Comparison Principles for State-Constrained Differential Inequalities with Applications to Time-Optimal Control

Seung-Jean Kim, Dong-Su Choi, and In-Joong Ha

School of Electrical Engineering

Seoul National University

San 56-1 Shinlim-Dong, Kwanak-Ku, Seoul 151-742

Korea

E-mail: ijha@asri.snu.ac.kr

Abstract:-In this paper, we develop some novel comparison principles that characterize maximal solutions of state-constrained differential inequalites. Then, we attempt to explicitly characterize the solution to the time-optimal control problem for a class of state-constrained second-order systems including robotic manipulators with geometric path constraints and single -degree-of-freedom (DOF) mechanical systems with friction. On the basis of the novel comparison principles, we show that the time-optimal trajectory is uniquely determined by two curves: the forward velocity limitation curve and the backward velocity limitation curve. These two curves can be constructed by solving two scalar ordinary differential equations. Finally, the method developed for solving the time-optimal control problem works regardless of the presence of the singular points and/or arcs and, moreover, works even when there exist an infinite number of switching points.

Key-Words : Optimal Control, Minimum Time, Singular Control, Robotic Manipulators, Differential Inequalities

1 Introduction

The problem of transferring the state of a given system from one state to another in minimum time is known as the *time-optimal control problem*, and it is one of the basic concerns of optimal control theory. During the past forty years, a series of fundamental results has been obtained by applications of the Pontryagin's maximum principle (PMP) [1] to time-optimal control of finite dimensional linear systems and low-order nonlinear systems [2]. As far as second-order systems without state constraints are concerned, extensive results on the structure and structural stability of time-optimal trajectories are now available in the literature [3]. On the contrary, in the case of state-constrained second-order systems, it is extremely difficult except for some special cases even to check whether the solution of the two-point boundary value problem (TPBVP) resulting from the PMP is indeed time-optimal.

Until now, differential and integral inequalities have been central in the study of uniqueness and asymptotic behavior of differential equation solutions, Lyapunov stability, and so on; See the vast literature in [10]. The main contribution of this paper is to show that some comparison principles for differential inequalities are useful to solve the time-optimal control problem for a class of state-constrained second-order systems including robotic manipulators with geometric path constraints [4], and single-degree-of-freedom (DOF) mechanical systems with friction [5]. To do so, we develop some novel comparison principles for state-constrained differential inequalites, which may be of independent interest. On the basis of the novel comparison principles, we then attempt to explicitly characterize the solution to the time-optimal control of state-constrained second-order systems. Specifically speaking, we show that the time-optimal trajectory is uniquely determined by two curves: the forward velocity limitation curve and the backward velocity limitation curve. These two curves can be constructed by solving two scalar ordinary differential equations. Finally, the method developed for solving the time-optimal control problem works regardless of the presence of the singular points and/or arcs and, moreover, works even when there exist an infinite number of switching points.

2 Comparison Principles

Consider the following two ordinary differential equations

$$v'(x) = h(x, v(x)) \in R^m,\ x \geq x_0,\ v(x_0) = v_0 \quad (1)$$
$$v'(x) = h(x, v(x)) \in R^m,\ x \leq x_f,\ v(x_f) = v_f \quad (2)$$

where $h : R^2 \to R$. Here and elsewhere, a. e. stands for "almost everywhere" and the derivative of a function $g : R^m \to R$ with respect to x will be denoted by g'. In our development, if the ordinary differential equation in (1) has the unique *solution* or *trajectory* in the classical sense [8], then it is denoted by $v_F(\cdot; x_0, v_0, h)$ in order to recall its dependence on the function h and the initial condition $v(x_0) = v_0$. On the other hand, if the ordinary differential equation in (2) has the unique solution or trajectory in the classical sense, then it is denoted by $v_B(\cdot; x_f, v_f, h)$.

