THE STATISTICAL PROPERTIES OF LONG-MEMORY AND EXPONENTIAL ACD MODELS

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Abstract

This paper examines some of the statistical properties of various Autoregressive Conditional Duration (ACD) specifications. More specifically, the statistical analysis of the long-memory, and exponential ACD models is presented. To allow for non-monotonic hazard functions we use either the generalized Gamma or the generalized F distributions. Conditions for the existence of the first two moments are established. We also provide analytical expressions of the autocorrelation function of the durations.

Keywords: Exponential ACD models, Long-memory, Generalized F distribution, Unconditional moments, Hypergeometric Functions.

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1 Introduction

Engle and Russell (1998) proposed a new econometric framework for the modelling of intertemporally correlated event arrival times, termed the Autoregressive Conditional Duration (ACD) model. A feature of Engle and Russell's linear ACD model with exponential or Weibull errors is that the implied conditional hazard functions are restricted to being either constant, increasing or decreasing. Bauwens and Veredas (1999), Hamilton and Jorda (2001), and Zhang et al. (2001) questioned whether this assumption is an adequate one. As an alternative to the Weibull distribution used in the original ACD model, Lunde (1999) employed an ACD model based on the generalized Gamma distribution, while Grammig and Maurer (2000) and Hautsch (2001) utilized the Burr and generalized F distributions respectively.

Recently, several extensions to Engle and Russell's (1998) basic model have been proposed. Lunde (1999) and Bauwens and Giot (2000, 2001a) model the effect of recent durations on the conditional mean with a logarithmic transformation. We call this model exponential ACD (EXACD) because of its resemblance to Nelson's EGARCH model (Nelson, 1991). For the IBM stock traded on the NYSE, Giot (2001) used an EXACD model to obtain a direct estimate of the intraday volatility. Dufour and Engle (2000) suggested an asymmetric EXACD specification which allows for an asymmetric response to innovation shocks. These models avoid some of the parameter restrictions postulated by the original ACD specification. Moreover, many authors modelled conditional heteroscedasticity in equidistant financial time series using long-memory models (e.g. Robinson and Zaffaroni, 1997, Robinson and Henry, 1999, Giraitis et al., 2000, Zaffaroni, 2000, Giraitis et al., 2002). To capture the long range time dependence in intertrade durations Jasiak (1998) proposed the fractionally integrated ACD (FIACD) model. Bauwens et al. (2000) review several duration models that have been proposed in the literature.

The objective of this paper is to investigate some of the statistical properties of three alternative ACD models that shed light on the dynamics of the transaction arrival process: the long-memory ACD (LMACD) model-which captures the long-term dependencies in the duration series and, in contrast to the FIACD model, is covariance stationary- and, two versions of the exponential ACD specification, which allow us to obtain analytically the unconditional moments implied by the model. In addition, since Engle and Russell (1998) find that the Weibull distribution suffers from remaining excess dispersion we use two alternative distributions which include the Weibull as a special case: the generalized F (GF) and the generalized Gamma (GG) (details on the properties of this distribution can be found in Bauwens and Giot, 2001b).

For all the aforementioned models we provide existence conditions of the second moments of the durations. We also derive analytical expressions for the autocorrelation function (ACF) of the durations[1]. Bauwens and Giot (2001a) and Bauwens et al. (2002) have also provided analytical expressions for the unconditional moments and ACF for the models belonging to the EXACD class as defined in Bauwens and Giot (2000). We should also mention that Carrasco and Chen (2002) provide sufficient conditions to ensure β-mixing and finite higher order moments for various linear and nonlinear GARCH, stochastic volatility, and ACD models.

To facilitate model identification, the results for the ACF of the durations can be applied so that properties of the observed data can be compared with the theoretical properties of the models. The significance of our results extends to the development of misspecification tests and estimation.

The rest of the paper is organized as follows. Section 2 briefly reviews the ACD model

[1] He et al. (2002), Demos (2002) and Karanasos and Kim (2003) studied the moment structure of EGARCH models, and Karanasos et al. (2003) examined the dependence structure of long-memory GARCH processes (see also Palma and Zevallos, 2003).

of Engle and Russell (1998) and considers the properties of the generalized Gamma and F distributions. Section 3 provides a detailed description of the long-memory, and exponential ACD models. In addition, it derives the ACF of the durations for all three models. Section 4 concludes the paper.

2 The ACD Model

Consider a stochastic point process that is simply a sequence of arrival times $\{t_0, t_1, \ldots, t_N\}$ with $0 \equiv t_0 < t_1 < \cdots < t_N$. Duration ($x_i$) is the time elapsed between two consecutive arrival times, i.e. $x_i \equiv t_i - t_{i-1}$.

The ACD model of Engle and Russell (1998) specifies the observed duration as a mixing process

$$x_i = \psi_i \varepsilon_i \quad (i \in \mathbb{Z}), \tag{2.1}$$

where $\{\varepsilon_i\}$ is a sequence of independent and identically distributed random variables with density $\xi(\epsilon_i; \phi) \equiv \xi(\varepsilon_i)$ (where ϕ is vector of parameters) and mean equal to one, and ψ_i denotes the conditional expectation of the ith duration. That is

$$\psi_i \equiv \psi_i(x_{i-1}, \ldots, x_1; \theta) = E(x_i | I_{i-1}),$$

where I_{i-1} denotes the conditioning information set generated by the durations preceding x_i and θ is a vector of parameters.

The flexibility of the model (2.1) lies in the rich host of candidates for the specification of the dynamic structure of ψ_i as well as the conditional density ξ.

Following Engle and Russell (1998), we express the conditional density of x_i as

$$g(x_i, \psi_i; \phi) \equiv g(x_i) = \frac{1}{\psi_i} \xi \left(\frac{x_i}{\psi_i} \right) \tag{2.2}$$

The specification in (2.2) can be generalized in many ways. The hazard functions can be given many parametric shapes. ACD specifications with exponential or Weibull errors imply that the conditional hazard functions (CHF) must either increase, decrease or stay constant during a time-spell. Grammig and Maurer (2000) investigated whether the restrictions concerning the CHF can imperil the successful application of ACD models. In a simulation study they show that the quasi maximum likelihood estimators of the basic ACD models tend to be biased and inefficient when the true data generating process required non-monotonic hazard functions. In addition, bias and inefficiency also affected the estimators of the parameters that were needed to predict expected durations. This entails severe consequences for the class of GARCH models for irregularly spaced data, recently introduced by Engle (2000) and Ghysels and Jasiak (1998), in which ACD models are employed to predict conditional expected durations that enter the conditional heteroskedasticity equation in the form of explanatory variables (abstracted from Grammig and Maurer, 2000).

Lunde (1999) and Grammig and Maurer(2000) -who used ACD models with generalized Gamma and Burr errors respectively- found that for price duration processes of NYSE traded stocks non-monotonic shapes of the hazard functions were indicated. Both studies, using standard likelihood ratio tests, rejected the exponential and Weibull ACD specifications in favor of the generalized Gamma and Burr ACD models. Accordingly, in what follows we examine two four-parameter general distributions that include as special or limiting cases many distributions considered in econometrics and finance. Expressions are reported that facilitate analysis of hazard functions, other distributional characteristics and parameter estimation.

2.1 The Generalized F Distribution

First, we examine the GF distribution. It is a particularly useful family of distributions which includes among others the Burr type 12, the Lomax, the Fisk, and the folded t. Its density is given by

$$\xi(\varepsilon_i) = \frac{a\varepsilon_i^{ap-1} q^q \varphi^{aq}}{B(p,q)(q\varphi^a + \varepsilon_i^a)^{p+q}} \quad (\varepsilon_i \geq 0, \quad i = 1, \ldots, N), \tag{2.3}$$

where $B(\cdot)$ is the beta function, $a > 0$, and the parameter φ is merely a scale parameter. Further, if the φ coefficient is

$$\varphi = \frac{B(p,q)}{q^{\frac{1}{a}} B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)},$$

then ε_i has mean equal to one. The GF distribution has integer moments of order up to 'μ' where -$p < \frac{\mu}{a} < q$ (see McDonald and Richards, 1987a). In a recent paper Hautsch (2002) used the generalized F distribution which allows for a wide range of possible hazard shapes. For volume durations he found that the GF distribution provides a significantly better fit than the exponential distribution. The mathematical expression for the CHF of the GF distribution is given in McDonald and Richards (1987b).

The conditional density of x_i can be written as

$$g(x_i) = \frac{a x_i^{ap-1} q^q (\varphi\psi_i)^{aq}}{B(p,q)[q(\varphi\psi_i)^a + x_i^a]^{p+q}},$$

Using the above expression we can write the log-likelihood function of the observations x_i, $i = 1, \ldots, N$ as

$$\mathcal{L} \equiv \sum_{i=1}^{N} \ln[g(x_i)] = c + \sum_{i=1}^{N} \{(ap-1)\ln(x_i) + aq\ln(\psi_i) - (p+q)\ln\left[q(\varphi\psi_i)^a + x_i^a\right]\},$$

where

$$c \equiv N\{\ln(a) + q\ln(q) + aq\ln(\varphi) - \ln[B(p,q)]\}$$

2.2 The Generalized Gamma Distribution

In this subsection we examine the GG distribution. This is a particularly useful family of distributions which includes among others the Weibull, the generalized Rayleigh, the Rayleigh, the Exponential, and the half normal. Its density is given by

$$\xi(\varepsilon_i) = \frac{a\varepsilon_i^{ap-1} e^{-\left(\frac{\varepsilon_i}{\varphi}\right)^a}}{\varphi^{ap}\Gamma(p)} \quad (\varepsilon_i \geq 0, \quad i = 1, \ldots, N), \tag{2.4}$$

where $\Gamma(\cdot)$ denotes the gamma function and the parameter φ is merely a scale parameter. In order to normalize the mean to be equal to one, the φ coefficient should be

$$\varphi = \frac{\Gamma(p)}{\Gamma(p + \frac{1}{a})}$$

The GG distribution has defined moments of order 'μ', where $\frac{\mu}{a} + p > 0$. Consequently, for $a > 0$, moments of positive integer order are defined. The CHF for the generalized Gamma function is examined in Glaser (1980). Lunde (1999), using a GG specification, found that the inverted U-shaped form was strongly supported by the data. As documented by Bauwens and Veredas (1999) the hazard function of several types of financial durations may be increasing for small durations and decreasing for long durations. To account for this stylized fact, Bauwens et al. (2000) used the GG distribution which has two shape parameters and breaks the one-to-one correspondence between the properties of overdispersion (underdispersion) and of decreasing (increasing) hazard.

Moreover, the conditional density of x_i can be written as

$$g(x_i) = \frac{a x_i^{ap-1} e^{-\left(\frac{x_i}{\varphi\psi_i}\right)^a}}{(\varphi\psi_i)^{ap}\Gamma(p)}$$

The log-likelihood function for the generalized Gamma ACD (GG-ACD) model is

$$\mathcal{L} = c + \sum_{i=1}^{N} \left\{ (ap-1)\ln(x_i) - \left(\frac{x_i \Gamma\left(p + \frac{1}{a}\right)}{\psi_i \Gamma(p)} \right)^a - ap\ln(\psi_i) \right\},$$

with

$$c \equiv N \left\{ \ln(a) - (1+ap)\ln[\Gamma(p)] + ap\ln\left[\Gamma\left(p + \frac{1}{a}\right)\right] \right\}$$

Lunde (1999) applied the generalized Gamma ACD (GG-ACD) model to a random sample consisting of seven stocks from the fifty stocks with the highest capitalization value on the NYSE. He found that the suggested generalization outperformed the model employed by Engle and Russell (1998).

