THE $\mathcal{H}_\infty$ DISCRETE MODEL MATCHING PROBLEM by STATIC STATE FEEDBACK

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Abstract

Our aim in this paper is to develop a new approach for solving the discrete Model Matching Problem (MMP) by the static state feedback in the sense of $\mathcal{H}_\infty$ optimality criterion by using Linear Matrix Inequalities (LMIs). The main contribution of this study could briefly be explained as to reformulate the discrete MMP in the formulation of LMI and to present the solvability conditions of the problem, and to give a design procedure for the static state feedback control law that provides the best performance of the discrete MMP in the sense of the $\mathcal{H}_\infty$ norm. It might be noticed that the formulation developed here makes it possible to deal with irregular static state feedback case as well as regular case.

Key-Words: Model Matching Problem, Linear Matrix Inequalities, State-Space $\mathcal{H}_\infty$ Control, Static State Feedback.

1 Introduction

The Model Matching Problem is one of the most familiar problems in control theory. This problem is of practical importance since it consists in compensating a given system so that the resultant system has a prespecified (model) transfer function. This is a brief formulation of many problems of practical interest, including the design of a servo with desired closed-loop dynamics, the realization of a transfer function using copies or approximates of a given dynamic component and the design of model-following control systems.

The discrete $\mathcal{H}_\infty$ MMP is to find a controller transfer matrix $R(z)$ as a precompensator, with property of stable and causal rational matrix, that is $R(z) \in \mathcal{RH}_\infty$, that minimizes the $\mathcal{H}_\infty$ norm of $T_1(z) - T_2(z)R(z)$ such that the stable and proper rational matrices $T_1(z)$ and $T_2(z)$ are given as the model and the system transfer matrices respectively. The $\mathcal{H}_\infty$ norm of a transfer matrix is

$$\|T(z)\|_\infty = \sup_{\omega \in [0,2\pi]} \sigma_{max}(T(e^{j\omega})) \qquad (1)$$

This means that the closed-loop performance of the controlled system, namely $T_2(z)R(z)$ that can easily be established by the dynamic state feedback, approximates the desired performance in given as $T_1(z)$, in the sense of the following criterion,

$$\gamma_{opt} = \inf_{R(z) \in \mathcal{RH}_\infty} \|T_1(z) - T_2(z)R(z)\|_\infty \qquad (2)$$

In the literature, there are some results on the $\mathcal{H}_\infty$ MMP. Two of them are based on Nevanlinna-Pick Problem (NPP) [2] and Nehari Problem (NP) [4,5]. In these studies, the $\mathcal{H}_\infty$ MMP has been reduced to the one of these problems and then by using the results on the solution of NPP or NP, first the value γ_{opt} defined in (2) is found and then the controller transfer matrix are obtained in the form of stable and proper rational matrix. A state-space solution of the $\mathcal{H}_\infty$ MMP that is based on canonical spectral factorizations and solutions of the algebraic Riccati equations (ARE), is given in [8].

In all these studies, the controller transfer matrix $R(z)$ is considered in the form of a precompensator given in (2) and $R(z)$ is found as a stable and proper rational matrix. Since the precompensators in the form of a proper and stable rational matrix can generally be established by the dynamic state feedback in the feedback configuration [9], none of these results on the standard $\mathcal{H}_\infty$ MMP cannot directly be used to solve the $\mathcal{H}_\infty$ MMP by the static state feedback.

In this study, a special formulation is developed to solve the discrete $\mathcal{H}_\infty$ MMP by the static state feedback. This formulation enables us to use the methods and results presented for the solution of the standard $\mathcal{H}_\infty$ OCP and so the problem is completely solved by the LMI-based parameterization. It should be noted that, in this paper, the static state feedback law that solves the discrete $\mathcal{H}_\infty$ MMP has not been limited to be regular, i.e. the control law can be obtained as an irregular static state feedback, if the optimal performance requires.

The paper is organized in the following way: In Chapter 2, we introduce some necessary preliminary background. In Chapter 3, we present some special formulation for the discrete MMP by the static state feedback in formulation of LMI. In Chapter 4, main result is given in Theorem 4.3 with including existence conditions of the problem. In Chapter 5, we construct the static state feedback controller by using Theorem 4.3 and the some conclusions are given in Chapter 6.

