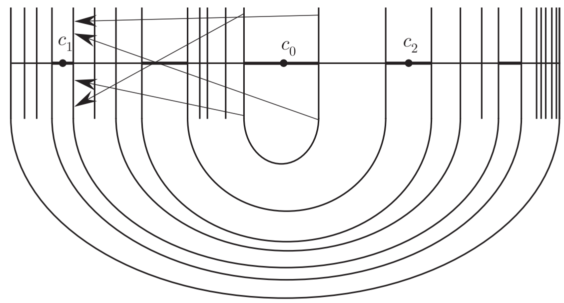

Fig. 4

Fig. 4 illustrates this construction. How should we continue the construction? In $\big((J_p \cap \mathbb{R}) \times I\big) \times \{1\}$ we need inverse images of the arcs added in the previous step; Fig. 5 illustrates how this is to be done. Note that this time some of these arcs do not join intervals to intervals. This is because points to the left of c_1 have no inverse images in the Cantor set $J_p \cap \mathbb{R}$.

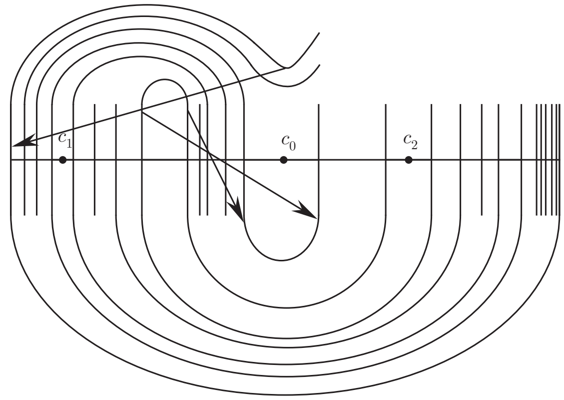

Fig. 5

Making these pictures is a bit addictive, and if one gets carried away, the result may look like Fig. 6.

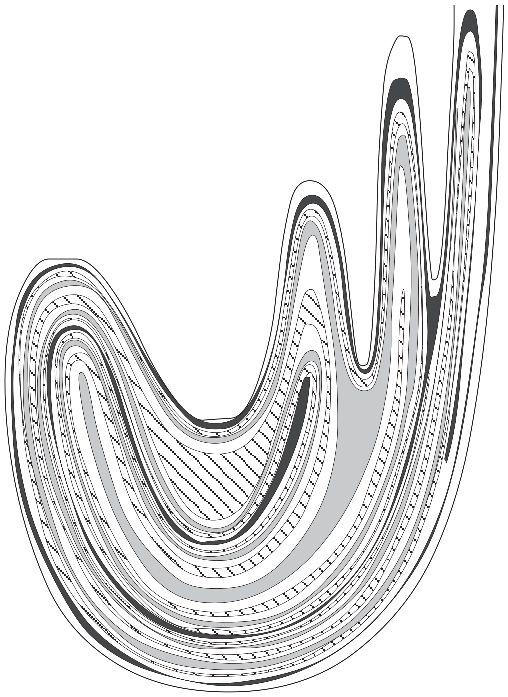

Fig. 6

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