3 ACD Models

The ACD model is closely related to the GARCH model and shares some of its features. Just as the simple GARCH model is often a good starting point, the simplest version of the ACD model seems like a natural starting point. However, as there are many alternative volatility models, there is a rich host of candidates for the dynamic specification of the conditional duration. Such specification includes models analogous to long-memory, exponential and many other GARCH models as possibilities. Fernandes and Grammig (2001) and Hautsch (2002) present a classification of different types of ACD models.

3.1 The Long-memory ACD Model

The standard ACD model accounts for short serial dependence in conditional durations and thus compels the pattern of the autocorrelation function to decay exponentially. In empirical applications of ACD models to high frequency intertrade durations the estimated coefficients on lagged variables sum up nearly to one. Such evidence indicates a potential misspecification that arises when an exponentially declining shape is fitted to a process showing an hyperbolic rate of decay. This would suggest that a more flexible structure allowing for longer term dependencies might improve the fit. In this respect, the motivation for using the long-memory ACD models is to capture the long-term dependencies in the duration series.

In the long-memory GARCH (LMGARCH) model of Robinson and Henry (1999) (see also Robinson, 1991 and Henry, 2001) the conditional variance of the process implies a slow hyperbolic rate of decay for the influence of the squared errors. Analogously to the LMGARCH process for the volatility, the long-memory ACD(n,d,m) [LMACD(n,d,m)] model for the duration is defined by

$$x_i = \omega + \frac{C(L)}{B(L)(1-L)^d}\eta_i, \tag{3.1}$$

for some $\omega, d \in (0,\infty)$ with

$$\begin{aligned} B(L) &\equiv 1 - \sum_{j=1}^{n} \beta_j L^j \equiv \prod_{j=1}^{n}(1-\lambda_j L), \\ C(L) &\equiv 1 - \sum_{l=1}^{m} c_l L^l, \end{aligned}$$

where $\eta_i \equiv x_i - \psi_i$ is a martingale difference sequence by construction, λ_j is the reciprocal of the jth root of $B(L)$, and d is the fractional differencing parameter. Further, we assume that all the roots of $C(L)$ and $B(L)$ lie outside the unit circle.

Lemma 1 *The duration $\{x_i\}$ can be written as an infinite sum of lagged values of η_i*

$$x_i = \omega + \sum_{j=0}^{\infty} \omega_j \eta_{i-j}, \tag{3.2a}$$

where

$$\omega_j \equiv \sum_{r=1}^{n} \lambda_r^+ \sum_{l=0}^{j} \binom{-d}{j-l} \pi_{rl} (-1)^{j-l}, \tag{3.2b}$$

with

$$\begin{aligned} \pi_{rl} &\equiv \sum_{j=0}^{\min\{l,m\}} \lambda_r^{l-j}(-c_j) \quad (c_0 \equiv -1), \\ \lambda_r^+ &\equiv \frac{\lambda_r^{n-1}}{\prod_{l=1,l\neq r}^{n}(\lambda_r - \lambda_l)} \end{aligned}$$

Proof. See Appendix A.

Moreover, ψ_i can be expressed as an infinite distributed lag of x_i terms:

$$\psi_i \equiv \left[1 - \frac{(1-L)^d B(L)}{C(L)}\right] x_i = \sum_{j=1}^{\infty} \phi_j x_{i-j} \tag{3.3}$$

with $\phi_1 > 0$, $\phi_j \geq 0$ $(j \geq 2)$ [2].

[2] The requirement $0 < \sum_{j=0}^{\infty} \omega_j^2 < \infty$ includes the case $\Omega(L) = \sum_{j=0}^{\infty} \omega_j L^j = C(L) \big/ [B(L)(1-L)^d]$, for $d \in (0, 0.5)$, and finite order polynomials $B(L)$ and $C(L)$ whose zeros are outside the unit circle in the complex plane. In this case the weights ϕ_j satisfy $\sum_{j=0}^{\infty} |\phi_j| < \infty$, $\phi_0 \equiv 1$. Under $\max_i \mathsf{E}(x_i^2) < \infty$ it follows that $\mathsf{E}(\eta_i^2) < \infty$ (see theorem 1 below), so that the innovations in (3.1) are square integrable martingale differences and x_i is well defined as a covariance stationary process and its autocorrelations $\mathsf{Corr}\,(x_i, x_{i-k}) \equiv \rho_k(x_i) = \sum_{j=0}^{\infty} \omega_j \omega_{j+k} \Big/ \sum_{j=0}^{\infty} \omega_j^2$ $(k \in \mathbb{N})$ can exhibit the usual long memory structure implied by $C(L) \big/ [B(L)(1-L)^d]$. Even if $\max_i \mathsf{E}(x_i^2) < \infty$ does not hold, the "autocorrelations" $\sum_{j=0}^{\infty} \omega_j \omega_{j+k} \Big/ \sum_{j=0}^{\infty} \omega_j^2$ are well defined under $0 < \sum_{j=0}^{\infty} \omega_j^2 < \infty$.

Jasiak (1998) presented empirical evidence for the presence of long memory, and proposed a fractionally integrated model for high frequency duration data. She applied the following specification to intertrade durations for the IBM and Alcatel stocks

$$B(L)(1-L)^d x_i = \omega + C(L)\eta_i \tag{3.4}$$

By analogy to the fractionally integrated GARCH (FIGARCH) model (the FIGARCH model was introduced by Baillie et al., 1996; see also Bollerslev and Mikkelsen, 1996) this process is called fractionally integrated ACD (FIACD) model. For $d > 0$, x_i has an unbounded first moment. However, when $d > 0$ the FIACD process governed by (3.4) is strictly stationary and ergodic, and the "autocorrelations" $\sum_{j=0}^{\infty} \omega_j \omega_{j+k} / \sum_{j=0}^{\infty} \omega_j^2$ are well defined under $\sum_{j=0}^{\infty} \omega_j^2 < \infty$.

3.2 Autocorrelation function of the LMACD model

Several previous articles dealing with financial market data-e.g. Dacorogna et al. (1993)- have commented on the behavior of the autocorrelation function of power transformed absolute returns, and the desirability of having a model which comes close to replicating certain stylized facts in the data (abstracted from Baillie and Chung, 2001). In this respect, one can apply the results in this section to check whether the long-memory ACD model can effectively replicate the observed pattern of autocorrelations of the durations.

Another potential motivation for the derivation of the results in this section is that the autocorrelations of the durations in (2.1) and (3.1) can be used to estimate the ACD parameters in (3.1). The approach is to use the minimum distance estimator (MDE), which estimates the parameters by minimizing the Mahalanobis generalized distance of a vector of sample autocorrelations from the corresponding population autocorrelations (see Baillie and Chung, 2001). One motivation for the MDE approach can be found in Jacquier et al. (1994) who, on examining the autocorrelations of transformations of fitted returns from MLE, have noted their discrepancy when compared with the autocorrelations of actual returns.

In this section we consider the LMACD(n, d, m) process defined by (2.1) and (3.1) with $d \in (0, 0.5)$ and the additional restriction that the roots of $B(z) = 0$ are simple.

Lemma 2 *The condition for the existence of the second moment of the duration is*

$$\left[1 - \frac{1}{\mathsf{E}(\varepsilon_i^2)}\right] \left(\sum_{j=0}^{\infty} \omega_j^2\right) < 1,$$

where ω_j is defined by (3.2b).

Proof. See Appendix A.

In the following theorem we establish a representation for the autocorrelation function of the durations of the LMACD(n, d, m) process, with $d \in (0, \frac{1}{2})$. Let F be the Gaussian hypergeometric function defined by $F(a, b; c; z) \equiv \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}$, where $(b)_j \equiv \prod_{i=0}^{j-1}(b+i)$ is Pochhammer's shifted factorial.

Theorem 1 *The autocorrelation function of $\{x_i\}$ can be expressed as*

$$\rho_k(x_i) = \frac{\sum_{j=0}^{\infty} \omega_j \omega_{j+k}}{\sum_{j=0}^{\infty} \omega_j^2} = \frac{z_k}{z_0}, \quad (k \in \mathbb{N}), \tag{3.5}$$

where

$$z_k \equiv \sum_{j=1}^{n} \sum_{l=0}^{m} \Psi_l \overline{\lambda}_j C(d, k, l, \lambda_j) \quad (k \geq 0),$$

with

$$\Psi_l \equiv \sum_{r=0}^{m-l} c_r c_{r+l} \quad (c_0 \equiv -1), \qquad \overline{\lambda}_j \equiv \frac{\lambda_j^+}{\prod_{r=1}^{n}(1-\lambda_j\lambda_r)},$$

$$\begin{aligned} C(d,k,l,\lambda_j) \equiv\ & \frac{\Gamma(d+k+l)}{\Gamma(1-d+k+l)} F(d+k+l,1;1-d+k+l;\lambda_j) + \\ & +1_l \frac{\Gamma(d+k-l)}{\Gamma(1-d+k-l)} F(d-k+l,1;1-d-k+l;\lambda_j) + \\ & \lambda_j \left[1_l \frac{\Gamma(d+k+1-l)}{\Gamma(2-d+k-l)} F(d+k+1-l,1;2-d+k-l;\lambda_j) + \right. \\ & \left. + \frac{\Gamma(d+k-1+l)}{\Gamma(k-d+l)} F(d-k+1-l,1;2-d-k;\lambda_j) \right], \end{aligned}$$

and

$$1_l \equiv \begin{cases} 0, & \text{if} \quad l = 0, \\ 1, & \text{if} \quad l \neq 0 \end{cases},$$

if and only if $\left[1 - \frac{1}{\mathsf{E}(\varepsilon_i^2)}\right] z_0 \leq 1$.

Proof. See Appendix A.

To illustrate the general result we consider the LMACD(1,d,1) process. In this case $1 - C(L) = c_1 L \equiv cL$ and $1 - B(L) = \beta_1 L \equiv \beta L$.

Corollary 1 *For the LMACD(1,d,1) model, with* $|c|, |\beta| < 1$ *and* $d \in (0, \frac{1}{2})$, *the autocorrelation function of* $\{x_i\}$ *is given by*

$$\rho_k(x_i) = \frac{z_k}{z_0} \quad (k \in \mathbb{N}), \tag{3.6}$$

where

$$\begin{aligned} z_k \equiv\ & \frac{\Gamma(1-2d)}{(1-\beta^2)\Gamma(d)\Gamma(1-d)} \left\{ \frac{\Gamma(d+k)}{\Gamma(1-d+k)} [(1+c^2-\beta c) \right. \\ & \times F(d+k,1;1-d+k;\beta) - c\beta F(d-k,1;1-d-k;\beta)] \\ & + \frac{\Gamma(d+k-1)}{\Gamma(k-d)} [\beta(1+c^2)-c] F(d-k+1,1;2-d-k;\beta) \\ & \left. - \frac{\Gamma(d+k+1)}{\Gamma(2-d+k)} c F(d+k+1,1;2-d+k;\beta) \right\} \quad (k \geq 0), \end{aligned}$$

if and only if $\left[1 - \frac{1}{\mathsf{E}(\varepsilon_i^2)}\right] z_0 \leq 1$.