We now clarify the concept of maximal solutions of differential inequalities. Consider the following differential inequality, defined on a subinterval $[x_0,\ x_f]$ of R.

$$\begin{aligned} F(x, v(x), v'(x)) &\leq 0, \text{ a. e. on } [x_0,\ x_f] \subset R & (3)\\ G(x, v(x)) &\leq 0, \ \forall x \in [x_0,\ x_f] & (4)\\ H(v(x_0), v(x_f)) &= 0 & (5) \end{aligned}$$

where F, G, and H are functions defined appropriately. A function $v : [x_0,\ x_f] \to R$ is said to be a *solution* of the above differential inequality, if it is absolutely continuous and satisfies (3)-(5). Let $\mathcal{S}$ denote the set of solutions of the differential inequalities in (3)-(5). Then, the differential inequality in (3)-(5) is said to have a *maximal solution*, if there exists a function $v^* \in \mathcal{S}$ such that

$$v \in \mathcal{S} \ \Rightarrow \ v(x) \ \leq \ v^*(x), \quad \forall x \in [x_0,\ x_f]. \qquad (6)$$

If the maximal solution exists, then it must be unique.

Consider the following two state-constrained differential inequalities.

$$\Sigma : \quad \begin{cases} v'(x) \leq f(x, v(x)), \text{a. e. on } [x_0,\ x_f] \\ \qquad\qquad \text{with } v(x_0) = v_0 \leq g(x_0) \\ v(x) \leq g(x),\ \forall x \in [x_0,\ x_f] \end{cases} \qquad (7)$$

$$\Omega : \quad \begin{cases} v'(x) \geq f(x, v(x)), \text{a. e. on } [x_0,\ x_f] \\ \qquad\qquad \text{with } v(x_f) = v_f \leq g(x_f) \\ v(x) \leq g(x),\ \forall x \in [x_0,\ x_f] \end{cases} \qquad (8)$$

where $f : R^2 \to R$ and $g : R \to R \cup \{\infty\}$. Note that the above two differential inequalities can take the form in (3)-(5) with functions F, G, and H defined approriately.

The well-known comparison principle [6] characterizes the maximal solution of the differential inequality in Σ when $g(x) \equiv \infty$. In what follows, we attempt to extend the comparison principle to the case of $g(x) < \infty$. Henceforth, we assume that g is a function from R into R. For given two functions $f : R^2 \to R$ and $g : R \to R$, we define two functions $\Sigma_{f,g} : R \times R \to R$ and $\Omega_{f,g} : R \times R \to R$ as follows.

$$\Sigma_{f,g}(x, v) \ \triangleq \ \begin{cases} f(x, v), & (x, v) \in A \\ \min\{f(x, v),\ g'(x)\}, & (x, v) \notin A \end{cases} \qquad (9)$$

$$\Omega_{f,g}(x, v) \ \triangleq \ \begin{cases} f(x, v), & (x, v) \in A \\ \max\{f(x, v),\ g'(x)\}, & (x, v) \notin A \end{cases} \qquad (10)$$

$$A \ \triangleq \ \{(x, v) \ : \ x \in R,\ v < g(x)\} \qquad (11)$$

As will be shown below, the maximal solutions of the differential inequalities in Σ and Ω are given, respectively, by the solutions of the scalar ordinary differential equations

$$\begin{aligned} v'(x) = \Sigma_{f,g}(x, v(x)), &\quad \text{a. e. on } x \geq x_0 & (12)\\ &\quad \text{with } v(x_0) = v_0, \\ v'(x) = \Omega_{f,g}(x, v(x)), &\quad \text{a. e. on } x \leq x_f & (13)\\ &\quad \text{with } v(x_f) = v_f. \end{aligned}$$

The following theorem states that, under quite natural assumptions on the functions f and g, each of the differential equations in (12) and (13) has the unique solution in classical sense, even though the functions $\Sigma_{f,g}$ and $\Omega_{f,g}$ are not necessarily continuous at all point $(x, v) \in R^2$.