The following notation will be used throughout the paper: $KerM$ and ImM for the null space and range of the linear operator associated with M respectively and N^* for the transpose conjugate of N matrix. Moreover $P > 0$ denotes that P matrix is positive definite.

2 Preliminaries and Problem Statement

Consider a causal discrete Linear Time-Invariant (LTI) generalized plant $P(z)$ described by the equation,

$$\begin{aligned} \underline{x}(k+1) &= \underline{A}\underline{x}(k) + B_1 w(k) + B_2 u(k) \\ z(k) &= C_1 \underline{x}(k) + D_{11} w(k) + D_{12} u(k) \quad (3) \\ y(k) &= C_2 \underline{x}(k) + D_{21} w(k) + D_{22} u(k) \end{aligned}$$

where $\underline{x}(k) \in \mathcal{R}^n$ is the state vector, $u(k) \in \mathcal{R}^{m_2}$ is the vector of control inputs, $w(k) \in \mathcal{R}^{m_1}$ is the vector of exogenous inputs, i.e. reference signals, disturbance signals, sensor noise, etc., $y(k) \in \mathcal{R}^{p_2}$ is the vector of measurements and $z(k) \in \mathcal{R}^{p_1}$ is the vector of output signals described to the performance of control system. The closed-loop system with the controller $K(z)$ is shown as in Figure 1:

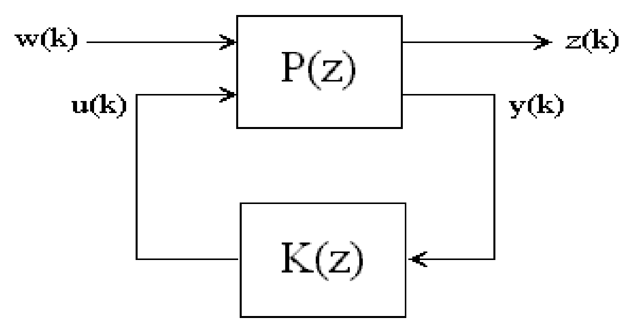

Figure 1

It is obvious that the plant $P(z)$ in Figure 1 can be given in the following form,

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zI - \underline{A})^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \quad (4)$$

about the state-space model of the plant and the transfer matrix from $w(k)$ to $z(k)$ in Figure 1 can be written as,

$$T_{zw}(z) = P_{11}(z) + P_{12}(z)K(z). (I - P_{22}(z)K(z))^{-1} P_{21}(z) \quad (5)$$

The discrete $\mathcal{H}_\infty$ OCP is to find all admissible controllers $K(z)$ such that $\|T_{zw}(z)\|_\infty$ is minimized. The following Lemma is known as **The Bounded Real Lemma** for discrete-time systems and can be used to turn the discrete $\mathcal{H}_\infty$ OCP into an LMI:

Lemma 2.1 *Consider a discrete-time transfer matrix $T(z)$ of (not necessarily minimal) realization $T(z) = D + C(zI - A)^{-1}B$. The following statements are equivalent:*
i) $\|D + C(zI - A)^{-1}B\|_\infty < \gamma$ *and A is Hurwitz in*

the discrete-time sense;
ii) *there exists a solution $X > 0$ to the LMI:*

$$\begin{bmatrix} -X^{-1} & A & B & 0 \\ A^* & -X & 0 & C^* \\ B^* & 0 & -\gamma I & D^* \\ 0 & C & D & -\gamma I \end{bmatrix} < 0. \quad (6)$$

Proof: [3]. □

3 Discrete Model Matching Problem in Formulation of LMI

In order to solve the discrete $\mathcal{H}_\infty$ MMP via LMI approach, the problem should be reformulated as a standard $\mathcal{H}_\infty$ OCP in state-space equations. For this aim, we shall consider a minimal realizations (A, B, C, D) of $T_2(z)$, namely controlled system, and (F, G, H, J) of $T_1(z)$, namely model system, so the state-space equations of these systems can be given as follows,

$$T_1(z): \quad \begin{aligned} q(k+1) &= Fq(k) + Gw(k) \\ y_1(k) &= Hq(k) + Jw(k) \end{aligned}$$

$$T_2(z): \quad \begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y_2(k) &= Cx(k) + Du(k) \end{aligned}$$

and the control input $u(k)$ can be generated by the static state feedback controller $K = \begin{bmatrix} L & M \end{bmatrix}$ such that:

$$\begin{aligned} u(k) &= Ky(k) = \begin{bmatrix} L & M \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ &= Lx(k) + Mw(k) \end{aligned} \quad (7)$$