Proof. The proof follows from lemma 2 and theorem 1, by setting $C(L) \equiv 1 - cL$ and $B(L) \equiv 1 - \beta L$. ■

Figure 1 plots the theoretical ACF of the duration of the above process (for various values of the three parameters c, β and d).

Remark 1. As expected, the autocorrelation function of the LMACD(1,1) process decays at a very slow rate. When $c = 0$, $\beta = .6$ and $d = .45$, for instance, the duration still has an autocorrelation coefficient around 0.5 at lag 1,000.

3.3 Exponential ACD Models

One limitation of the linear ACD specification results from the nonnegativity constraints on the parameters which are imposed to ensure that ψ_i remains nonnegative for all i. These constraints imply that an increasing $f(\varepsilon_i)$ in any period increases ψ_{i+j} for all $j \geq 1$, ruling out oscillatory behavior in the ψ_i process. Alternatively, one can use a model in which the logarithm of the conditional duration follows an ACD-like process. This is analogous to Nelson's (1991) exponential GARCH model for the conditional variance.

Consider the following ACD(n, m)

$$B(L)\ln(\psi_i) = \omega + C(L)f(\varepsilon_i), \tag{3.7a}$$

with

$$B(L) \equiv -\sum_{j=0}^{n} \beta_j L^j = \prod_{j=1}^{n}(1 - \lambda_j L) \quad (\beta_0 \equiv -1), \tag{3.7b}$$

$$C(L) \equiv \sum_{j=1}^{m} c_j L^j, \tag{3.7c}$$

where λ_j is the reciprocal of the jth root of the autoregressive polynomial $B(L)$. Without loss of generality, assume that β_n and c_m are both not equal to zero.

We will call the models in the class where

$$f(\varepsilon_i) = \ln(\varepsilon_i), \tag{3.8}$$

and the innovations ε_i follow the GG distribution, generalized Gamma-logarithmic EXACD (GG-LG-EXACD). When the conditional distribution of the durations is the generalized F, these models will be called generalized F-logarithmic EXACD (GF-LG-EXACD).

Furthermore, models in the class where

$$f(\varepsilon_i) = \varepsilon_i^a, \tag{3.9}$$

and the innovations are drawn from the generalized Gamma distribution[3], will be called generalized Gamma-exponential ACD (GG-EXACD).

Bauwens and Giot (2000) proposed a version of the EXACD model, with $f(\varepsilon_i) = \varepsilon_i$ and Weibull errors, under the name logarithmic ACD model. Bauwens and Giot (2000) applied this ACD specification to price durations relative to the bid-ask quote process of three securities listed on the NYSE. Lunde (1999), using duration data for seven stocks traded on the NYSE, estimated a version of the GG-EXACD(1,1) model with $f(\varepsilon_i) = \varepsilon_i$. Bauwens and Giot (2001a), using duration data for several stocks traded on the NYSE, estimated the EXACD(1,1) and LG-EXACD(1,1) models with an exponential distribution for the error term. Giot (2000) used an EXACD(1,1) model to compute the Value-at-Risk for three stocks traded on the NYSE.

[3] In (3.9) a is one of the parameters of the generalized Gamma distribution (see eq. 2.4).

3.4 Moments of Exponential ACD models

Durations between stock market events are often characterized by overdispersion. Another important stylized fact is the shape of the ACF of the durations, which usually decreases slowly from a relatively low positive first-order autocorrelation. It is essential that, for some parameter values, the ACD models can accommodate such stylized facts. In the light of this, it is important (a) to know whether the general shape of the sample autocorrelations is captured by the estimated model and (b) to find the model for which the estimated theoretical ACF is closest to the ACF of the data and, therefore, has the best fit as far as the ACF is concerned. To gain further insight into how well an ACD model fits the data we need to check whether the unconditional moments computed from the analytical formulae are in line with the empirical ones. Therefore, analytical results on the moments and autocorrelations of the durations for the logarithmic and exponential ACD models can indicate whether these specifications provide a better alternative to the standard ACD model.

Assumption 1. The polynomials $B(L)$ and $C(L)$ in (3.7) have no common roots.

Assumption 2. $B(z) = 0$ and $C(z) = 0$ have no roots in the closed disc $\{z : |z| \leq 1\}$.

In what follows we only examine the case where the roots of the autoregressive polynomial $B(L)$ are distinct.

Lemma 3 *Under assumptions 1 and 2, the* μth power *of the duration for the ACD model in (2.1) and (3.7) can be expressed as*

$$x_i^{\mu} \;=\; e^{\frac{\mu\omega}{B(1)}}\varepsilon_i^{\mu} \times \prod_{l=1}^{\infty}[e^{\mu_l f(\varepsilon_{i-l})}] \qquad (\mu \in \mathbb{R}_+), \tag{3.10a}$$

with

$$\mu_l \;\equiv\; \mu\delta_l \equiv \mu \sum_{f=1}^{n} \zeta_f z_{fl}, \tag{3.10b}$$

and

$$\zeta_f \;\equiv\; \frac{\lambda_f^{n-1}}{\prod_{j=1, j\neq f}^{n}(\lambda_f - \lambda_j)},$$

$$z_{fl} \;\equiv\; \begin{cases} \sum_{j=0}^{l-1} c_{l-j}\lambda_f^j, & \text{if} \quad l \leq m, \\ z_{fm}\lambda_f^{l-m}, & \text{if} \quad l > m \end{cases},$$

where c_l, λ_f are defined in (3.7).

Proof. See Appendix B.

Lemma 4 *Let assumptions 1 and 2 hold. Suppose further that* $\mathsf{E}(\varepsilon_i^{2\mu})$, $\mathsf{E}(\mathrm{e}^{2\mu_l f(\varepsilon_i)})$, *and* $\mathsf{E}(\varepsilon_i^{\mu}\mathrm{e}^{\mu_l f(\varepsilon_i)})$ *are finite for all l. Then the 2μth moment of the duration and the* kth ($k \in \mathbb{N}$) *autocorrelation of the μth power of the duration, for the ACD model in (2.1) and (3.7), have the form*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}}\mathsf{E}(\varepsilon_i^{2\mu}) \times \prod_{l=1}^{\infty}[\mathsf{E}(e^{2\mu_l f(\varepsilon_{i-l})})], \tag{3.11a}$$

$$\begin{aligned}\rho_k(x_i^{\mu}) = {} & \mathsf{E}(\varepsilon_i^{\mu})\left\{\prod_{l=1}^{k-1}[\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})] \times \mathsf{E}[\varepsilon_{i-k}^{\mu} e^{\mu_k f(\varepsilon_{i-k})}] \times \prod_{l=1}^{\infty}[\mathsf{E}(e^{(\mu_{k+l}+\mu_l) f(\varepsilon_{i-k-l})})]\right. \\ & \left. - \left[\prod_{l=1}^{\infty}[\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})]\right]^2 \times \mathsf{E}(\varepsilon_i^{\mu})\right\} \\ & \times \left\{\mathsf{E}(\varepsilon_i^{2\mu})\prod_{l=1}^{\infty}\left[\mathsf{E}(e^{2\mu_l f(\varepsilon_{i-l})})\right] - [\mathsf{E}(\varepsilon_i^{\mu})]^2\left[\prod_{l=1}^{\infty}[\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})]\right]^2\right\}^{-1},\end{aligned} \tag{3.11b}$$

Note that, when $k = 1$, the first product term in (3.11b) is replaced by 1.

Proof. See Appendix B.

Remark 3. For the practical computation of the moments in lemma 4 , the infinite products that appear in the expression (3.11) can be truncated after a sufficiently large number of terms since μ_l tends to zero. In practice, we found that for first and second moments, truncation after 1000 terms was more than sufficient to get a high accuracy (see also Bauwens and Giot, 2001a and Bauwens et al., 2002).

The 2μth moment and the kth autocorrelation of the duration have been independently obtained by Bauwens et al. (2002). When considering the ACD(n, m) model in (3.7), theorems 1 and 2 in Bauwens et al. (2002) and (3.11) are equivalent.

Fernandes and Grammig (2001) propose an exponential ACD model with $f(\varepsilon_i) = [|\varepsilon_i - b| + \gamma(\varepsilon_i - b)]^{\upsilon}$, under the name asymmetric Box-Cox ACD (ABC-ACD) model. This specification provides a flexible functional form that permits the logarithm of the conditional duration to respond in distinct manners to small and large shocks. For the ABC-ACD(1,1) model they report expressions for the 2μth moment and kth autocovariance of the duration which are similar to our equation (3.11).

Theorem 2 *Suppose that assumptions 1 and 2 hold and that both $\frac{2\mu}{a}, \frac{2\mu_l}{a} \in (-p, q)$. Then the 2μth moment of the duration and the* kth *autocorrelation of the μth power of the duration, for the GF-LG-EXACD(n, m) model in (3.7)-(3.8), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}} \times B_{2\mu} \times \prod_{l=1}^{\infty}(B_{2\mu_l}), \tag{3.12a}$$

$$\rho_k(x_i^{\mu}) = \frac{B_{\mu}\left[\prod_{l=1}^{k-1}(B_{\mu_l}) \times B_{\mu+\mu_k} \times \prod_{l=1}^{\infty}(B_{\mu_{k+l}+\mu_l}) - \left[\prod_{l=1}^{\infty}(B_{\mu_l})\right]^2 \times B_{\mu}\right]}{B_{2\mu} \times \prod_{l=1}^{\infty}(B_{2\mu_l}) - (B_{\mu})^2 \times \left[\prod_{l=1}^{\infty}(B_{\mu_l})\right]^2}, \tag{3.12b}$$

with

$$B_{\mu_l} \equiv \frac{[B(p,q)]^{(\mu_l-1)} B\left(p + \frac{\mu_l}{a}, q - \frac{\mu_l}{a}\right)}{\left[B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)\right]^{\mu_l}},$$

where $B(\cdot)$ is the beta function, μ_l is defined in lemma 3, and *p, q, a are the parameters of the GF distribution (see eq. 2.3). Note that, when $k = 1$, the first product term in the numerator of (3.12b) is replaced by 1.*

Proof. See Appendix B.

To illustrate the preceding general theory we consider the LG-EXACD(1,1) process when the distribution of the innovations is either Lomax or Fisk (these models will be called Lomax-LG-EXACD (L-LG-EXACD) and Fisk-LG-EXACD (F-LG-EXACD) respectively). In these cases $\mu_l \equiv \mu c\beta^{l-1}$, where $c \equiv c_1$, $\beta \equiv \beta_1$. We have the following corollaries.

