Abstract: In the present work we consider the higher order linear parabolic equation in a rectangle with initial and boundary conditions. We establish new a priori estimates for the solutions to this problem in general Hölder anisotropic norms, under the assumption that the coefficients and the independent term are continuous functions in the rectangle, they satisfy the general Hölder condition in the rectangle of exponent $a(l)$, with respect to the space variable only and they satisfy the general Hölder condition on the boundary of exponent $b(l)$, with respect to all variables. In this connection, however, we also obtain an estimate for the modulus of continuity with respect to the time of the higher derivatives with respect to $x$ of the corresponding solutions.

On the basis of our new a priori estimates for the solution to this problem, we establish the corresponding theorem on the solvability in general Hölder anisotropic spaces.

We apply our results in the linear theory to establish the local solvability with respect to the time, in general Hölder anisotropic spaces, for the nonlinear parabolic equation, with the same type of initial and boundary conditions.

Key words: estimates, solvability, equations, parabolicity, solution, problem.

1. Introduction

In the present work we consider the higher order linear parabolic equation

$$Lu + (-1)^n \sum_{i=0}^{2m} a_r(t,x)D_x^i u = f(t,x),$$

in the rectangle $\mathcal{R}_T = [0, T] \times [-\rho, \rho]$ with the initial condition

$$u|_{t=0} = \varphi_0(x),$$

and the boundary conditions on $S_T = \{(t,x) : 0 \leq t \leq T, x = \pm \rho\}$

$$b_{\rho}^p(D_x u) |_{x = -\rho} = \sum_{q=0}^{m_p} b_{\rho q} D_x^q u |_{x = -\rho} = \varphi_\rho(t),$$

where $0 \leq t \leq T$, $p = 1, \ldots, m$; $0 \leq m_p \leq 2m - 1$, $b_{\rho q}$ - real numbers satisfying the complementary condition with respect to the operator $Lu$, $p = 1, \ldots, m$; $q = 0, 1, \ldots, m_p$ (See [1]).

Here $x$ is a point in $[-\rho, \rho]$, $t \in [0, T]$, $u_t = \frac{\partial u}{\partial t}$, $D_x^r u = \frac{\partial^r u}{\partial x^r}$, $r = 0, \ldots, m$, $D_0^0 u = u$.

We establish new a priori estimates for the solutions to the problem (1), (2), (3) in general Hölder anisotropic norms, under the assumption that the coefficients $a_r(t,x)$ and the independent term $f(t,x)$ are continuous functions in the rectangle $\mathcal{R}_T = [0, T] \times [-\rho, \rho]$, they
satisfy the general Hölder condition in the rectangle \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) of exponent \( \beta(l), l > 0 \) with respect to the space variable only and they satisfy the general Hölder condition on the boundary \( S_T \) of exponent \( \alpha(l), l > 0 \) with respect to the variables \((t, x)\) (See Theorem 1). In this connection, however, we also obtain an estimate for the modulus of continuity with respect to the time \( t \) of the derivatives \( D_t^r u, r = 0, \ldots, 2m, \) but not \( u_t \) (See Theorem 1).

Note that in the works [1]-[13] and in many others, the a priori estimates are obtained under the fulfillment of a (general) Hölder condition with respect to the totality of variables \((t, x)\).

On the basis of new a priori estimates for the solutions to the problem (1), we establish the corresponding theorem on the solvability for this problem in general Hölder anisotropic spaces (See Theorem 2). We assume that the coefficients of Equation (1) satisfy the uniform parabolicity condition: for any \((t, x) \in \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) (4)

We apply our results in the linear theory (for the problem (1),(2),(3)) to establish the local solvability with respect to the time \( t \), in general Hölder anisotropic spaces, for the nonlinear parabolic equation

\[
 u_t = A(t, x, u, D_xu, \ldots, D_x^{2m}u),
\]

in \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) with the initial condition (2) and the boundary conditions (3) (See Theorem 3).

In the present work, the equation (5) is linearized directly. No conditions are imposed here on the nature of the growth of the nonlinearity of the function \( A(t, x, p^0, p^1, \ldots, p^{2m}) \), where \( p^r \)-scalar, \( 0 \leq r \leq 2m \), which is defined for \((t, x) \in \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) and any \( p^r, 0 \leq r \leq 2m \).

The main assumption concerning the function \( A(t, x, p^0, p^1, \ldots, p^{2m}) \) is the parabolicity condition: for any \((t, x) \in \mathcal{R}_T \) and \( p^0, p^1, \ldots, p^{2m} \)

\[
(-1)^{m+1} A_2(t, x, p^0, p^1, \ldots, p^{2m}) > 0 \quad (6)
\]

We require less smoothness conditions from the functions \( A(t, x, p^0, p^1, \ldots, p^{2m}) \) and \( \phi(x) \) than in the works [8], [10] and [11] (See theorem 3).

In almost all the work we suppose that in the equation (1), the function \( f = f_1 + f_2 \); the functions \( a_i(t, x), 0 \leq r \leq 2m \) and \( f_1 \) satisfy the general Hölder condition in \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) of exponent \( \beta(l), l > 0 \) with respect to the space variable only and they satisfy the (complementary) condition general Hölder condition on the boundary \( S_T \) of exponent \( \beta(l), l > 0 \) with respect to the variables \((t, x)\), \( f_2 \) satisfies the general Hölder condition in \( \mathcal{R}_T = [0, T] \times [\rho, -\rho] \) of exponent \( \alpha(l), l > 0 \) with respect to the space variable only and it satisfies the (complementary) general Hölder condition on the boundary \( S_T \) of exponent \( \alpha(l), l > 0 \) with respect to the variable \((t, x)\) (See [14]).