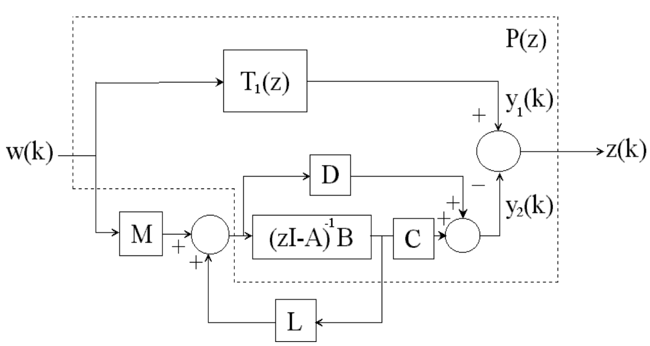

Figure 2

It must be noted that the static state feedback is said to be **regular**, if M is nonsingular and it is said to be **irregular** if M is singular. It is easily seen as in Figure 2 that discrete $\mathcal{H}_\infty$ MMP can be considered as in the discrete $\mathcal{H}_\infty$ OCP with $\|T_{zw}(z)\|_\infty < \gamma$.

We propose a plant $P(z)$ described by,

$$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} w(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \quad (8)$$

$$z(k) = y_1(k) - y_2(k) = \begin{bmatrix} -C & H \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + Jw(k) - Du(k) \quad (9)$$

$$y(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w(k) \quad (10)$$

where $q(k) \in \mathcal{R}^{n_1}$ and $x(k) \in \mathcal{R}^{n_2}$ are the state vectors of the model and system given in $T_1(z)$ and $T_2(z)$, respectively.

As a result of above analysis, the following Remark can be given:

Remark: *The discrete $\mathcal{H}_\infty$ MMP for the model and system given in $T_1(z)$ and $T_2(z)$, respectively, is equivalent to the discrete $\mathcal{H}_\infty$ OCP for the plant described by equation (8), (9) and (10). Figure 3 shows this idea:*

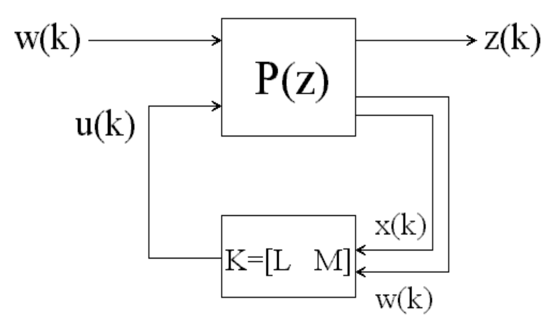

Figure 3

We have assumed $D_{22} = 0$ because of removing more complicated formulas. So, a realization (not necessarily minimal) of the closed-loop transfer matrix from $w(k)$ to $z(k)$ is obtained as

$$T_{zw}(z) = D_{cl} + C_{cl}(zI - A_{cl})^{-1} B_{cl} \quad (11)$$

where

$$
\begin{aligned}
A_{cl} &= \underline{A} + B_2 K C_2 && (12)\\
B_{cl} &= B_1 + B_2 K D_{21} && (13)\\
C_{cl} &= C_1 + D_{12} K C_2 && (14)\\
D_{cl} &= D_{11} + D_{12} K D_{21} && (15)
\end{aligned}
$$

The following Lemma can be given on the internal stability of the closed-loop system:

Lemma 3.1 *For the system described in (8), (9) and (10), there exists any compatible matrix K such that the matrix $A_{cl} = \underline{A} + B_2 K C_2$ is Hurwitz, if and only if the pair (A, B) is stabilizable and the matrix F is Hurwitz.*

Proof: When one considers $\underline{A}$, B_2, C_2 and K in A_{cl}, the following relation is obtained:

$$
\begin{aligned}
A_{cl} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} L & M \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A + BL & 0 \\ 0 & F \end{bmatrix} \qquad (16)
\end{aligned}
$$

The matrix A_{cl} is Hurwitz if and only if the matrix F is Hurwitz and there exists any compatible matrix L such that the matrix $A + BL$ is Hurwitz, i.e. the pair (A, B) is stabilizable. □

In order to guarantee the existence of the static state feedback law such that the closed-loop system is internally stable, throughout the paper, we assume that the pairs (A, B) is stabilizable and the matrix F is Hurwitz.