Corollary 2a *For the L-LG-EXACD(1,1) model, with $q = 3$, $-.5 < c < 1$ $(c \neq 0)$, $|\beta| < 1$ and $-.5 < c\beta$, the kth autocorrelation of the duration is given by (3.12b) with*

$$B_{\mu_l} \equiv \Gamma(1+\mu_l)\Gamma(3-\mu_l)2^{(\mu_l-1)}, \quad \mu_l \equiv c\beta^{l-1}$$

Proof. The proof follows from theorem 2, by setting $p = a = \mu = 1$, $q = 3$ and $\mu_l \equiv c\beta^{l-1}$. ■

Corollary 2b *For the F-LG-EXACD(1,1) model, with $a = 4$ and $|c|, |\beta| < 1$, the kth autocorrelation of the duration is given by (3.12b) with*

$$B_{\mu_l} \equiv \frac{\Gamma\left(1+\frac{\mu_l}{4}\right)\Gamma\left(1-\frac{\mu_l}{4}\right)}{\left[\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^{\mu_l}}, \quad \mu_l \equiv c\beta^{l-1}$$

Proof. The proof follows from theorem 2, by setting $p = q = \mu = 1$, $a = 4$ and $\mu_l \equiv c\beta^{l-1}$. ■

Figures 2 and 3 plot the theoretical ACF of the duration of the above processes with $(c, \beta) \in \{(0.05, 0.995), (0.8, 0.995), (0.05, 0.98), (0.8, 0.98)\}$ (we used Maple to evaluate the autocorrelations).

Theorem 3 *Let assumptions 1 and 2 hold and $\frac{2\mu}{a}, \frac{2\mu_l}{a} > -p$. Then the 2μth moment of the duration and the* kth *autocorrelation of the μth power of the duration, for the GG-LG-EXACD(n, m) model in (2.1) and (3.7)-(3.8), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}} \times \Gamma_{2\mu} \times \prod_{l=1}^{\infty}(\Gamma_{2\mu_l}), \tag{3.13a}$$

$$\rho_k(x_i^{\mu}) = \frac{\Gamma_\mu \left[\prod_{l=1}^{k-1}(\Gamma_{\mu_l}) \times \Gamma_{\mu+\mu_k} \times \prod_{l=1}^{\infty}(\Gamma_{\mu_{k+l}+\mu_l}) - \left[\prod_{l=1}^{\infty}(\Gamma_{\mu_l})\right]^2 \times \Gamma_\mu\right]}{\Gamma_{2\mu} \times \prod_{l=1}^{\infty}(\Gamma_{2\mu_l}) - (\Gamma_\mu)^2 \times \left[\prod_{l=1}^{\infty}(\Gamma_{\mu_l})\right]^2}, \tag{3.13b}$$

with

$$\Gamma_{\mu_l} \equiv \frac{\Gamma\left(p+\frac{\mu_l}{a}\right)[\Gamma(p)]^{(\mu_l-1)}}{[\Gamma\left(p+\frac{1}{a}\right)]^{\mu_l}},$$

where $\Gamma(\cdot)$ is the gamma function, μ_l is *defined in lemma 3, and p, a are the parameters of the GG distribution (see eq. 2.4). Note that, when $k = 1$, the first product term in the numerator of (3.13b) is replaced by 1.*

Proof. See Appendix B.

To illustrate the general result we consider the LG-EXACD(1,1) process with innovations that are drawn from either the Rayleigh or the Exponential distribution (these models will be called Rayleigh-LG-EXACD (R-LG-EXACD) and Exponential-LG-EXACD (E-LG-EXACD) respectively). In these cases $\mu_l \equiv \mu c\beta^{l-1}$. We have the following corollaries.

Corollary 3a *For the R-LG-EXACD(1,1) model satisfying $|c|, |\beta| < 1$, the kth autocorrelation of the duration is given by (3.13b) with*

$$\Gamma_{\mu_l} \equiv \frac{\Gamma\left(1+\frac{\mu_l}{2}\right)}{\left[\Gamma\left(\frac{3}{2}\right)\right]^{\mu_l}}, \quad \mu_l \equiv c\beta^{l-1}$$

Proof. The proof follows from theorem 3, by setting $p = \mu = 1$, $a = 2$ and $\mu_l \equiv c\beta^{l-1}$. ■

Corollary 3b *For the E-LG-EXACD(1,1) model satisfying* $-.5 < c < 1$ $(c \neq 0)$, $|\beta| < 1$ *and* $-.5 < c\beta$ *the kth autocorrelation of the duration is given by (3.13b) with*

$$\Gamma_{\mu_l} \equiv \Gamma(1 + \mu_l), \quad \mu_l \equiv c\beta^{l-1}$$

Proof. The proof follows from theorem 3, by setting $p = a = \mu = 1$, and $\mu_l \equiv c\beta^{l-1}$. ■

For the E-LG-EXACD(1,1) model Bauwens and Giot (2001a) illustrate the variation of $\rho_1(x_i)$ as a function of c (from 0 to 0.2) and β (from 0.8 to 0.98).

Figures 4 and 5 plot the theoretical ACF of the duration of the above processes with $(c, \beta) \in \{(0.05, 0.995), (0.8, 0.995), (0.05, 0.98), (0.8, 0.98)\}$.

Theorem 4 *Suppose that assumptions 1 and 2 hold and* $\frac{2\mu}{a} > -p$, $2\mu_l < \varphi^{-a}$. *Then the* 2μ*th moment of the duration and the kth autocorrelation of the* μ*th power of the duration, for the GG-EXACD(m, n) model in (2.1), (3.7) and (3.9), are given by*

$$\mathsf{E}(x_i^{2\mu}) = e^{\frac{2\mu\omega}{B(1)}} \times \Gamma_{2\mu} \times \prod_{l=1}^{\infty}(\Delta_{2\mu_l}), \tag{3.14a}$$

$$\rho_k(x_i^{\mu}) = \frac{\Gamma_\mu \left[\prod_{l=1}^{k-1}(\Delta_{\mu_l}) \times \Delta_{\mu,\mu_k} \times \prod_{l=1}^{\infty}(\Delta_{\mu_{k+l}+\mu_l}) - \left[\prod_{l=1}^{\infty}(\Delta_{\mu_l})\right]^2 \times \Gamma_\mu\right]}{\Gamma_{2\mu} \times \prod_{l=1}^{\infty}(\Delta_{2\mu_l}) - (\Gamma_\mu)^2 \times \left[\prod_{l=1}^{\infty}(\Delta_{\mu_l})\right]^2}, \tag{3.14b}$$

with

$$\Delta_{\mu,\mu_k} \equiv \frac{\Gamma\left(p + \frac{\mu}{a}\right)[\Gamma(p)]^{(\mu-1)}[\Gamma\left(p + \frac{1}{a}\right)]^{pa}}{\left\{\left[\Gamma(p + \frac{1}{a})\right]^a - \mu_k[\Gamma(p)]^a\right\}^{(p+\frac{\mu}{a})}},$$

$$\Delta_{\mu_l} \equiv \frac{[\Gamma\left(p + \frac{1}{a}\right)]^{pa}}{\left\{\left[\Gamma(p + \frac{1}{a})\right]^a - \mu_l[\Gamma(p)]^a\right\}^p},$$

where μ_l, Γ_μ *are defined in lemma 3 and theorem 3 respectively, and* p, a, φ *are the parameters of the GG distribution. Note that, when* $k = 1$, *the first product term in the numerator of (3.14b) is replaced by 1.*

As an illustration we consider the GG-EXACD(1,1) model. Let Φ be the basic hypergeometric series denoted by ${}_1\Phi_0(a; b, z) \equiv \sum_{j=0}^{\infty} \frac{(a;b)_j}{(b;b)_j} z^j$, where $(a; b)_j$ is the b-shifted factorial.

Corollary 4a *For the GG-EXACD(1,1) model satisfying* $|\beta| < 1$, $-1 < c < \frac{\varphi^{(-a)}}{2\mu}$ $(c \neq 0)$ *and* $c\beta < \frac{\varphi^{(-a)}}{2\mu}$ *the kth autocorrelation of the* μth power of the *duration is given by*

$$\frac{\Gamma_\mu \left\{\left[{}_1\Phi_0(\beta^{k-1}; \beta, c^*)\right]^p \times \Delta_{\mu,\mu_k} \times [{}_1\Phi_0(0; \beta, c_k)]^p - [{}_1\Phi_0(0; \beta, c^*)]^p \, \Gamma_\mu\right\}}{\Gamma_{2\mu} \, [{}_1\Phi_0(0; \beta, 2c^*)]^p - (\Gamma_\mu)^2 \, [{}_1\Phi_0(0; \beta, c^*)]^{2p}}, \tag{3.15}$$

with

$$c^* \equiv \frac{c\mu \, [\Gamma(p)]^a}{\left[\Gamma(p + \frac{1}{a})\right]^a}, \quad c_k \equiv c^*(1 + \beta^k), \quad \mu_l \equiv \mu c\beta^{l-1},$$

where Γ_μ *and* Δ_{μ,μ_k} *are defined in lemma 3 and theorem 4 respectively.*

Proof. The proof follows from theorem 4, by setting $\mu_l \equiv \mu c\beta^{l-1}$. ■

As a further illustration we consider the Rayleigh-EXACD (R-EXACD) and Exponential EXACD (E-EXACD) processes of order (1,1). We have the following corollaries.

Corollary 4b *For the R-EXACD(1,1) model satisfying* $|\beta| < 1$, $-1 < c < .393$ $(c \neq 0)$, $c\beta < .393$ *the kth autocorrelation of the duration is given by (3.15), with* $p = \mu = \Gamma_\mu = 1$ *and*

$$\Delta_{1,\mu_k} \equiv \frac{1}{\{1 - 1.273\mu_k\}^{\frac{3}{2}}}, \quad c^* \equiv 1.273c$$

Proof. The proof follows from corrolary 4a, by setting $p = \mu = 1$, and $a = 2$. ■

Figure 6 plots the theoretical ACF of the duration of the above process with $(c, \beta) \in \{(0.05, 0.995), (0.3, 0.995), (0.05, 0.98), (0.3, 0.98)\}$.

Corollary 4c *For the E-EXACD(1,1) model satisfying* $|\beta| < 1$, $-1 < c < 0.5$ $(c \neq 0)$ *and* $c\beta < .5$ *the kth autocorrelation of the duration is given by (3.15), with* $p = \mu = \Gamma_\mu = 1$ *and*

$$\Delta_{1,\mu_k} \equiv \frac{1}{(1 - \mu_k)^2}, \quad c^* \equiv c$$

Proof. The proof follows from corollary 4a, by setting $p = a = \mu = 1$. ■

For the E-EXACD(1,1) model Bauwens and Giot (2001a) illustrate the variation of $\rho_1(x_i)$ as a function of c (from 0 to 0.2) and β (from 0.8 to 0.98).

Figure 7 plots the theoretical ACF of the duration of the above process with $(c, \beta) \in \{(0.05, 0.995), (0.4, 0.995), (0.05, 0.98), (0.4, 0.98)\}$.

For all the models in corollaries 2-4 the parameters c and β control the decay of the theoretical autocorrelation function. We have the following remarks.

Remark 4a. In all cases it is seen that the autorrelations start higher and decrease more rapidly when the value of c is high than when it is low.

Remark 4b. In all cases, when the value of c is low, the model based autocorrelations seem to start higher and decrease slower when the value of β is high than when it is low.

Remark 4c. In all cases, when the value of c is high, it is seen that the autocorrelations start higher and decrease faster when the value of β is high than when it is low.

Until the present contribution and the papers by Bauwens and Giot (2001a), and Bauwens et al. (2002) the unconditional moments implied by the logarithmic and exponential ACD models were not available analytically. Bauwens and Giot (2000) relied therefore on numerical simulations to compute these moments.