All the coefficients and the independent terms are continuous in the rectangle \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \). Some close results have been established in [15], [16], [17], [18], [19], [20] and [22].

2. Basic Notations. Auxiliary Propositions.

We shall say that the function \( u(t, x) \) defined in the rectangle \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) satisfies the general Hölder condition of exponent \( \beta(l), l > 0 \) in \( \mathcal{R}_T = [0, T] \times [-\rho, \rho] \) with respect to the space variables if there exists a constant \( C > 0 \) such that \( |u(t, x) - u(t, y)| \leq C|x - y|^{\beta(l)}, \quad (t, x), (t, y) \in \mathcal{R}_T. \)
The function $\beta(\ell)$ is defined and continuous in $0 < \ell < \infty$. Moreover it has the following properties:

la. $\beta(\ell) \to \sigma \in [0, 1]$ if $\ell \to 0^+$ or $\ell \to +\infty$.

lb. $\beta'(\ell) \ln \ell \to 0$ if $\ell \to 0^+$ or $\ell \to +\infty$.

lc. if $\sigma = 0$ then $\beta(\ell) \ln \ell \to -\infty$ for $\ell \to 0^+$ and $(\beta(\ell) + \beta'(\ell)) \ln \ell > 0$ for $\ell \in ]0, \ell_0[$, where $\ell_0$ is sufficiently small number (we suppose that the derivative $\beta'(\ell)$ exists and it is a continuous function in $R_0 = ]0, \ell_0[ \cup ]/\ell_0, +\infty[)$.

Note that the condition lc. introduces a new set of functions (the functions $u(t, x)$ that satisfy the general Hölder condition with respect to $x$ only). In this new set of functions we will obtain the corresponding existence and uniqueness theorems for the solutions to the problems (1), (2), (3) and (5), (2), (3).

It follows from the conditions la, lb, lc that

Ila. $\beta(R)$ is a monotonically increasing function for $\ell \in R_0$

Ilb. $\frac{(k!)^{\beta(\ell)-\sigma}}{\ell^{\beta(\ell)-\sigma}} \to 1$ if $\ell \to 0^+$ or $\ell \to +\infty$.

uniformly respect to $k$, $0 < a \leq k \leq b < \infty$.

We denote by $\Gamma^i, i = 1, 2$, the set of functions $\beta(\ell)$ for which

$\Gamma^i \beta(\ell) \equiv \Gamma_{\beta(\ell)} \equiv \int_0^\ell \beta'(\ell) \, dt < \infty \quad ; \quad \Gamma^2 \beta(\ell) \equiv \Gamma_{\Gamma^1 \beta(\ell)} < \infty$

For the functions $\beta(\ell) \in \Gamma^1$ ( $\beta(\ell) \in \Gamma^2$ ) we introduce the functions

$B_1^\beta(\ell) = B(\ell) = \frac{1}{\ln \ell} \ln \frac{\Gamma_{\beta(\ell)}}{\Gamma_{\beta(\ell-1)}}$;

$B_2^\beta(\ell) = B_{\beta(\ell)}$.

The function $B_1^\beta(\ell)$ is a function of the type $\beta(\ell)$, $j = 1, 2$.

See the examples of functions of the type $\beta(\ell)$ in [21]

For the functions $u(t, x)$ defined and Hölder continuous (in the general sense) of exponent $\beta(\ell)$, $\ell > 0$ in the rectangle $\mathcal{R}_T = [0, T] \times [-\rho, \rho]$ with respect to the space variable, we introduce the following norms for $0 < t < T$

$$
|u|^i_{0, 0, \ell} = \sup_{0 \leq \tau \leq t} |u|^i_{0, \ell}, \quad |u|^i_{0, 0, \beta(\ell)} = \sup_{0 \leq \tau \leq t} |u|^i_{0, 0, \beta(\ell)}, \quad \beta(\ell)
$$

where

$$
|u|^i_{0, 0, \beta(\ell)} = \sup_{x \in [-\rho, \rho]} |u(t, x)|, \quad |u|^i_{0, 0, \beta(\ell)} = |u|^i_{0, 0} + H_{1, \beta(\ell)}(u),
$$

$$
H_{1, \beta(\ell)}(u) = \sup_{x \in [-\rho, \rho]} \frac{|u(t, x)|}{|x-y|^i_{0, 0, \beta(\ell)}}.
$$

For the functions $u(t, x)$ that have continuous derivatives with respect to $x$ up to the order $m$ ($m = 1, 2, ...$) inclusively in the rectangle $\mathcal{R}_T = [0, T] \times [-\rho, \rho]$ and satisfying the general Hölder condition of exponent $\beta(\ell)$, $\ell > 0$, with respect to space variables in the rectangle $\mathcal{R}_T = [0, T] \times [-\rho, \rho]$ we define the norms

$$
|u|^i_{m, \beta(\ell)} = \sum_{m=0}^m [D^m u]^i_{0, 0, \beta(\ell)} \quad ; \quad |u|^i_{m, 0, \beta(\ell)} = \sup_{0 \leq \tau \leq t} |u|^i_{m, 0, \beta(\ell)}, \quad m = 0, 1, 2, ..., D^0 u = u
$$

We will denote by $C^i_{m, \beta(\ell)}(\mathcal{R}_t)$, $m = 0, 1, 2, ...$ the Banach space of functions $u(t, x)$ that are continuous in $\mathcal{R}_t = [0, t] \times [-\rho, \rho]$ together with all derivatives respect to $x$ up to the order $m$, $m = 0, 1, 2, ...$ inclusively and have a finite norms (10).