Theorem 1 Suppose that the function f is piecewise continuous with respect to the first argument and locally Lipschitz with respect to the second argument and that the function g is piecewise continuously differentiable. Then, the differential equation in (12) (respectively, (13)) has the unique solution in the classical sense, that is $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ (respectively, $v_B(\cdot; x_f, v_f, \Omega_{f,g})$) is well defined. ■

The proof is given in Appendix A.

The following theorem can be viewed as a natural extension of the well-known comparison principle [6] for the differential inequalities without state constraints to those with state constraints in Σ and Ω.

Theorem 2 Suppose that the hypotheses of Theorem 1 are satisfied. Suppose further that there is no finite escape phenomenon over the interval $[x_0,\ x_f]$ in each of the the differential equations in (12) and (13). Then, the maximal solutions of the state-constrained differential inequalities in Σ and Ω are given, respectively, by $v_F(\cdot; x_0, v_0, \Sigma_{f,g})$ and $v_B(\cdot; x_f, v_f, \Omega_{f,g})$. ■

The proof is given in Appendix B.

3 An Application to Time-Optimal Control

Consider the second-order system

$$\ddot{x} = u \qquad (14)$$

subject to the control input constraint:

$$u_m(x, \dot{x}) \ \leq \ u \ \leq \ u_M(x, \dot{x}). \qquad (15)$$

Here and elsewhere, the function $\dot{f}$ denotes the time derivative of f.

We denote by $\mathcal{X}$ the set of all trajectories $(\tilde{x}, \dot{\tilde{x}}) : [0,\ \infty) \to R$ satisfying the constraint

$$u_m(\tilde{x}(t), \dot{\tilde{x}}(t)) \leq \ddot{\tilde{x}}(t) \ \leq \ u_M(\tilde{x}(t), \dot{\tilde{x}}(t)), \text{a. e. on } t \geq 0. \qquad (16)$$

along with the initial condition

$$(\tilde{x}(0), \dot{\tilde{x}}(0)) = (x_0, 0). \tag{17}$$

For each trajectory $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{X}$, we denote $t_f(\tilde{x}, \dot{\tilde{x}})$ by the traversal time of $(\tilde{x}, \dot{\tilde{x}})$ from the initial state $(x_0, 0)$ to the final state $(x_f, 0)$, that is,

$$t_f(\tilde{x}, \dot{\tilde{x}}) = \inf\{t > 0 \; : \; (\tilde{x}(t), \dot{\tilde{x}}(t)) = (x_f, 0)\}. \tag{18}$$

Here, we set $t_f(\tilde{x}, \dot{\tilde{x}}) = \infty$, if the trajectory $(\tilde{x}, \dot{\tilde{x}})$ does not arrive at $(x_f, 0)$ within a finite time. Then, we define the subset $\mathcal{X}_f$ of $\mathcal{X}$ as the set of the trajectories reaching $(x_f, 0)$ within a finite time. In what follows, we only consider the case of

$$x_0 < x_f \tag{19}$$

since the other case of $x_0 > x_f$ can be transformed into $-x_0 < -x_f$ via the transform $s : x \to -x$. Then, we impose the following state constraint on the system in (14).

$$0 \leq \dot{\tilde{x}}(t) \leq \alpha(\tilde{x}(t)), \quad \forall t \geq 0 \tag{20}$$

Here, we assume that the function $\alpha : R \to R$ is piecewise continuously differentiable and satisfies that

$$\alpha(x) > 0, \quad \forall x \in R. \tag{21}$$

In view of (19), it is natural to assume that the admissible velocity is always non-negative. Besides, the constraints in (20) requires that the admissible velocity is bounded above.

In our development, we make the following assumption.