4 Main Result

In order to present a synthesis theorem on the LMI-based characterization of the discrete $\mathcal{H}_\infty$ MMP, let us give the following Lemmas that will be used to prove of the Theorem that will be presented later.

Lemma 4.1 *Suppose P, Q and H are matrices and that H is symmetric. The matrices N_P and N_Q are full rank matrices satisfying $Im N_P = Ker P$ and $Im N_Q = Ker Q$. Then there exists a matrix J such that*

$$H + P^* J^* Q + Q^* J P < 0 \qquad (17)$$

if and only if the inequalities

$$N_P^* H N_P < 0 \qquad \text{and} \qquad N_Q^* H N_Q < 0 \qquad (18)$$

both hold.

Proof: See [6]. □

Lemma 4.2 *The block matrix*

$$\begin{bmatrix} P & M \\ M^* & N \end{bmatrix} < 0 \qquad (19)$$

if and only if $N < 0$ and $P - MN^{-1}M^* < 0$. In the sequel, $P - MN^{-1}M^*$ will be referred to as the Schur Complement of N.

Proof: See [1]. □

We can now present the synthesis theorem on the LMI-based solution of the discrete $\mathcal{H}_\infty$ MMP with the static state feedback:

Theorem 4.3 *A controller $K = \begin{bmatrix} L & M \end{bmatrix}$ which solves the discrete $\mathcal{H}_\infty$ MMP with the static state feedback and the closed-loop system is internally stable, if and only if there exists a symmetric matrix $X_{cl} = \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} > 0$ such that,*

$$F^* X_3 F - X_3 + \frac{1}{\gamma} H^* H < 0 \qquad (20)$$

$$
\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* - X_{cl}^{-1} & \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & -\gamma I + \begin{pmatrix} -C & H \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (21)
$$

where N_c is a full rank matrix with

$$Im N_c = Ker \begin{bmatrix} B^* & 0 & -D^* \end{bmatrix} \qquad (22)$$

and a minimal realizations (A, B, C, D) of $T_2(z)$, namely controlled system, and (F, G, H, J) of $T_1(z)$, namely model system.

Proof: From the Bounded Real Lemma, $K = \begin{bmatrix} L & M \end{bmatrix}$ is the static state feedback controller in Figure 3 if and only if the LMI

$$\begin{bmatrix} -X_{cl}^{-1} & A_{cl} & B_{cl} & 0 \\ A_{cl}^* & -X_{cl} & 0 & C_{cl}^* \\ B_{cl}^* & 0 & -\gamma I & D_{cl}^* \\ 0 & C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \qquad (23)$$

holds for some $X_{cl} > 0$ in $\mathcal{R}^{n\times n}$. Using the expressions (12), (13), (14) and (15) of A_{cl}, B_{cl}, C_{cl} and D_{cl}, this LMI can also be written as:

$$H_{X_{cl}} + P^*KQ + Q^*K^*P < 0 \tag{24}$$

where

$$H_{X_{cl}} = \begin{bmatrix} -X_{cl}^{-1} & \underline{A} & B_1 & 0 \\ \underline{A}^* & -X_{cl} & 0 & C_1^* \\ B_1^* & 0 & -\gamma I & D_{11}^* \\ 0 & C_1 & D_{11} & -\gamma I \end{bmatrix} \tag{25}$$

$$Q = \begin{bmatrix} 0 & C_2 & D_{21} & 0 \end{bmatrix} \tag{26}$$

$$P = \begin{bmatrix} B_2^* & 0 & 0 & D_{12}^* \end{bmatrix} \tag{27}$$

We can use Lemma 4.1 to eliminate the matrix K in (24). Therefore, (24) holds for some K if and only if

$$N_P^* H_{X_{cl}} N_P < 0 \quad and \quad N_Q^* H_{X_{cl}} N_Q < 0 \tag{28}$$

where

$$Im N_P = Ker P \ , \ Im N_Q = Ker Q \ , \ X_{cl} > 0.$$

Meanwhile, from (27) it follows that bases of $Ker P$ are of the form

$$N_P = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ V_2 & 0 & 0 \end{bmatrix} \tag{29}$$