For the functions $u(t, x)$ that have continuous derivatives with respect to $x$ up to the order $m$ ($m = 0, 1, 2, ...$) inclusively in the layer $\overline{\Pi}_t = [0, t] \times E_1$, $0 \leq t \leq T$, $E_1 = ]-\infty, +\infty[$ (or in the
half layer $\overline{\mathcal{R}}_l = \{(x,t) \mid 0 \leq t \leq T, \quad x \geq 0\}$ and satisfying the general Hölder condition of exponent $\beta(i), \ell > 0$, with respect to space variables in the layer $\overline{\mathcal{R}}_l$ (or in the half layer $\overline{\mathcal{R}}^n_l$) we define similarly the norms (7) ... (10) and the corresponding spaces $C^{[l]}_{m,\beta(i)}(\overline{\mathcal{R}}_l)$ (or the spaces $C^{[l]}_{m,\beta(i)}(\overline{\mathcal{R}}^n_l)$).

We define the parabolic distance between each two points $P = (\theta, x) \in \mathcal{R}_T$, $Q = (\tau, y) \in \mathcal{R}_T$, by the magnitude

$$ d(P, Q) = \left( |\theta - \tau|^\frac{1}{2} + |x - y|^2 \right)^{1/2}. $$

(11)

For the functions $u(t, x)$ that have continuous derivatives with respect to $x$ up to the order $m$ ($m = 0, 1, 2, \ldots$) inclusively in the rectangle $R_i = ]0, t] \times ]-\rho, \rho[$ and satisfying the general Hölder condition of exponent $\beta(i), \ell > 0$, with respect to the variables $(t, x)$ in the rectangle $R_i = ]0, t] \times ]-\rho, \rho[$ with we define the norms for $0 < t < T$

0 $\leq t \leq T$ with respect to the variables $(t, x)$, we introduce the following norms:

$$ |u|^{R_i}_{0,0} = \sup_{(t,x) \in R_i} |u(t,x)|, \quad H^{R_i}_{\beta(i)}(u) = \sup_{P, Q \in R_i} \frac{\|u(P) - u(Q)\|}{d(P, Q)^{[\beta(i)]}} \quad (12) $$

$$ |u|^R_{0,\beta(i)} = |u|^{R_i}_{0,0} + H^{R_i}_{\beta(i)}(u), \quad (13) $$

$$ |u|^R_{m,\beta(i)} = \sum_{r=0}^{m} |D^r_x u|^{R_i}_{0,\beta(i)} \quad m = 0, 1, 2, \ldots, \quad D^0_x u = u \quad (14) $$

We will denote by $C_{m,\beta(i)}(R_i), \quad m = 0, 1, 2, \ldots, 0 < t < T$ the Banach space of functions $u(t, x)$ that are continuous in $R_i = [0, t] \times [-\rho, \rho[$ together with all derivatives respect to $x$ up to the order $m, m = 0, 1, 2, \ldots$ inclusively and have a finite norm (14) With respect to the coefficients of the equation (1) we assume that $a_i(t, x) \in C^{[l]}_{0,\beta(i)}(\overline{\mathcal{R}}_T), \quad 0 \leq r \leq 2m$ and

$$ \sum_{|\beta| = 0}^{2m} a_k^{[r]}_{0,\beta(i)} = B < \infty, \quad \sum_{|\beta| = 0}^{2m} a_k^{[r]}_{0,\beta(i)} = C < \infty \quad (15) $$

For equation (5) we consider in addition to the parabolicity condition (6) that there exists a domain $H_M = \{(t, x) \in \overline{\mathcal{R}}_T, |u| \leq M, |p^r| \leq M, \ 1 \leq r \leq 2m, \ M = \text{const.}\}$ in which the function $A(t, x, p^0, p^1, \ldots, p^{2m})$, and its derivatives with respect to $p^0, p^1, \ldots, p^{2m}$, up to the second order inclusively are continuous, satisfy the Lipschitz condition with respect to $p^0, p^1, \ldots, p^{2m}$, the general Hölder condition of exponent $\beta(i)$ with respect to $x$ and with the constant $B_M$. They also satisfy the general Hölder condition of exponent $\frac{B_0}{2m}$ with respect to $t$ on the boundary $x = \pm \rho, \quad 0 < t < T$ with the constant $B_M$. Moreover

$$ A(t, x, 0, 0, 0) \in C^{[l]}_{0,\beta(i)}(\overline{\mathcal{R}}_T), \quad |A(t, x, 0, 0, 0)|_{0,\beta(i)} \leq B_0 \quad (16) $$

All the mentioned derivatives are bounded in $H_M$ by the constant $B_M$.


Now we shall consider the equation (1) in the rectangle $\mathcal{R}_T$ with the initial zero condition

$$ u \mid_{t=0} = 0 \quad (17) $$

and the boundary conditions (3)

We define

$$ \|\varphi_p\|^{[0]}_{\beta_p(i)} = |\varphi_p|^{[0]}_{0,\beta_p(i);2m-m_p}, \quad \text{for} \quad 1 \leq m_p, $$

$$ \|\varphi_p\|^{[0]}_{\beta_p(i)} = |D_p \varphi_p|^{[0]}_{0,\beta_p(i);2m} + |\varphi_p|^{[0]}_{0,\beta_p(i);2m}, \quad \text{for} \ m_p = 0. \quad (18) $$
Theorem 1. Let \( \beta_p(l) = \frac{\beta(l) + 2m - m_p}{2m}, \quad p = 1, \ldots, m \). The functions \( \beta_p^+(l) = \frac{\beta(l) + 2m - m_p}{2m}, \quad p = 1, \ldots, m \), are functions of the type \( \beta(l) \), where \( \beta(l) \to \sigma \in [0,1] \) if \( l \to 0^+ \) or \( l \to +\infty \). Note that

\[
\beta_p(l) \to \beta_p^+(l) \to \frac{\sigma + 2m - m_p}{2m} \quad \in \quad ]0,1[ \quad \text{if} \quad l \to 0^+ \text{or} \quad l \to +\infty.
\]