(A.1) The functions u_m and u_M are locally Lipschitz and satisfy that

$$u_m(x, \dot{x}) < u_M(x, \dot{x}), \quad \text{if} \;\; 0 \leq \dot{x} < \alpha(x) \tag{22}$$

$$u_m(x, 0) < 0 < u_M(x, 0), \quad \forall x \in [x_0, \; x_f]. \tag{23}$$

The inequality in (22) is quite natural. On the other hand, in the case of single-DOF mechanical systems, the inequality in (23) simply means that the mass can stand still at any position between $x = x_0$ and $x = x_f$.

The time-opimal control problem we attempt to solve can be stated as follows:

$$(\mathrm{P}) : (x^*, \dot{x}^*) \;=\; \arg \min_{(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}} \; t_f(\tilde{x}, \dot{\tilde{x}}) \tag{24}$$

where the subset $\mathcal{P}$ consists of the trajectories $(\tilde{x}, \dot{\tilde{x}})$ in $\mathcal{X}_f$ satisfying the state constraint in (20). In what follows, the trajectory $(x^*, \dot{x}^*)$ is called the time-optimal trajectory or solution.

We begin by establishing an important property of the time-optimal trajectory.

Lemma 1 Let $\hat{x} \in \mathcal{P}$ and suppose that a trajectory $(\hat{x}, \dot{\hat{x}})$ has an intermediate zero-velocity point before arriving at the final state $(x_f, 0)$. Then, the trajectory $(\hat{x}, \dot{\hat{x}})$ is not time-optimal. ■

The proof is omitted because of limited space.

As the direct consequence of Lemma 1, the time-optimal control problem (P) is reduced to the following:

$$(\bar{\mathrm{P}}) : (x^*, \dot{x}^*) \;=\; \arg \min_{(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}_n} \; t_f(\tilde{x}, \dot{\tilde{x}}) \tag{25}$$

where $\mathcal{P}_n$ denotes the set of time functions $(\tilde{x}, \dot{\tilde{x}}) \in \mathcal{P}$ with no intermediate zero-velocity points before arriving at the final state $(x_f, 0)$.

To present the time-optimal solution, we need to develop a comparison principle for the following differential inequality.

$$\Lambda : \begin{cases} a_m(x, v(x)) \leq v'(x) \leq a_M(x, v(x)), \\ \text{a. e. on } [x_0, \; x_f] \\ 0 < v(x) \leq \alpha(x) \;, \; \forall x \in (x_0, \; x_f) \\ v(x_0) = v(x_f) = 0 \end{cases} \tag{26}$$

where the functions $a_m : R \times R \to R$ and $a_M : R \times R \to R$ are defined as

$$a_m(x, v) \triangleq \frac{u_m(x, v)}{v}, \quad a_M(x, v) \triangleq \frac{u_M(x, v)}{v}. \tag{27}$$

The following theorem states the comparison principle for the differential inequality in Λ.

Theorem 3 The differential inequality in Λ has the maximal solution v^* given by

$$v^*(x) \triangleq \min\{v_F(x; x_0, 0, \Sigma_{a_M, \alpha}), \; v_B(x; x_f, 0, \Omega_{a_m, \alpha})\}, \quad \forall x \in [x_0, \; x_f]. \tag{28}$$

■

We now clarify the relationship between the time-optimal solution $(x^*, \dot{x}^*)$ and the maximal solution v^* of the differential inequality in Λ.

Theorem 4 The time-optimal trajectory $(x^*, \dot{x}^*)$ traverses in the $x - \dot{x}$ phase plane along the maximal solution v^* of the differential inequality in Λ with the minimum traversal time t_f^* given by

$$t_f^* = \int_{x_0}^{x_f} \frac{1}{v^*(x)} dx. \tag{29}$$

■

The proofs of Theorem 3 and Theorem 4 are omitted because of limited space.

4 Conclusion

In this paper, we have developed some variants of the conventional comparison principle for state-constrained differential inequalites. Then, on the basis of the novel comparison principles, we have characterized completely the solution to the time-optimal control problem for a class of state-constrained second-order systems.