4 Conclusions

This paper has provided a detailed description of the long-memory, and exponential ACD models. We also investigated the properties of the generalized Gamma and generalized F distributions which allow for non-monotonic hazard functions. For all ACD specifications we derived analytical expressions of the autocorrelation function of the durations. Conditions for the existence of the first two moments of the durations were also established. The derivation of the autocorrelations of the durations and their comparison with the corresponding sample equivalents will help the investigator (a) to decide which is the most appropriate method of estimation (e.g. maximum likelihood estimation (MLE), minimum distant estimator (MDE)) for a specific model, (b) to chose, for a given estimation technique, the model (e.g. LMACD, LG-EXACD, EXACD) that best replicates certain stylized facts of the data and, (c) in conjunction with the various model selection criteria, to identify the optimal order of the chosen specification.

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Figure 1. Autocorrelation function of x_i. FIACD(1,1) model.

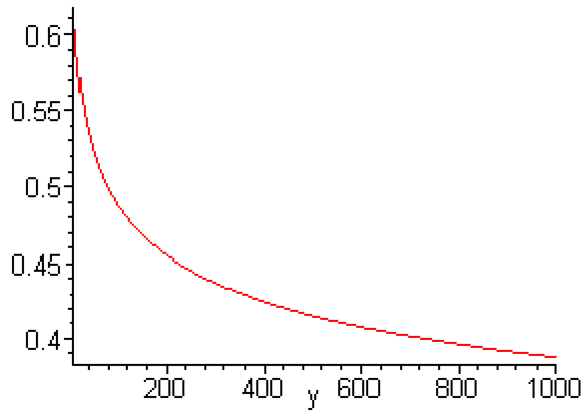

Figure 1a: $c = 0.2$, $\beta = 0.3$, $d = 0.45$.

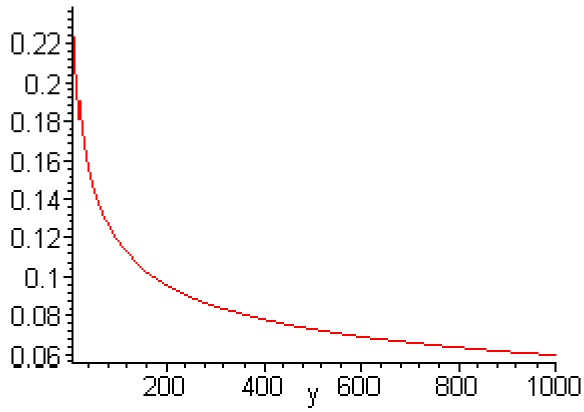

Figure 1b: $c = 0.2$, $\beta = 0.3$, $d = 0.35$.

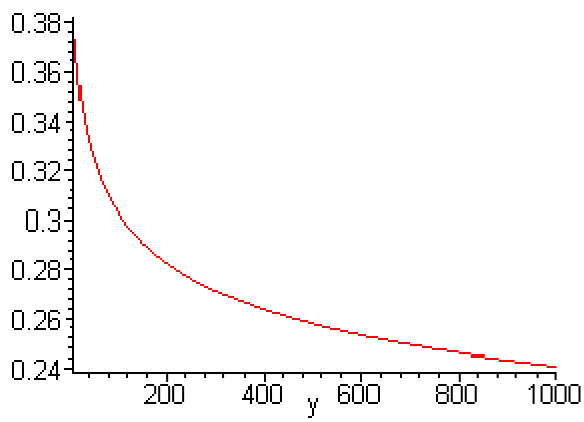

Figure 1c: $c = 0.2$, $\beta = 0.6$, $d = 0.45$.

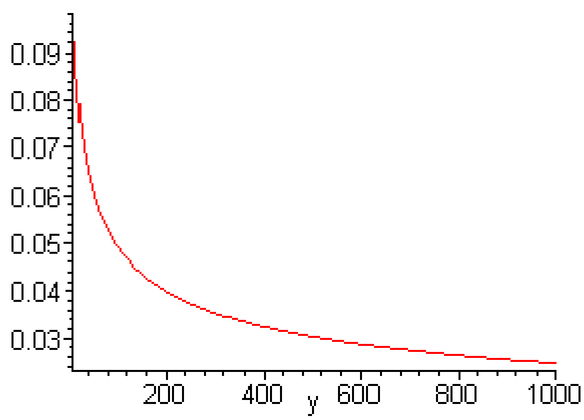

Figure 1d: $c = 0.2$, $\beta = 0.6$, $d = 0.35$.

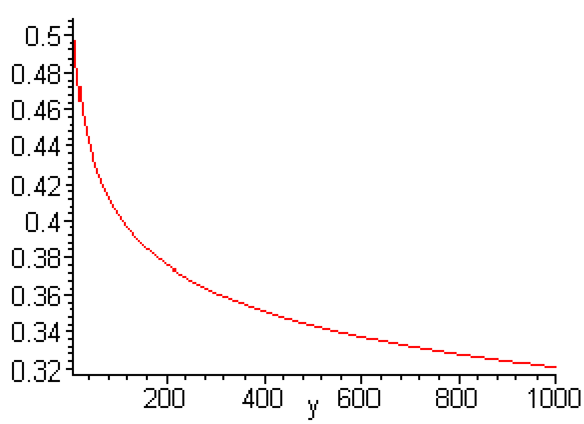

Figure 1e: $c = 0.4$, $\beta = 0.6$, $d = 0.45$.

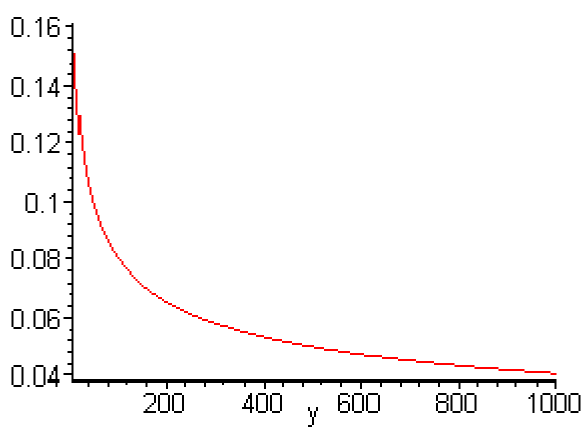

Figure 1f: $c = 0.4$, $\beta = 0.6$, $d = 0.35$.

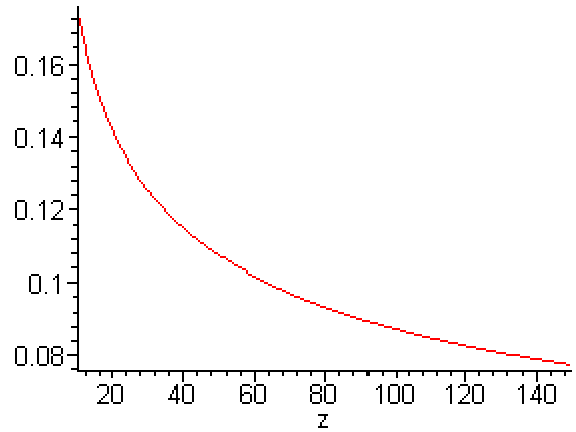

Figure 1g: $c = 0$, $\beta = 0.3$, $d = 0.45$.

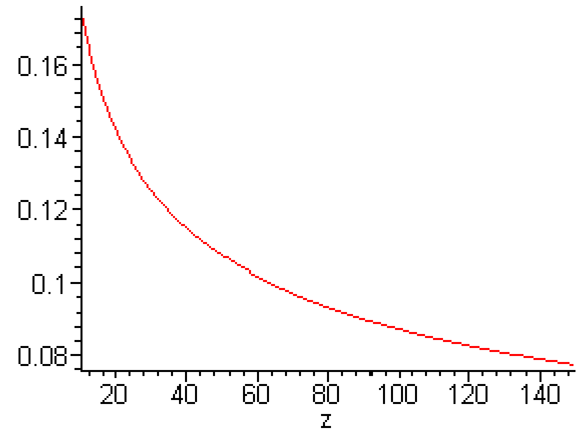

Figure 1h: $c = 0$, $\beta = 0.3$, $d = 0.35$.

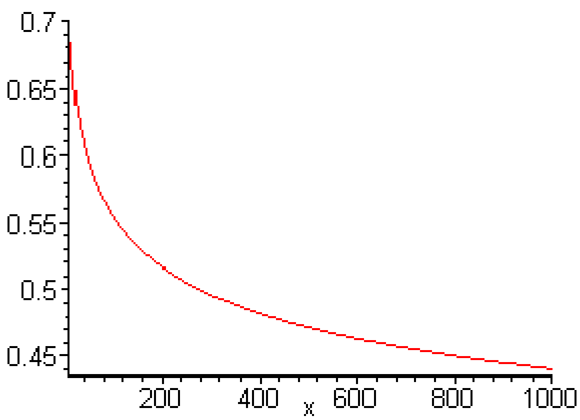

Figure 1i: $c = 0.2$, $\beta = 0$, $d = 0.45$.

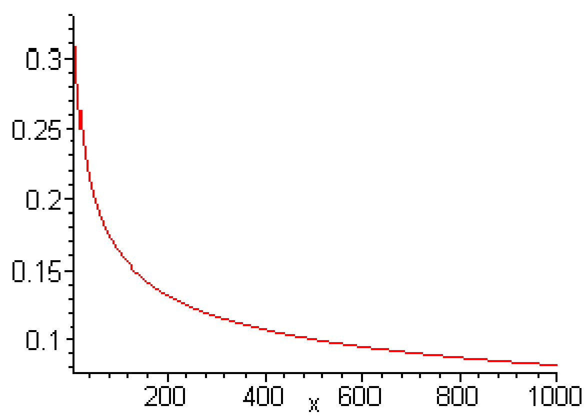

Figure 1j: $c = 0.2$, $\beta = 0$, $d = 0.35$.

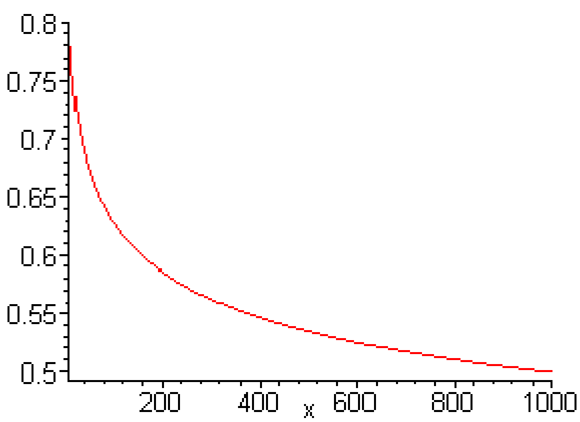

Figure 1k: $c = 0.6$, $\beta = 0$, $d = 0.45$.

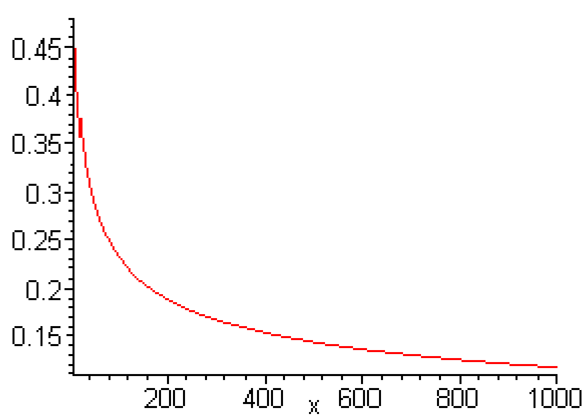

Figure 1l: $c = 0.6$, $\beta = 0$, $d = 0.35$.