In all the work we suppose that

\[
\left\| \phi_p^+ \right\|_{[0,T]}^{[\beta_p(l)]} < \infty, \quad p = 1, \ldots, m
\] (19)

Theorem 1. Let \( u(t,x) \in C_{2m,B_p(\mathcal{R}_T)}^{[\eta]}(\mathcal{R}_T) \) be a solution to the problem (1), (17), (3) in the rectangle \( \mathcal{R}_T \). Assume that \( f_1 \in C_{0,\beta}^{[\eta]}(\mathcal{R}_T) \), \( f_2 \in C_{\alpha(x)}^{[\eta]}(\mathcal{R}_T) \), \( \beta(l) \), \( \alpha(l) \in \Gamma^2 \), \( \beta(l) \to \sigma_1 \), \( \alpha(l) \to \sigma_2 \) if \( l \to 0^+ \) or \( l \to +\infty \), \( 0 \leq \sigma_1 < \sigma_2 < 1 \), \( \|f_1(\mathcal{S}_\alpha(l))\| < \infty \), \( \|f_2(\mathcal{S}_\beta(l))\| < \infty \).

Furthermore the conditions (4), (15) and (19) hold. Then there exists a constant \( K \), depending only on \( n, m, \lambda, B, C, T, \alpha(l), \beta(l), B_p(l) \) and on the constants of the complementary condition in \( S_T \), and in \( t = 0 \) such that for \( 0 \leq t \leq T \) the next estimate holds

\[
|u|_{2m,B_p(\mathcal{R}_T)}^{[\eta]} \leq K \left[ |f_1|_{0,\beta(l)}^{[\eta]} + |f_1|_{0,\beta(l)}^{[\eta]} + \sum_{m=1}^{m=2m-1} \left( \|\phi_p^{+}\|_{0,\beta_p(l)}^{[\eta]} + \|\phi_p^{+}\|_{0,\beta_p(l)}^{[\eta]} \right) \right].
\] (20)

Proof. Let \( x_o \in [-\rho, \rho] \) and \( d \in ]0,1[ \). We introduce the function \( \eta(x) = \mu(|x-x_o|) \), where \( \mu(l) \) is a decreasing infinitely differentiable function on \( l > 0 \), \( \mu(l) = 1 \) for \( 0 \leq l \leq d \); \( \mu(l) = 0 \) for \( l \geq 2d \) and satisfying the inequality

\[
\sum_{x=1}^{x=2d} d^x |\mu^{(x)}(l)| \leq K
\] (21)

The function \( \omega(t,x) = \eta(x)u(t,x) \) satisfies the equation

\[
L \omega = \omega + (-1)^m a_{2m}(t,x) D^2 \omega = [a_{2m}(t,x) - a_{2m}(t,x_0)]D^2 \omega + \hat{f}_1(t,x) + \eta(x)f_2(t,x)
\] (22)

in the rectangle \( \mathcal{R}_T \), where

\[
\hat{f}_1(t,x) = \left[ (-1)^m \sum_{r=0}^{2m-1} a_r(t,x) D^r u + f_1(t,x) \right] \eta(x) + (-1)^m a_{2m}(t,x) \sum_{r=0}^{2m-1} C^r \eta^{(2m-r)} D^r u,
\]

\( C^r \) – binomial coefficients

The function \( \omega(t,x) = \eta(x)u(t,x) \) \( \left( \omega \in C_{2m,B_p(\mathcal{R}_T)}^{[\eta]}(\mathcal{R}_T) \right) \) satisfies the initial zero condition (17) and the boundary conditions ( for \( x_0 = -\rho \))

\[
\sum_{x=-\rho}^{x=\rho} d^x |\mu^{(x)}(l)| \leq K
\] (23)

We next find a bound for the solution \( \omega(t,x) = \eta(x)u(t,x) \) to the problem (22), (17), (23) in \( [-\rho, \rho + 2d] \).

The mapping \( y = x - (-\rho) \) \( (y = x + \rho) \) \( , x \in [-\rho, \rho] \) transforms the equation (22) into the equation

\[
L \omega \equiv \omega_t + (-1)^m a_{2m}(t,0) D^2 \omega = [a_{2m}(t,y) - a_{2m}(t,0)]D^2 \omega + \hat{f}_1(t,y) + \eta(y)\hat{f}_2(t,y)
\] (24)

where
\[\varpi(t, y) = \eta(y)\overline{u}(t, y) = \eta(y - \rho)u(t, y - \rho), \quad y_0 = x_0 + \rho = 0, \overline{u}_{2m}(t, y_0) = a_{2m}(t, 0),\]
\[\tilde{f}_1(t, y) = \tilde{f}_1(t, y - \rho); \quad \tilde{f}_2(t, y) = \tilde{f}_2(t, y - \rho);\]

The function \(\varpi(t, y)\) satisfies the equation (24) in the rectangle \(\mathcal{R}^+_T = [0, T] \times [0, 2\rho]\), the initial zero condition
\[\varpi \mid_{t=0} = 0\] (25)
and the boundary conditions
\[b_p(D_z\varpi) \mid_{y=0} = \sum_{q=0}^{m_p} b_{p^q} D^q_z \varpi \mid_{y=0} = \varphi_p^-(t), \quad p = 1, \ldots, m; \quad 0 \leq m_p \leq 2m - 1\] (26)

We next find a bound for the solution \(\varpi(t, y) = \eta(y)\overline{u}(t, y)\) to the problem (24),(25),(26) in the rectangle \(\bar{\mathcal{R}}_{t, 2d} = [0, t] \times [0, 2d + \rho], 0 \leq t \leq T\).