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Appendix A. Proof of Theorem 1

The proof of Theorem 1 will require some notational convention about set-valued maps, which can be found in [9].

Define two set-valued maps $E_{f,g} : R^2 \to R$ and $F_{f,g} : R^2 \to R$ as follows:

$$E_{f,g}(x,v) \tag{30}$$

$$\triangleq \begin{cases} f(x,v), & \text{if } v < g(x) \\ \min\{f(x,g(x)),\ g'(x)\},\ f(x,g(x)), & \text{if } v = g(x) \\ \min\{f(x,v),\ g'(x)\}, & \text{if } v > g(x) \end{cases} \tag{31}$$

$$F_{f,g}(x,v) \tag{32}$$

$$\triangleq \begin{cases} f(x,v), & \text{if } v < g(x) \\ f(x,g(x)),\ \max\{f(x,g(x)),\ g'(x)\}, & \text{if } v = g(x) \\ \max\{f(x,v),\ g'(x)\}, & \text{if } v > g(x) \end{cases} \tag{33}$$

Here and elsewhere, a singleton $\{a\}$ is denoted by a by abuse of notation. Observe that the functions $\Sigma_{f,g}$ and $\Omega_{f,g}$ in (12) and (13) can be embedded, respectively, into the set-valued map $E_{f,g}$ and $F_{f,g}$ in the sense that

$$\Sigma_{f,g}(x,v) \in E_{f,g}(x,v),\quad \Omega_{f,g}(x,v) \in F_{f,g}(x,v),\quad \forall (x,v) \in R^2. \tag{34}$$

Then, we consider the following two differential inclusions:

$$v'(x) \in E_{f,g}(x,v(x)),\quad \text{a. e. on } x \geq x_0 \text{ with } v(x_0) = v_0 \tag{35}$$

$$v'(x) \in F_{f,g}(x,v(x)),\quad \text{a. e. on } x \leq x_f \text{ with } v(x_f) = v_f \tag{36}$$

To prove Theorem 1, we need to establish several lemmas.

Lemma 2 Suppose that the function f is continuous and the function g is continuously differentiable. the set-valued maps $E_{f,g}$ and $F_{f,g}$ are upper semicontinuous. ■

The proof is omitted because of limited space.

Lemma 3 Under the hypotheses of Lemma 2, there exists a function v defined on $[x_0,\ x_1)$ (x_1 could be ∞), which is absolutely continuous on each compact subset of $[x_0,\ x_1)$ and satisfies (35) a. e. on $[x_0,\ x_1)$.

Proof: Note from the definition of $E_{f,g}$ in (31) that at each $(x^*, v^*) \in R^2$, the set $E_{f,g}(x^*, v^*)$ is compact and convex. This along with Lemma 2 implies that all the hypotheses of Theorem 2.4 in [9], which states necessary conditions for the existence of solutions in differential inclusions, are satisfied. Hence, the assertion in this lemma is true. ■

Lemma 4 Under the hypotheses of Lemma 2, there exists a function v defined on $(x_2,\ x_f])$ (x_2 could be $-\infty$), which is absolutely continuous on each compact subset of $(x_2,\ x_f]$ and satisfies (36) a. e. on $(x_2,\ x_f]$.