Figure 2. Autocorrelation function of x_i; L-LG-EXACD(1,1), $q = 3$.

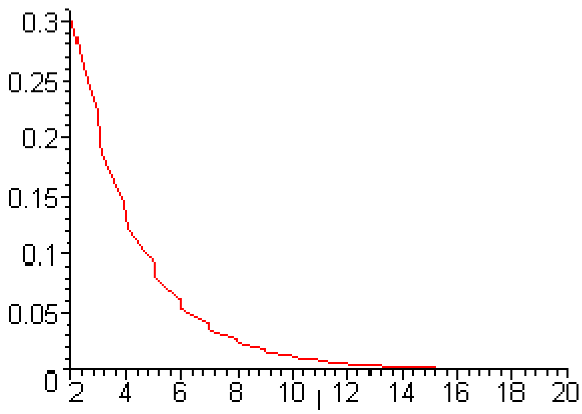

Figure 2a: $c = 0.05$, $\beta = 0.995$.

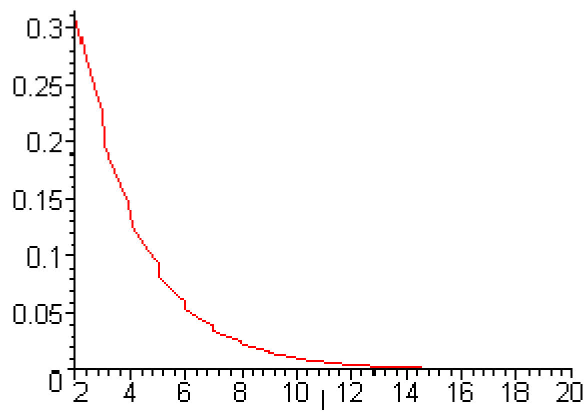

Figure 2b: $c = 0.8$, $\beta = 0.995$.

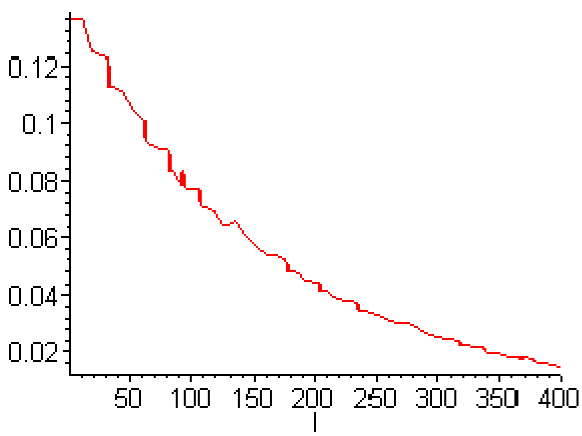

Figure 2c: $c = 0.05$, $\beta = 0.98$.

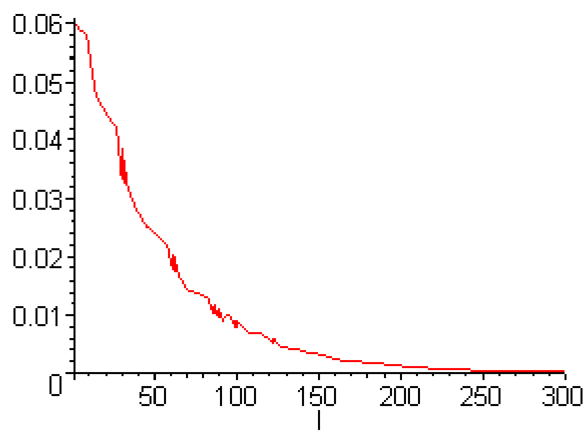

Figure 2d: $c = 0.8$, $\beta = 0.98$.

Figure 3. Autocorrelation function of x_i; F-LG-EXACD(1,1), $a = 4$.

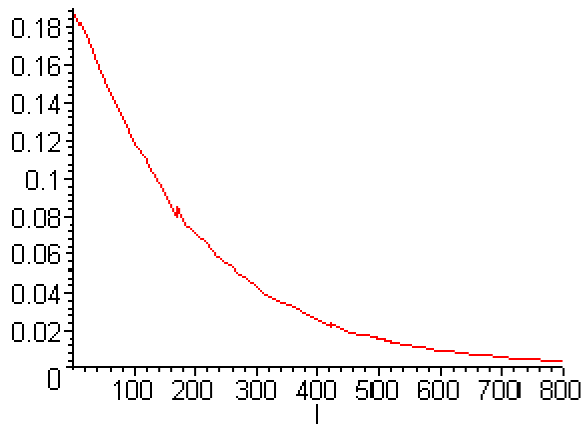

Figure 3a: $c = 0.05$, $\beta = 0.995$.

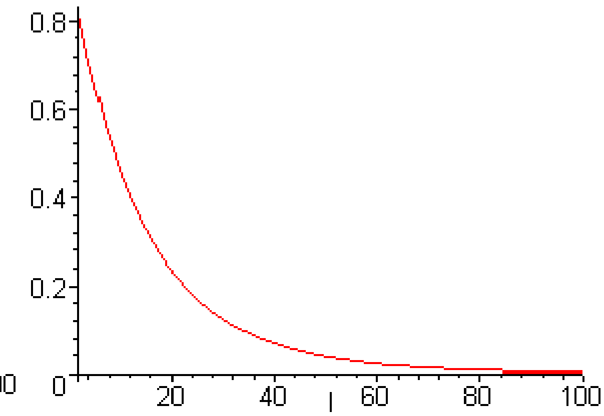

Figure 3b: $c = 0.8$, $\beta = 0.995$.

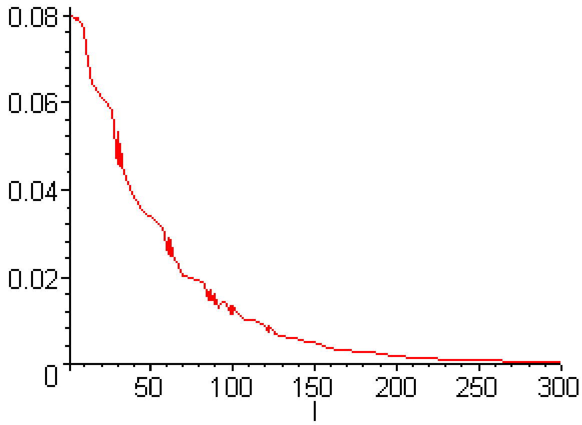

Figure 3c: $c = 0.05$, $\beta = 0.98$.

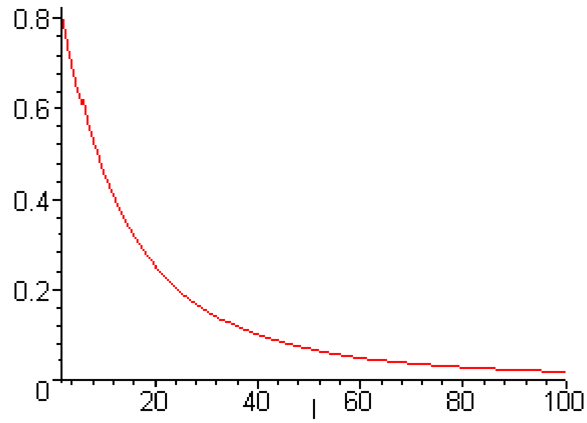

Figure 3d: $c = 0.8$, $\beta = 0.98$.

Figure 4. Autocorrelation function of x_i; R-LG-EXACD(1,1).

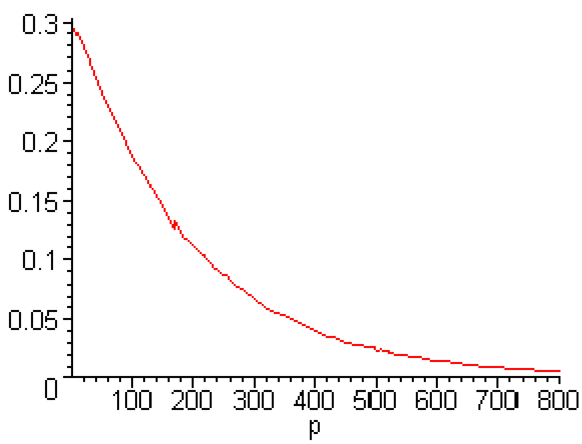

Figure 4a: $c = 0.05$, $\beta = 0.995$.

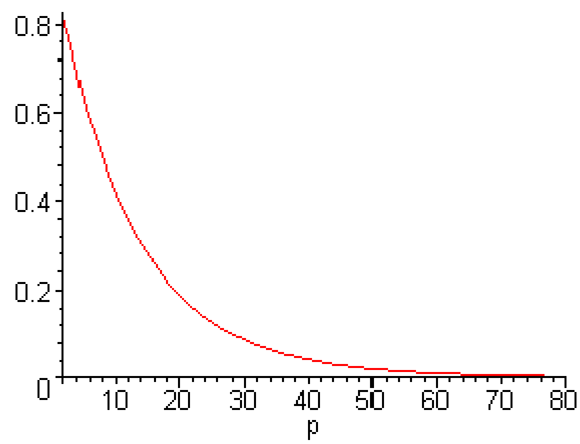

Figure 4b: $c = 0.8$, $\beta = 0.995$.

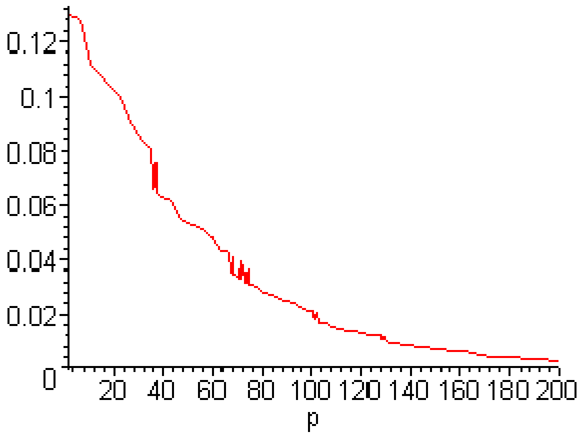

Figure 4c: $c = 0.05$, $\beta = 0.98$.

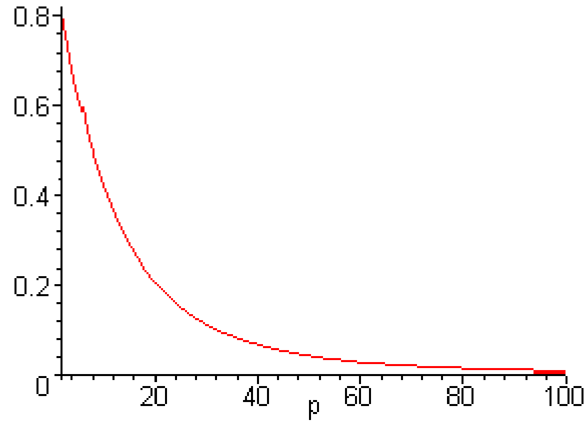

Figure 4d: $c = 0.8$, $\beta = 0.98$.

Figure 5. Autocorrelation function of x_i; E-LG-EXACD(1,1).