The function \(\varpi(t, y) \in C^{[T]}_{2m, B_{\eta}(0)}(\Pi_t^+\bar{\mathcal{R}})\) satisfies the equation (24) in the half layer \(\Pi_t^+ = \{ (t, y) : \ 0 \leq t \leq T, \ y \geq 0 \}\) (observe that the functions \(\eta(y)\) and \(\varpi(t, y)\) vanishes for \(\left|y\right| \geq 2d, \ 2d < \rho\), the initial zero condition (25) and the boundary conditions (26).

We can represent the solution to the problem (25),(26),(27) in the form
\[\varpi(t, y) = u^1 + u^2\] (27)
\[u^1(t, z) = \int_0^t \int_{E_1} G_{n}(t, \tau; z - \xi) \sum_{j=1}^{2} F_j^j(t, \xi) \ d\xi d\tau\] (28)

where
\[G_{n}(t, \tau; y - \xi)\] is the fundamental solution to the operator
\[L_0, \ F_1(t, \xi) = [\overline{a}_{2m}(t, \xi) - \overline{a}_{2m}(t, 0)]D^2_{z\xi}\overline{u} + \tilde{f}_1(t, \xi), \quad F_2(t, \xi) = \eta(\xi)\tilde{f}_2(t, \xi)\]
The function \(u^2 = \varpi(t, y) - u^1\) is a solution to the problem
\[L_0(u^2) = 0\] (29)
\[u^2 \mid_{t=0} = 0\] (30)
\[b_p(D_z u^2) \mid_{y=0} = \psi_p(t)\] (31)

Where
\[\psi_p(t) = \left[ \eta(0)\varphi_p^-(t) - b_p(D_z)u^1(t, 0) \right], \quad p = 1, \ldots, m\]
The function \(u^1(t, z) \in C^{[T]}_{2m, B_{\eta}(0)}(\Pi_t^+)\) is a solution to the Cauchy problem for the equation
\[L_0 u^1 = u^1 + (-1)^m \overline{a}_{2m}(t, 0)D^2_{z\xi} u^1 = [\overline{a}_{2m}(t, z) - \overline{a}_{2m}(t, 0)]D^2_z u^1 + \tilde{f}_1(t, z) + \eta(z)\tilde{f}_2(t, z)\] (32)
with the initial zero condition (17) in the layer \(\Pi_T = [0, T] \times E_1, \ E_1 = [-\infty, \infty]\).

Reasoning as in the proof of theorem 1 in [19], choosing \(0 < d_1 < d\) small enough ( see [19] ), such that \(d_1 B_{\eta}(0) < \frac{1}{2k\varrho}\) and \(d < \frac{\varrho}{2}\) we obtain that the solution \(u^1(t, z)\) to the Cauchy problem (32),(17) satisfies the next inequality for \(0 \leq t \leq T, (\varpi(t, y) \in C^{[T]}_{2m, B_{\eta}(0)}(\Pi_t^+\bar{\mathcal{R}})\))
\[|u^1|^{[2]}_{2m, B_{\varrho}(0)} \leq K \left[ d^2_{(m+1)} ||u||^{[1, 2d]}_{2m-1, B_{\varrho}(0)} + d^{2(m-1)}_1 \left( \int_{E_1} |f|^{[1, 2d]}_{0, B_{\varrho}(0)} + t^{\frac{d-\varrho}{2d}} \int_0^{[1, 2d]}_{0, B_{\varrho}(0)} \tilde{f}_2^{[1, 2d]} \right) \right].\] (33)

where the symbol \([t], 2d_1\) means that the norms in the right hand side of (33) are consider in the rectangle \([0, t] \times [0, 2d_1]\).

Now we shall consider the solution \(u^2(t, x)\) to the problem (29) - (31). At first we next find a
bound for the norm \( \|\psi_p(t)\|_{B_{\beta B}(t)}^{0,r} \), \( p = 1,\ldots,m \); \( 0 \leq t \leq T \). Reasoning as in the proof of theorem 3 in [19], we find the next estimate for \( 0 \leq t \leq T \)

\[
\|\psi_p\|_{B_{\beta B}(t)}^{0,r} < K \left\{ \left| d_1^{-(2m+1)} \right| \left| d_1^{(2m-1)} \| f_1 \|_{0,\beta B}^{0,r} \right| + \left| \tilde{f}_1 \right|_{0,\beta B}^{0,r} + \left| \tilde{f}_2 \right|_{0,\beta B}^{0,r} \right\} +
\]

\[
+ d_1^{-(2m-1)} \left| \int_J f_1 \right|_{0,\beta B}^{0,r} + \sum_{r=0}^{2m-1} \left| \int J \Pi(t,0) \left| \left| \phi \right|_{0,\beta B}^{0,r} \right| \right| + \| \phi \|_{B_{\beta B}(t)}^{0,r}, \quad p = 1,\ldots,m
\]

(34)

where \( \tilde{f}_j(t) = f_j(t,0) \), \( j = 1,2 \)

Now we consider the equation

\[
L_0(v) = v + (-1)^m D_2^{2m} v = 0;
\]

(35)

The functions

\[
v_p(t,y) = \int_0^t G_p(t,0 \rightarrow y) \psi_p(t) d\tau, \quad p = 1,\ldots,m
\]

(36)

where

\[
G_p(t,0 \rightarrow y) - \text{component of the Green function to the problem (35), (30), (31).}
\]