Proof: Through some arguments similar to those used to prove Lemma 4, the proof can be done. ■

Lemma 5 Suppose that the hypotheses of Lemma 2 hold. Let v, defined on $[x_0,\ x_1)$, be any solution of the differential inclusion in (35) with $v(x_0) \leq g(x_0)$. Then,

$$v(x) \leq g(x),\quad \forall x \in [x_0,\ x_1) \tag{37}$$

On the other hand, let v, defined on $(x_2,\ x_f]$, be any solution of the differential inclusion in (36) with $v(x_f) \leq g(x_f)$. Then,

$$v(x) \leq g(x),\quad \forall x \in (x_2,\ x_f]. \tag{38}$$

Proof: We only present the proof for (37), since that for (38) is similar. Let v be any solution of the differential inclusion in (35) with $x_0 \leq g(x_0)$. Define the set A by

$$A \triangleq \{x_0 < x < x_1\ :\ v(x) > g(x)\}. \tag{39}$$

Suppose that the set A is non-empty. Then, the set A is open and, hence, consists of countably many disjoint open subintervals $\{(a_i,\ b_i)\ :\ i = 1, 2, \cdots\}$ of the interval $(x_0,\ x_1)$ [7] such that

$$v(a_i) = g(a_i), \quad v(b_i) = g(b_i), \quad i = 1, 2, \cdots. \tag{40}$$

Fix one of the subintervals, say, $(a_k,\ b_k)$ and define $\bar{x} \triangleq (a_k + b_k)/2$. Then, it holds that

$$v(\bar{x}) > g(\bar{x}). \tag{41}$$

Here, note from the definition of the set-valued map $E_{f,g}$ in (31) that

$$v'(x) = \min\{f(x, v(x)),\ g'(x)\} \le g'(x), \quad \forall x \in A. \tag{42}$$

This along with (40) implies that

$$\begin{aligned} v(\bar{x}) &= v(a_k) + \int_{a_k}^{\bar{x}} v'(s)ds \le v(a_k) + \int_{a_k}^{\bar{x}} g'(s)ds \\ &= g(a_k) + \int_{a_k}^{\bar{x}} g'(s)ds = g(\bar{x}). \end{aligned} \tag{43}$$

However, this is self-contradictory to the (41). Accordingly, we have shown that (37) holds. ■

Let h be a function from a closed subset D of R into R. Then, we define the set $L_h(a)$ by

$$L_h(a) \triangleq \{x \in R : h(x) = a\}, \quad a \in R. \tag{44}$$

The following two lemmas will be central to the proof of Theorem 1.

Lemma 6 Suppose that the function $h : D \to R$ is absolutely continuous. Then, the derivative h' is zero a. e. on each set $L_h(a)$, $a \in R$.

Proof: Let $a \in R$. Then, the set $L_h(a)$ is closed. It suffices to consider the case where the set $L_h(a)$ is uncountable. Then, the Cantor-Bendixon theorem [7] states that $L_h(a)$ contains a perfect subset E_p and a countable subset E_c such that

$$L_h(a) = E_p \cup E_c, \quad E_p \cap E_c = \emptyset. \tag{45}$$

On the other hand, it follows from the absolute continuity of the function h that there exist two subsets $A, B \subset L_h(a)$ such that h always has a derivative on A; B is a measure-zero set; and

$$L_h(a) = A \cup B, \quad A \cap B = \emptyset, \tag{46}$$

It then follows from (45) and (46) that

$$L_h(a) = E_p \cup E_c = (E_p \cap A) \cup (E_p \cap B) \cup E_c. \tag{47}$$

Here, it is clear that

$$E_p \cap B, \text{ and } E_c \text{ are sets of measure zero.} \tag{48}$$

As will be seen soon,

$$h'(x) = 0, \quad \forall x \in E_p \cap A. \tag{49}$$

Accordingly, the assertion in this lemma is true.

Finally, we show that (49) holds. Suppose that $x \in E_p \cap A$. By the definition of the perfect set, there exists a sequence of real numbers $x_n \in E_p$ satisfying $x_n \neq x$, $\forall n \in N$ and $\lim_{n\to\infty} x_n = x$. Hence, $h'(x)$ exists for all $x \in E_p \cap A$ and

$$h'(x) = \lim_{n\to\infty} \frac{h(x_n) - h(x)}{x_n - x} = \lim_{n\to\infty} \frac{a - a}{x_n - x} = 0, \quad \forall x \in E_p \cap A.$$

■

Lemma 7 Suppose that the function f is continuous with respect to the first argument and locally Lipschitz with respect to the second argument and the function g is continuously differentiable. Then, each of the differential equations in (12) and (13) has the unique solution.