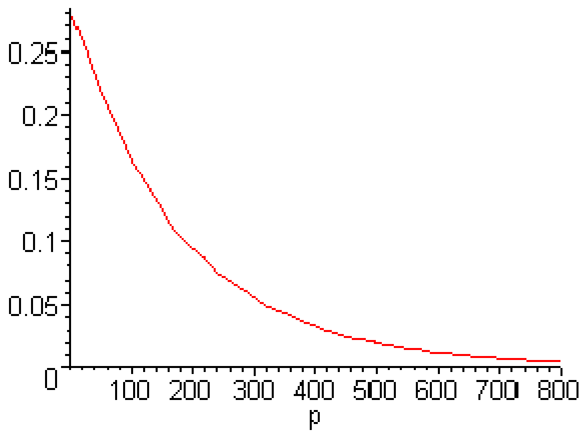

Figure 5a: $c = 0.05$, $\beta = 0.995$.

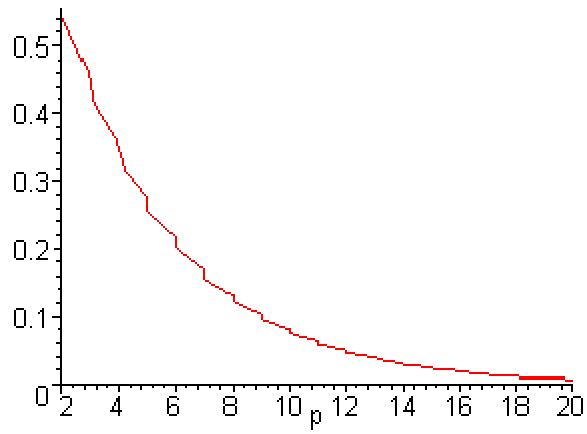

Figure 5b: $c = 0.8$, $\beta = 0.995$.

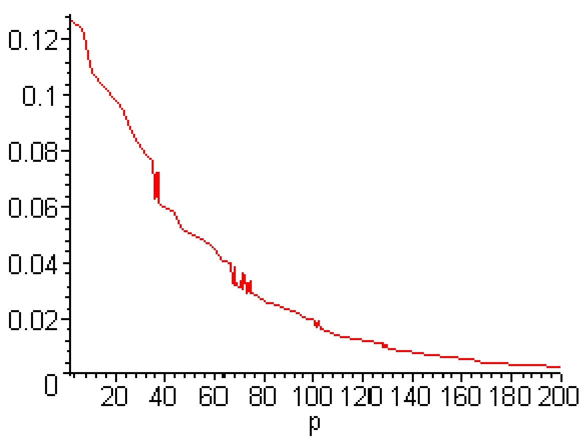

Figure 5c: $c = 0.05$, $\beta = 0.98$.

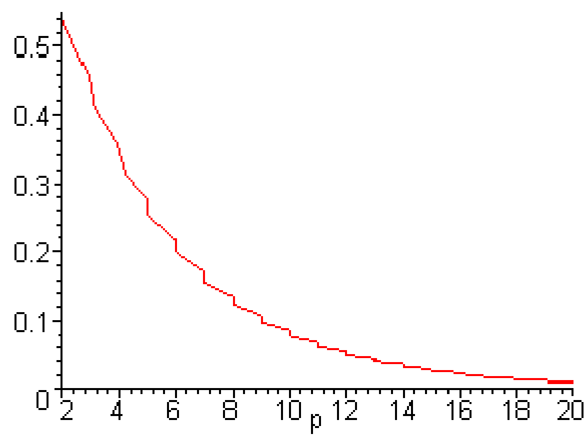

Figure 5d: $c = 0.8$, $\beta = 0.98$.

Figure 6. Autocorrelation function of x_i; R-EXACD(1,1).

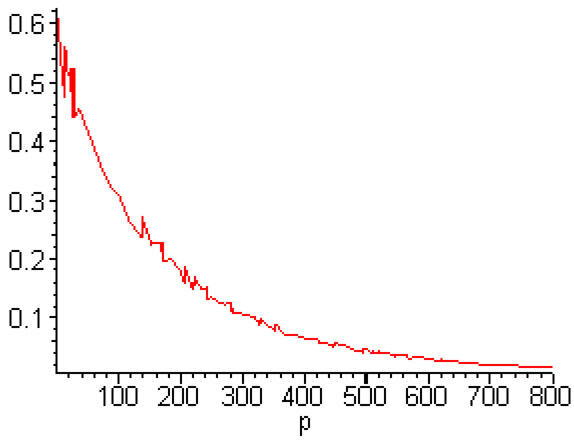

Figure 6a: $c = 0.05$, $\beta = 0.995$.

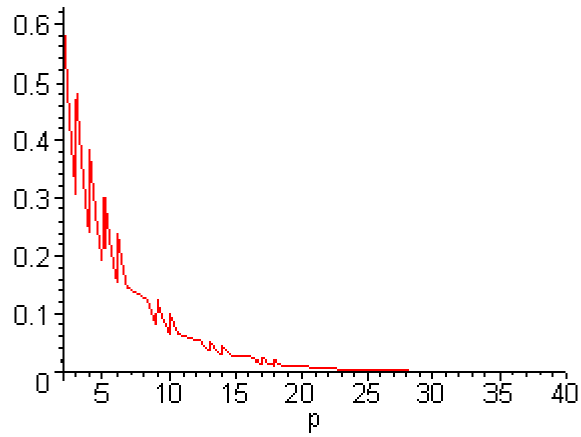

Figure 6b: $c = 0.3$, $\beta = 0.995$.

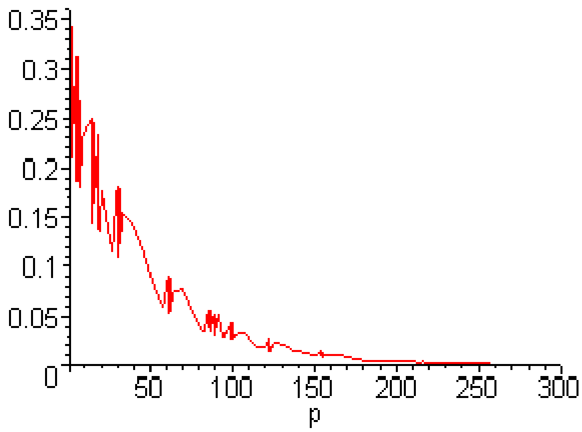

Figure 6c: $c = 0.05$, $\beta = 0.98$.

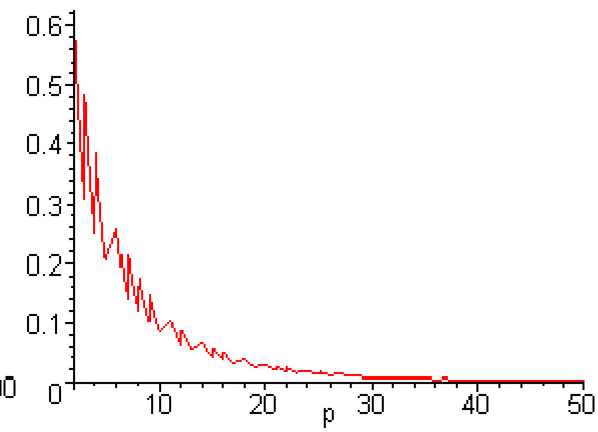

Figure 6d: $c = 0.3$, $\beta = 0.98$.

Figure 7. Autocorrelation function of x_i; E-EXACD(1,1).

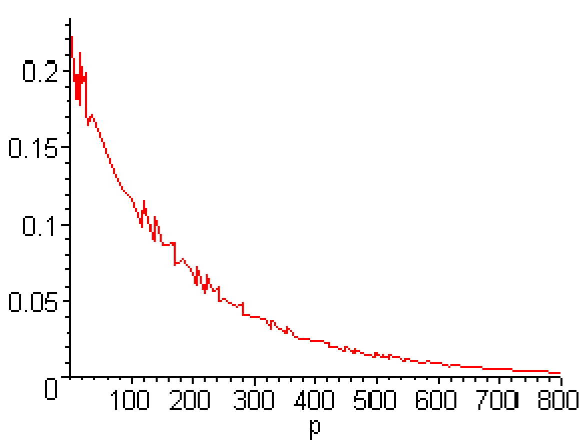

Figure 7a: $c = 0.05$, $\beta = 0.995$.

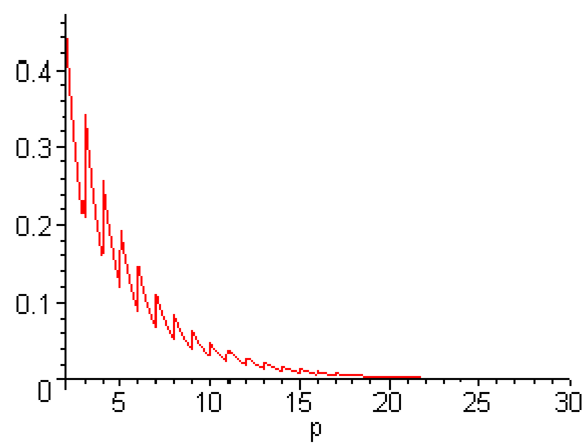

Figure 7b: $c = 0.4$, $\beta = 0.995$.

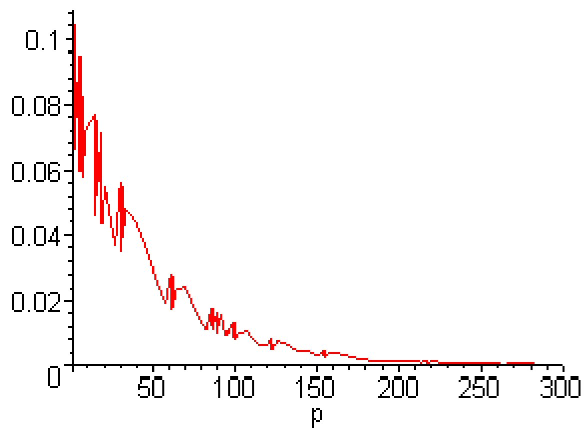

Figure 7c: $c = 0.05$, $\beta = 0.98$.

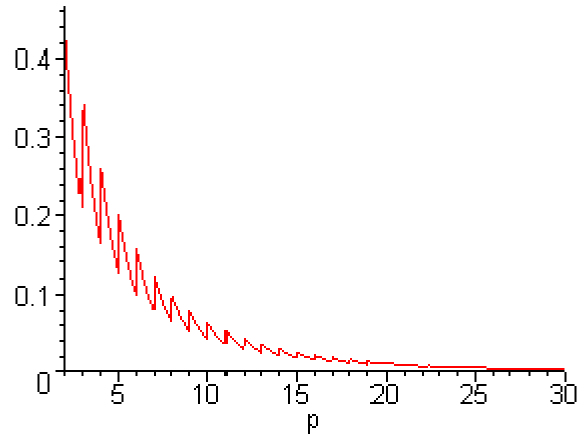

Figure 7d: $c = 0.4$, $\beta = 0.98$.