We can represent the solution \( v(t,y) \in C_{2m,\beta B}(\Pi_T), \Pi_T = \{(t,x) ; 0 \leq t \leq T, x \geq 0\} \) to the problem (35),(30),(31) in the form (See [8],[10])

\[
v(t,y) = \sum_{p=1}^m v_p(t,y)
\]

(37)

and the next inequality holds

\[
\|v_p\|_{2m,\beta B}^{0,r} \leq K \|\psi_p\|_{B_{\beta B}(t)}^{0,r}, \quad p = 1,\ldots,m
\]

(38)

\( K \) is a constant depending on \( n, m, \lambda, B, T, C, \beta(l), B_{\beta B}(t) \), and on the constants of the complementary condition in \( S_T = \{(t,x) ; 0 \leq t \leq T, x = x \geq \rho\} \) and in \( t = 0 \).

We can represent the solution \( u^2(t,y) \) to the problem (29) - (31) in the form (37) and applying the inequalities (38) and (34), we obtain the next estimate (by going back to the variable \( t \)) for \( 0 \leq t \leq T \):

\[
\left| u^2 \right|_{2m,\beta B}^{0,r} \leq K \left\{ \left| d_1^{-(2m+1)} \right| \left| d_1^{(2m-1)} \| f_1 \|_{0,\beta B}^{0,r} \right| + \left| \tilde{f}_1 \right|_{0,\beta B}^{0,r} \right\} +
\]

\[
+ d_1^{-(2m-1)} \left| \int J f_1 \right|_{0,\beta B}^{0,r} + \sum_{r=0}^{2m-1} \left| \int J \Pi(t,0) \left| \left| \phi \right|_{0,\beta B}^{0,r} \right| \right| + \sum_{p=1}^m \| \phi_p \|_{B_{\beta B}(t)}^{0,r}, \quad (39)
\]

Combining the inequalities (33) and (39) we can get from (27) the estimate (\( 0 \leq t \leq T \))
\[
|\sigma|_{2m, B_{\rho(t)}^0} \leq K\left\{ |d_1^{(2m+1)}|^2_{2m-1, B_{\rho(t)}^0} + t^{\frac{s_0-s_1}{2m}} \left( |d_1^{(2m-1)}|^2_{0, B_{\rho(t)}^0} + \left| \bar{f}_1 \right|^2_{0, \frac{\beta_0(t)}{2m}} \right) \right\} + \\
+ d_1^{(2m-1)} \left| \bar{f}_1 \right|^2_{0, B_{\rho(t)}^0} + \sum_{p=0}^m \| \phi_p \|^2_{B_{\rho(t)}^p} \right\} (40)
\]

By other hand from the estimates of the moduli of continuity with respect to the time for the derivatives of the solution to the Cauchy problem (32), (17) in the layer \( \Pi_T = [0, T] \times E_1, E_1 = ]-\infty, \infty[ \) and from the estimates of the moduli of continuity with respect to the time for the derivatives \( D_y^{2m-1} u^2(t,y), j = 0, 1, ..., 2m - 1, y \geq 0, 0 \leq t_2 < t_1 \leq T \) of the solution \( u^2(t,y) \) to the problem (29), (30), (31) in the half layer \( \Pi_T = \{(t,y) ; 0 \leq t \leq T, y \geq 0 \} \) (See [10]) and from the estimates (34), (37), (39) we can get from (27) the estimate

\[
\left| D_y^{2m-j} \sigma(t_1,y) - D_y^{2m-j} \sigma(t_2,y) \right| \leq K\left\{ |d_1^{(2m+1)}|^2_{2m-1, B_{\rho(t)}^0} + \\
+ t^{\frac{s_0-s_1}{2m}} \left( |d_1^{(2m-1)}|^2_{0, B_{\rho(t)}^0} + \left| \bar{f}_1 \right|^2_{0, \frac{\beta_0(t)}{2m}} \right) \right\} + \\
+ d_1^{(2m-1)} \left| \bar{f}_1 \right|^2_{0, B_{\rho(t)}^0} + \sum_{p=0}^m \| \phi_p \|^2_{B_{\rho(t)}^p}, (41)
\]

Choosing \( \alpha < d = d_0 \) small enough (See [19]), such that \( d^2_{\rho(t)} < \frac{1}{2K} \) and \( d = \frac{1}{2} \min \{1, \frac{\rho}{2} \} \) we find that (by going back to the space variable \( \chi \))

\[
|u|_{2m, B_{\rho(t)}^0} \leq |u|_{2m-1, B_{\rho(t)}^0} \leq K\left\{ |u|_{2m-1, B_{\rho(t)}^0} + t^{\frac{s_0-s_1}{2m}} \left( |f_2|_{0, B_{\rho(t)}^0} + |f_1|_{0, \frac{\beta_0(t)}{2m}} \right) \right\} + \\
+ |f_1|_{0, B_{\rho(t)}^0} + \sum_{p=1}^m \| \phi_p \|^2_{B_{\rho(t)}^p}, (42)
\]

For \( x_0 = \rho \) we obtain similarly the next estimate

\[
|u|_{2m, B_{\rho(t)}^0} \leq K\left\{ |u|_{2m-1, B_{\rho(t)}^0} + t^{\frac{s_0-s_1}{2m}} \left( |f_2|_{0, B_{\rho(t)}^0} + |f_1|_{0, \frac{\beta_0(t)}{2m}} \right) \right\} + \\
+ |f_1|_{0, B_{\rho(t)}^0} + \sum_{p=1}^m \| \phi_p \|^2_{B_{\rho(t)}^p}, (43)
\]

where the norms in the right hand side are given in the rectangle \([0, t] \times [\rho - 2d_0, \rho] \).