Proof: We first present the proof of the assertion for the differential equations in (12). Let $v : [x_0,\ x_1) \to R$ be the solution of the differential inclusion in (35), whose existence is guaranteed by Lemma 3. It suffices to show that for any $\bar{x}$, $x_0 \le \bar{x} \le x_1$,

$$v'(x) = \Sigma_{f,g}(x, v(x)), \quad \text{a.e. on } [x_0,\ \bar{x}]. \tag{50}$$

Choose a point $\bar{x} \in (x_0,\ x_1)$ and define the function $h : [x_0,\ \bar{x}] \to R$ by

$$h(x) \triangleq v(x) - g(x). \tag{51}$$

Clearly, the funciton h is absolutely continuous.

Note that the closed interval $[x_0,\ \bar{x}]$ can be partitioned as follows.

$$[x_0,\ \bar{x}] = E_n \cup E_0, \tag{52}$$

where E_n and E_0 are the subsets of the closed interval $[x_0,\ \bar{x}]$ defined by

$$\begin{aligned} E_n &\triangleq \{x \in [x_0,\ \bar{x}] : h(x) \neq 0\}, \\ E_0 &\triangleq \{x \in [x_0,\ \bar{x}] : h(x) = 0\}. \end{aligned}$$

Here, it is easy to see from Lemma 5 that whenever $x \in E_n$, the set $E_{f,g}(x, v(x))$ is the singleton $\{\Sigma_{f,g}(x, v(x))\}$ and, therefore, that

$$v'(x) = \Sigma_{f,g}(x, v(x)), \quad \text{a. e. on } E_n. \tag{53}$$

On the other hand, note that the set E_0 is closed since v is continuous. Then, by Lemma 6, we can see that

$$v'(x) = g'(x), \quad \text{a. e. on } E_0. \tag{54}$$

This, then, implies that

$$g'(x) \in E_{f,g}(x, g(x)), \quad \text{a. e. on } E_0. \tag{55}$$

Note from the definition of the set-valued map $E_{f,g}$ in (31) that

$$g'(x) \le f(x, g(x)), \quad \text{a. e. on } E_0. \tag{56}$$

and from the definititon of the function $\Sigma_{f,g}$ in (9) that

$$\Sigma_{f,g}(x, g(x)) = g'(x), \quad \text{a. e. on } E_0. \tag{57}$$

This along with (54) implies that

$$v'(x) = \Sigma_{f,g}(x, g(x)), \quad \text{a. e. on } E_0. \tag{58}$$

Finally, we can see from (53) and (58) that the solution of any differential inclusion in (35) always satisfies the differential equation in (12).

We now turn to the unqueness of the solution of the differential equation in (12). Let v_1 and v_2 be two solutions of the differential equation in (12) and let $[x_0,\ x_1)$ be the common interval of existence of v_1 and v_2. Define two subsets M and N of the interval $(x_0,\ x_1)$ by

$$M \triangleq \{x_0 < x < x_1 \ : \ v_1(x) = g(x)\} \tag{59}$$

$$N \triangleq \{x_0 < x < x_1 \ : \ v_1(x) < g(x)\}. \tag{60}$$

Then, it is trivial to see from Lemma 5 that

$$v_2(x) \le v_1(x), \quad \forall x \in M. \tag{61}$$

Next, we show that

$$v_2(x) \le v_1(x), \quad \forall x \in N. \tag{62}$$

Observe that the set N consists of countably many disjoint open subintervals $\{(c_i,\ d_i) : i = 1, 2, \cdots\}$ of the interval $(x_0,\ x_f)$. Consequently, there exists a subinterval, say, $(c_k,\ d_k)$ such that

$$\bar{x} \in (c_k,\ d_k), \quad v_1(c_k) = g(c_k). \tag{63}$$

Note from the definition of the function $\Sigma_{f,g}$ in (9) that

$$v_1'(x) = f(x, v_1(x)), \quad \text{a. e. on } (c_k,\ d_k) \text{ with } v_1(c_k) = g(c_k). \tag{64}$$