A Appendix

Proof. [Lemma 1] From (3.1) we have that

$$x_i = \omega + (1-L)^{-d}\left[\prod_{j=1}^{n}(1-\lambda_j L)\right]^{-1} C(L)\eta_i,$$

where the operator $(1-\gamma L)^{-d}$ is defined as

$$(1-\gamma L)^{-d} \equiv \sum_{j=0}^{\infty}\binom{-d}{j}(-\gamma)^j L^j$$

Hence, on account of

$$\prod_{j=1}^{n}(1-\lambda_j L) = \sum_{j=1}^{n}\frac{\lambda_j^{n-1}}{\prod_{l=1,l\neq j}^{n}(\lambda_j-\lambda_l)(1-\lambda_j L)},$$

and

$$(1-\lambda_j L)^{-1}C(L) = \sum_{f=0}^{\min\{l,m\}}\sum_{l=0}^{\infty}\lambda_j^{l-f}(-c_f)L^l,$$

we obtain (3.2a). ■

Proof. [Lemma 2] Rewriting equation (3.2a) we have

$$x_i = \omega + \sum_{j=0}^{i}\omega_j\eta_{i-j}, \tag{A.1}$$

where ω_j is defined in (3.2b).

From (A.1) it follows that

$$\mathsf{Var}(x_i) = \left(\sum_{j=0}^{\infty}\omega_j^2\right)\mathsf{E}(\eta_i^2) \tag{A.2}$$

Using the fact that

$$\mathsf{E}(\eta_i^2) = \mathsf{E}(x_i^2)\left[1-\frac{1}{\mathsf{E}(\varepsilon_i^2)}\right],$$

we obtain the second moment of the duration

$$\mathsf{E}\left(x_i^2\right) = \frac{[\mathsf{E}\,(x_i)]^2}{1-\left[1-\frac{1}{\mathsf{E}(\varepsilon_i^2)}\right]\left(\sum_{j=0}^{\infty}\omega_j^2\right)}$$

■

Proof. [Theorem 1] Applying (3.2b) yields

$$\rho_k(x_i) = \frac{\sum_{j=0}^{\infty}\omega_j\omega_{j+k}}{\sum_{j=0}^{\infty}\omega_j^2} = \frac{\sum_{r=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{r,k-j}v_j}{\sum_{r=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi_{r,-j}v_j},$$

where

$$\phi_{r,k-j} \equiv \binom{-d}{|k-j|+r}\binom{-d}{r}(-1)^{|k-j|},$$
$$v_j \equiv \sum_{l=1}^{n}\sum_{s=1}^{n}\sum_{f=0}^{\infty}\lambda_l^{+}\lambda_s^{+}\pi_{lf}\pi_{s,f+|j|},$$

with

$$\pi_{lf} \equiv \sum_{r=0}^{\min\{f,m\}} \lambda_l^{f-r}(-c_r),$$
$$\lambda_l^{+} \equiv \frac{\lambda_l^{n-1}}{\prod_{r=1,r\neq l}^{n}(\lambda_l-\lambda_r)}$$

But since

$$\sum_{r=0}^{\infty}\phi_{r,k-j} \equiv \frac{\Gamma(1-2d)\Gamma(d+|k-j|)}{\Gamma(d)\Gamma(1-d)\Gamma(1-d+|k-j|)}, \quad \text{and} \quad \sum_{r=1}^{n}\frac{\lambda_l^{+}\lambda_r^{+}}{1-\lambda_l\lambda_r} \equiv \overline{\lambda}_l,$$

it follows that

$$\begin{aligned}
\rho_k(x_i) &= \frac{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|k-j|)}{\Gamma(1-d+|k-j|)}\sum_{r=1}^{n}\sum_{l=0}^{m}\Psi_l\overline{\lambda}_r(\mathbf{1}_l\lambda_r^{||j|-l|}+\lambda_r^{|j|+l})}{\sum_{j=-\infty}^{\infty}\frac{\Gamma(d+|-j|)}{\Gamma(1-d+|-j|)}\sum_{r=1}^{n}\sum_{l=0}^{m}\Psi_l\overline{\lambda}_r(\mathbf{1}_l\lambda_r^{||j|-l|}+\lambda_r^{|j|+l})}\\
&= \left\{\sum_{l=0}^{m}\sum_{j=0}^{\infty}\Psi_l\sum_{r=1}^{n}\overline{\lambda}_r\left[\frac{\mathbf{1}_l\Gamma(d+|k-l-j|)}{\Gamma(1-d+|k-l-j|)}+\frac{\Gamma(d+k+l+j)}{\Gamma(1-d+k+l+j)}\right.\right.\\
&\quad\left.\left.+\left(\frac{\Gamma(d+|k-1+l-j|)}{\Gamma(1-d+|k-1+l-j|)}+\frac{\mathbf{1}_l\Gamma(d+|k+1-l+j|)}{\Gamma(1-d+|k+1-l+j|)}\right)\lambda_r\right]\lambda_r^j\right\}\\
&\quad\times\left\{\sum_{l=0}^{m}\sum_{j=0}^{\infty}\Psi_l\sum_{r=1}^{n}\overline{\lambda}_r\left[\frac{\mathbf{1}_l\Gamma(d+|-l-j|)}{\Gamma(1-d+|-l-j|)}+\frac{\Gamma(d+l+j)}{\Gamma(1-d+l+j)}\right.\right.\\
&\quad\left.\left.+\left(\frac{\Gamma(d+|-1+l-j|)}{\Gamma(1-d+|-1+l-j|)}+\frac{\mathbf{1}_l\Gamma(d+|1-l+j|)}{\Gamma(1-d+|1-l+j|)}\right)\lambda_r\right]\lambda_r^j\right\}^{-1},
\end{aligned}$$

where

$$\Psi_l \equiv \sum_{r=0}^{m-l}c_rc_{r+l} \quad (c_0 \equiv -1), \qquad \overline{\lambda}_l \equiv \frac{\lambda_l^{+}}{\prod_{r=1}^{n}(1-\lambda_l\lambda_r)}$$

Hence, upon observing that

$$\sum_{j=0}^{\infty}\frac{\Gamma(d+k+l+j)}{\Gamma(1-d+k+l+j)}\lambda_r^j = \frac{\Gamma(d+k+l)}{\Gamma(1-d+k+l)}F(d+k+l,1;1-d+k+l;\lambda_r),$$

(3.5) is obtained. ■

B Appendix

Proof. [Lemma 3] The logarithm of ψ_i in (3.7) can be expressed as an infinite distributed lag of $f(\varepsilon_i)$ terms:

$$\ln(\psi_i) = \frac{\omega}{B(1)} + \frac{C(L)}{B(L)} f(\varepsilon_i)$$

The above expression can be written as

$$\ln(\psi_i) = \frac{\omega}{B(1)} + \sum_{l=1}^{\infty} \delta_l f(\varepsilon_{i-l}), \tag{B.1}$$

where

$$\delta_l = \sum_{f=1}^{n} \zeta_f z_{fl},$$

with

$$\begin{aligned} \zeta_f &= \frac{\lambda_f^{n-1}}{\prod_{j=1, j \neq f}^{n} (\lambda_f - \lambda_j)}, \\ z_{fl} &\equiv \begin{cases} \sum_{j=0}^{l-1} c_{l-j} \lambda_f^j, & \text{if} \quad l \leq m, \\ z_{fm} \lambda_f^{l-m}, & \text{if} \quad l > m \end{cases} \end{aligned}$$

From (B.1) it follows that

$$\psi_i = e^{\frac{\omega}{B(1)}} \prod_{l=1}^{\infty} [e^{\delta_l f(\varepsilon_{i-l})}]$$

Raising both sides of the above equation to power μ and using the fact that $x_i^\mu = \psi_i^\mu \varepsilon_i^\mu$ yields (3.10). ■

Proof. [Lemma 4] Rewriting (3.10a) we have

$$x_i^\mu = e^{\frac{\mu\omega}{B(1)}} \varepsilon_i^\mu \times \prod_{l=1}^{\infty} [e^{\mu_l f(\varepsilon_{i-l})}], \tag{B.2a}$$

or

$$x_{i-k}^\mu = e^{\frac{\mu\omega}{B(1)}} \varepsilon_{i-k}^\mu \times \prod_{l=1}^{\infty} [e^{\mu_l f(\varepsilon_{i-k-l})}] \tag{B.2b}$$

Multiplying (B.2a) by (B.2b) and taking expectations yields

$$\begin{aligned} \mathsf{E}(x_i^\mu x_{i-k}^\mu) &= e^{\frac{2\mu\omega}{B(1)}} \mathsf{E}(\varepsilon_i^\mu) \mathsf{E}(\psi_i^\mu \varepsilon_{i-k}^\mu \psi_{i-k}^\mu) = \\ &= e^{\frac{2\mu\omega}{B(1)}} \mathsf{E}(\varepsilon_i^\mu) \times \prod_{l=1}^{k-1} [\mathsf{E}(e^{\mu_l f(\varepsilon_{i-l})})] \times \mathsf{E}(\varepsilon_{i-k}^\mu e^{\mu_k f(\varepsilon_{i-k})}) \times \prod_{l=1}^{\infty} [\mathsf{E}(e^{(\mu_{k+l} + \mu_l) f(\varepsilon_{i-k-l})})], \end{aligned}$$

where $\mu_l = \mu\delta_l$. Using the above expression and the fact that $\rho_k(x_i^\mu) = \frac{\mathsf{E}(x_i^\mu x_{i-k}^\mu) - [\mathsf{E}(x_i^\mu]^2}{\mathsf{E}(x_i^{2\mu}) - [\mathsf{E}(x_i^\mu)]^2}$ we obtain (3.11b). ■

Proof. [Theorem 2] Recall that for the LG-EXACD model, we have

$$f(\varepsilon_i) = \ln(\varepsilon_i) \tag{B.3}$$

In addition, the μth moment for the GF distribution is

$$\mathsf{E}(\varepsilon_i^{\mu}) = \frac{B(p+\frac{\mu}{a}, q-\frac{\mu}{a})B(p,q)^{\mu-1}}{B\left(p+\frac{1}{a}, q-\frac{1}{a}\right)^{\mu}} \quad \forall\ i, \tag{B.4}$$

where p, q and a are the parameters of the GF distribution. Inserting (B.3) and (B.4) into (3.11) yields (3.12). ∎

Proof. [Theorem 3] When the innovations $\{\varepsilon_i\}$ are drawn from the GG distribution, we have

$$\mathsf{E}(\varepsilon_i^{\mu}) = \frac{\Gamma(p+\frac{\mu}{a})\Gamma(p)^{\mu-1}}{\Gamma\left(p+\frac{1}{a}\right)^{\mu}} \quad \forall\ i, \tag{B.5}$$

where p and a are the parameters of the GG distribution. Inserting (B.3) and (B.5) into (3.11) yields (3.13). ∎

Proof. [Theorem 4] Recall that for the GG-EXACD model, we have

$$f(\varepsilon_i) = \varepsilon_i^a, \tag{B.6}$$

where a is one of the parameters of the GG distribution. Further, by direct computation we obtain

$$\mathsf{E}(\varepsilon_i^{\mu} e^{\mu_k \varepsilon_i^a}) = \frac{\Gamma(p+\frac{\mu}{a})[\Gamma(p)]^{(\mu-1)}\left[\Gamma\left(p+\frac{1}{a}\right)\right]^{pa}}{\left\{\left[\Gamma\left(p+\frac{1}{a}\right)\right]^{a} - \mu_k[\Gamma(p)]^a\right\}^{(p+\frac{\mu}{a})}} \quad \forall\ i \tag{B.7}$$

Note that (B.7), when $\mu_k = 0$, gives (B.5). Inserting (B.5)-(B.7) into (3.11) yields (3.14). ∎