For \( x_0 \in R_{\rho(t)}^{d_0} = [-\rho + \frac{d_0}{2}, \rho - \frac{d_0}{2}] \) we derive similarly the next interior estimate in the rectangle \( R_{\rho(t)}^{d_0} = [0, t] \times [-\rho + \frac{d_0}{2}, \rho - \frac{d_0}{2}] \), \( 0 \leq t \leq T \)

\[
|u|_{2m, B_{\rho(t)}^0} \leq K\left\{ |f_1|_{0, B_{\rho(t)}^0} + t^{\frac{s_0-s_1}{2m}} |f_2|_{0, B_{\rho(t)}^0}, b. + |u|_{2m-1, B_{\rho(t)}^0} \right\} (44)
\]

where the symbol \([t], b. \) means that the norms in the right hand side of (44) are considered in
the rectangle \([0, t] \times \left[-\rho + \frac{d_0}{2}, \rho - \frac{d_0}{2}\right]\).

Now we can get from (42)-(44) the following estimate in the rectangle \(\mathcal{R}_t, 0 \leq t \leq T\):

\[
|u|_{2m, B_{\beta(t)}} \leq K \left\{ f_1 \left( \frac{S_t}{2m} \right) + |f_1|_{0, B_{\beta(t)}} + |u|_{2m-1, B_{\beta(t)}} + t^{-\frac{d_2 - 1}{2}} \begin{pmatrix}
\left| f_2 \left( \frac{S_t}{2m} \right) \right| + |f_2|_{0, B_{\beta(t)}} + \\
+ \sum_{p=1}^{m} \left( \|\phi_p^+\|_{0, \beta_p(t)} + \|\phi_p^-\|_{0, \beta_p(t)} \right)
\end{pmatrix} \right\}
\]

Applying the interpolation inequality (See [19]) with \(\varepsilon\) small enough and arguing as in the last part of the proof of theorem 1 in [16] to eliminate \(|u|_{0,0}^R\) we get the estimate (21).

Remark 1 We can reduce the boundary problem (1), (2), (18) with non-zero initial condition \(u \mid_{x=0} = \varphi_0(x)\) to the boundary value problem (1), (17), (18) by means of the transformation \(u = u(t, x) - \varphi_0(x)\), where \(\varphi_0(x) \in C_{2m, \beta(t)}([-\rho, \rho])\).

4. Existence and uniqueness theorems.

Theorem 2. Suppose that all conditions of theorem 1 are true. Assume furthermore that the following consistency conditions hold

\[
\sum_{q=0}^{m} b_{qp}^2 \varphi_0(x) \mid_{x=\rho} = \varphi_p(0), \text{if } m_p \geq 1, \quad p = 1, \ldots, m; \quad b_{qp}^2 \varphi_0(x) \mid_{x=\rho} = \varphi_p(0), \text{if } m_p = 0.
\]

Then there exists a unique solution \(u(t, x) \in C^{[T]}_{m, \beta(t)}(\mathcal{R}_T)\) to the boundary problem (1), (2), (3) with continuous derivatives \(u_t\) in \(\mathcal{R}_T\).

We can get the proof of this theorem on the basis of the new a priori estimates established in this work and with the aid of the method of continuity in a parameter (see [9] and [24]).

We proceed now to formulate the local existence theorem for solutions to the non-linear boundary value problems for the equation (5).

( with the initial zero condition (17) and the boundary conditions (3) )

Let \(\beta_p(t) = \frac{\beta(t) + 2m - m_p}{2m}\). We will suppose that

\[
\left\| \varphi_p \right\|_{\beta_p(t)}^{[0, T]} < \infty
\]

Here we consider that the function

\[ A(t, x, u, D_x u, \ldots, D_x^{2m} u) = L(u) + F(t, x, u, D_x u, \ldots, D_x^{2m} u) + L(u) + A(t, x, 0, 0, \ldots, 0) \]

where

\[ L(u) = A \rho^0 (t, x, 0, 0, \ldots, 0) u + A \rho^1 (t, x, 0, 0, \ldots, 0) D_x u + \ldots + A \rho^{2m} (t, x, 0, 0, \ldots, 0) D_x^{2m} u, \]

\[ F(t, x, u, D_x u, \ldots, D_x^{2m} u) = A(t, x, u, D_x u, \ldots, D_x^{2m} u) - L(u) - A(t, x, 0, 0, \ldots, 0) \]

Theorem 3. Suppose that all assumptions with respect to the function \(A(t, x, p^0, p^1, \ldots, p^{2m})\) hold. Moreover \(0 \leq \sigma_1 < \sigma_2 < 1\) and the following consistency conditions hold

\[
\varphi_p(0) = 0, \quad \text{if } m_p \geq 1, \quad \varphi_p(0) = 0
\]
\[
\begin{pmatrix}
\varphi_p \\
\end{pmatrix}
(0) = b_0 \varphi \quad A(0,x,0,...,0) \quad \text{if} \quad m_p = 0
\]  

(46)

Then there exists \( t_0 \in (0, T) \), determined by the above assumptions, such that the problem , (5), (17), (3) has in the rectangle 
\( \mathcal{R}_{t_0} = [0,t_0] \times [-\rho, \rho] \) a unique solution \( u(t,x) \in C_{2m,B_{\rho(t)}}(\mathcal{R}_{t_0}) \) with a continuous derivative \( u \) in \( \mathcal{R}_{t_0} = [0,t_0] \times [-\rho, \rho] \).