Then, it holds that

$$v_2(c_k) \le v_1(c_k) = g(c_k). \tag{65}$$

On the other hand, it is easy to see from the definition of the function $\Sigma_{f,g}$ in (9) that

$$\Sigma_{f,g}(x, v) \le f(x, v), \quad \forall (x, v) \in R^2$$

and, hence, that the function v_2 satisfies the differential inequality on the interval $(c_k,\ d_k)$

$$v_2'(x) \le f(x, v_2(x)), \quad \text{a. e. on } (c_k,\ d_k). \tag{66}$$

This along with (65), (64), and the comparison principle [6] implies that

$$v_2(x) \le v_1(x), \quad \forall x \in [c_k,\ d_k). \tag{67}$$

So far, we have shown that

$$v_2(x) \le v_1(x), \quad \forall x \in [x_0,\ x_1). \tag{68}$$

Through some arguments similar to those used to show the above inequality, we can also show that

$$v_2(x) \ge v_1(x), \quad \forall x \in [x_0,\ x_1). \tag{69}$$

Hence, the two solutions are identical on the common interval of existence and, hence, the differential equation in (12) has the unque solution.

Finally, through some arguments similar to those used to show that the differential equation in (13) has the unique solution, it can be shown that the differential equation in (13) also has the unique solution. ■

We are now ready to prove Theorem 1

Proof of Theorem 1: From the piecewise continuity of the functions f and g, there exist a finite number of points $a_k \in [x_0,\ x_1]$, $k = 1, 2, \cdots, p$ with $a_1 = x_0 < a_2 < \cdots < a_{p-1} < a_p = x_1$ such that f is continuous with respect to the first argument and locally Lipschitz with respect to the second argument on each of these subintervals $(a_k,\ a_{k+1})$, $k = 1, 2, \cdots, p-1$, while the function g is continuously differentiable on each of these subintervals $(a_k,\ a_{k+1})$, $k = 1, 2, \cdots, p-1$. Then, it can be easily seen from Lemma 7 that the assertion of this theorem holds. ■

Appendix B. Proof of Theorem 2

We present only the proof for the differential inequality in Σ, since that for the differential inequality in Ω is similar. For notational brevity, we temporarily write

$$\overline{v}(x) \triangleq v_F(x; x_0, v_0, \Sigma_{f,g}). \tag{70}$$

Note that

$$v \le g(x) \ \Rightarrow \ \Sigma_{f,g}(x, v) \le f(x, v). \tag{71}$$

This along with Lemma 5 implies that the differential inequality in Σ is satisfied with $v = \overline{v}$.

Define two subsets B and C of the interval $(x_0,\ x_f)$ by

$$B \triangleq \{x_0 < x < x_f \ : \ \overline{v}(x) = g(x)\} \tag{72}$$

$$C \triangleq \{x_0 < x < x_f \ : \ \overline{v}(x) < g(x)\}. \tag{73}$$

Let a function $\tilde{v} : [x_0,\ x_f] \to R$ satisfy the differential inequality in Σ. Here, it is trivial to see that

$$\tilde{v}(x) \le \overline{v}(x), \quad \forall x \in B. \tag{74}$$

Through some arguments similar to those used to show that the solution of the differential equation in (12) has the unique solution, we can show that

$$\tilde{v}(x) \le \overline{v}(x), \quad \forall x \in C. \tag{75}$$

Finally, it follows from (74) and (75) that $\overline{v}$ is the maximal solution of the differential inequality in Σ. ■