where

$$N_c = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{30}$$

is any basis of the null space of $\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}$. So $N_P^* H_{X_{cl}} N_P < 0$ can be reduced to

$$\begin{bmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ V_2 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} -X_{cl}^{-1} & \underline{A} & B_1 & 0 \\ \underline{A}^* & -X_{cl} & 0 & C_1^* \\ B_1^* & 0 & -\gamma I & D_{11}^* \\ 0 & C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ V_2 & 0 & 0 \end{bmatrix} < 0 \tag{31}$$

and

$$\begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix}^* \begin{bmatrix} \underline{A}X_{cl}^{-1}\underline{A}^* - X_{cl}^{-1} & B_1 & \underline{A}X_{cl}^{-1}C_1^* \\ B_1^* & -\gamma I & D_{11}^* \\ C_1 X_{cl}^{-1}\underline{A}^* & D_{11} & -\gamma I + C_1 X_{cl}^{-1} C_1^* \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix} < 0 \tag{32}$$

or equivalently

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \underline{A}X_{cl}^{-1}\underline{A}^* - X_{cl}^{-1} & \underline{A}X_{cl}^{-1}C_1^* & B_1 \\ C_1 X_{cl}^{-1}\underline{A}^* & -\gamma I + C_1 X_{cl}^{-1} C_1^* & D_{11} \\ B_1^* & D_{11}^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \tag{33}$$

On the other hand,

$$Im N_Q = Ker \begin{bmatrix} 0 & C_2 & D_{21} & 0 \end{bmatrix} = Ker \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \tag{34}$$

and

$$N_Q = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{35}$$

so, the condition $N_Q^* H_{X_{cl}} N_Q < 0$ is equivalent to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^* \begin{bmatrix} -X_{cl}^{-1} & \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} & 0 \\ \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & -X_{cl} & 0 & \begin{pmatrix} -C^* \\ H^* \end{pmatrix} \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & 0 & -\gamma I & J^* \\ 0 & \begin{pmatrix} -C & H \end{pmatrix} & J & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{36}$$

Then one can easily see that the reduced form of above inequality is in the form of a Lyapunov inequality as follows:

$$F^* X_3 F - X_3 + \frac{1}{\gamma} H^* H < 0 \tag{37}$$

by using Lemma 4.2. Finally, it can easily be derived the condition (21) when one considers the relations (8), (9) and (10) in (33). □

5 Controller Construction

Although the Theorem 4.3 is about the solvability conditions of the discrete $\mathcal{H}_\infty$ MMP with the static state feedback, it also provides a construction procedure:

Step 1: Find the matrix $X_{cl}^{-1} > 0$ for γ_{opt} with satisfying the LMI given in (21) by using The LMI Control Toolbox [7].

Step 2: Find X_3 by using $X_{cl}^{-1} = \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1}$ obtained in the previous step and verify the condition (20) for γ and X_3.

Step 3: If the LMI (20) is satisfied, find the static state feedback control law as $K = \begin{bmatrix} L & M \end{bmatrix}$ by solving the LMI given in (24) and stop the controller construction procedure, else increase γ.

Step 4: Find the matrix $X_{cl}^{-1} > 0$ for new γ with satisfying the LMI given in (21) and go to Step 2.

Since the matrix F is Hurwitz, the LMI given (20) has a positive definite solution for all value of $\gamma \in \mathcal{R}^+$. So we need to solve only one LMI (21) to obtain the optimal solution of the discrete $\mathcal{H}_\infty$ MMP by the static state feedback.

6 Conclusions

In this study, the discrete $\mathcal{H}_\infty$ MMP by the static state feedback has been investigated in state-space formulation and an LMI-based solution of the problem has been presented. The existence conditions of the static state feedback controller have been given in Theorem 4.3 and a construction procedure has been proposed in Section 5. The results on the discrete $\mathcal{H}_\infty$ MMP by the static state feedback, presented in this paper, might be regarded as the general results, since the static state feedback law that solves the discrete $\mathcal{H}_\infty$ MMP has not been limited to be regular, i.e. the control law could be obtained as an irregular static state feedback, if the optimal performance requires.

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