Proof. We will prove Theorem 3 by means of an iterative process in which one successively solves the equations

\[ u_i^\nu = L(u^{\nu}) + F(t,x,u^{\nu-1},D_xu^{\nu-1},...,D_x^{2m}u^{\nu-1}) + A(t,x,0,...,0) \quad \text{, } \nu = 1,2,..., 
\]  

(47)

with the zero initial condition (17 ) and with the boundary conditions

\[
 b^+_p(D_xu) \bigg|_{x=-\rho} = \varphi^-_p(t), \quad p = 1,...,m
\]  

(48)

Furthermore

\[ u^0(t,x) = 0
\]  

(49)

We will show that there exists such a sequence of functions \( u^\nu(t,x), \nu = 1,2,..., \) defined in some rectangle \([0,t_0] \times [-\rho, \rho]\) with \( t_0 \) small enough. Moreover they are bounded in \( C_{2m,B_{\rho(t)}}(\mathcal{R}_{t_0}) \).

We assume that there exists the functions \( u^\mu(t,x), \mu \leq \nu - 1 \) and they satisfy the inequality

\[
|u^\mu|_{2m,B_{\rho(t)}^2} \leq q = \text{const.} = \min(1,M)
\]  

(50)

where the numbers \( t_0 \) and \( \eta \) are still to be determined.

Now we can consider the equation (47) as a linear equation of type (1), where

\[
 f_1(t,x) = F(t,x,u^{\nu-1},D_xu^{\nu-1},...,D_x^{2m}u^{\nu-1}), \quad f_2(t,x) = A(t,x,0,...,0).
\]

Reasoning as in the proof of theorem 1, using lemma1 and theorem 2, we conclude that there exists \( u^\nu(t,x) \in C_{2m,B_{\rho(t)}}(\mathcal{R}_{t_0}) \) and it satisfies the inequality

\[
|u^\nu|_{2m,B_{\rho(t)}^2} \leq K_1 \left[ |u^{\nu-1}|_{2m,B_{\rho(t)}^2} + |u^{\nu-1}|_{2m,B_{\rho(t)}^2} + t_0^\nu \right]
\]  

(51)

We will use \( K_1, K_2, K_3, ..., \) to denote constants, depending on \( n, m, \lambda, \alpha(\cdot), \beta(\cdot), B_{\beta(t)}^2, C, B_0, B_M \) but not depending on \( t_0 \). From the inequality (55) it follows the estimates

\[
\omega^\nu \equiv |u^\nu - u^{\nu-1}|_{2m,B_{\rho(t)}^2} \leq K_2 \left[ |u^{\nu-1}|_{2m,B_{\rho(t)}^2} + |u^{\nu-2}|_{2m,B_{\rho(t)}^2} \right] \cdot |u^{\nu-1} - u^{\nu-2}|_{2m,B_{\rho(t)}^2} + \\
+ \left( |u^{\nu-1}|_{2m,B_{\rho(t)}^2} + |u^{\nu-2}|_{2m,B_{\rho(t)}^2} \right) \cdot |u^{\nu-1} - u^{\nu-2}|_{2m,B_{\rho(t)}^2}
\]  

(52)

Putting \( 2K_1 q \leq \frac{1}{2}, \quad 2K_1 t_0^\nu = \frac{1}{2} q \) we find that \( |u^\nu|_{2m,B_{\rho(t)}^2} \leq \frac{1}{2} q + \frac{1}{2} q = q \).

In the last inequalities we have selected

\[
 q = \min \left( \frac{1}{4K_1}, \frac{1}{8K_2}, 1, M \right), \quad t_0 = \left( \frac{q}{4K_1} \right)^{\frac{1}{4}}
\]  

(53)

From (50) and (53) it follows that \( \omega^\nu \leq 4K_2 q \omega^{\nu-1} \), then

\[
\omega^\nu \leq \frac{1}{4} \omega^{\nu-1} \quad \omega^\nu \leq \frac{1}{2} \omega^{\nu-1}
\]

(54)

We have determined the number \( t_0 \) by (53), then the sequence \( u^\nu(t,x) \) converges in the
Assume that there exists two solutions \( u^1(t,x), u^2(t,x) \) of this problem in the corresponding rectangles \( R_{\ell_0} \) and consider the function \( u = u^1(t,x) - u^2(t,x) \). This function satisfies the equation (in the smaller of the corresponding rectangles \( R_{\ell_0} \))

\[
Lu = \int_0^1 A_p(t,x,\tau z^1 + (1-\tau)z^2)d\tau u + \left( \int_0^1 A_p(t,x,\tau z^1 + (1-\tau)z^2)d\tau, D_xu \right) + \\
+ \left( \int_0^1 A_{2m}(t,x,\tau z^1 + (1-\tau)z^2)d\tau, D_x^{2m}u \right)
\]

(54)

Here the components of the vector function \( z^r = (u^r,D_xu^r,\ldots,D_x^{2m}u^r)^T \), \( r = 1,2 \) belong to the space \( C_{0,\beta_{\rho(1)}}(R_{\ell_0}) \) and the coefficients of the operator \( L \) also belong to this space. Since \( u(t,x) \) satisfies the zero initial condition (17) and the boundary conditions

\[
b_p(D_xu) \mid _{x=\rho} = 0, \quad p = 1, \ldots, m
\]

(55)

it follows that \( u(t,x) \equiv 0 \). This completes the proof of Theorem 3.

Remark 2. We can reduce the boundary problem (5), (2), (3) with non-zero initial condition \( u \mid _{t=0} = \varphi_0(x) \) to the boundary problem (5),(17),(3) with the zero initial condition \( u \mid _{t=0} = 0 \) by means of the transformation \( \bar{u} = u(t,x) - \varphi_0(x) \), where \( \varphi(x) \in C_{2m,\beta(1)}(E_n) \).

References


