Error Analysis of Alternative Virtual Work Formulation Based on Mean and Deviation Potentials

ESSAM S HAMDI And SONJA T LUNDMARK
Department of Electrical Power Engineering
Chalmers University of Technology
412 96 Gothenburg
SWEDEN

Abstract: - In permanent-magnet devices operating at sufficiently low saturation, torque can be calculated by reformulating the vector potential problem so as to perform subtraction of energies analytically, before problem discretization. One such a technique, known as the mean and deviation potentials method, has been experimentally evaluated elsewhere [1]. This paper presents a mathematical evaluation of the error behavior of the method.

Key-Words: - Permanent-magnet motors, Torque calculations, Virtual work method.

1 Introduction

Cogging torque is a saliency effect that arises from the interaction between a salient pole on one member of an electric machine (rotor or stator) and the teeth on the other member. The interaction implies a magnetic field distribution that is a function of rotor position. Cogging torque is thus defined as the non-uniform torque, function of rotor position, which arises when only an excitation field is present [2].

The calculation of cogging torque in electro-mechanical devices is a difficult task because the effects sought are dependent on small changes in the magnetic fields in the machine. Two problems arise: mesh artifact and modeling precision. Mesh artifact can often be minimized by enforcing symmetry on the finite element mesh either explicitly or by averaging complementary meshes.

Difficulties arising from modeling precision can usually be traced to subtraction of nearly equal quantities. These can at times be eliminated by reformulating the problem in such a way that the critical subtractions are performed in preliminary analytic development, where only exact quantities arise, leaving comparatively stable (though sometimes more time-consuming) operations to be carried out in subsequent numerical approximation.

The conventional torque calculation based on virtual work method can be reformulated. Instead of explicitly subtracting two separately calculated system energies, the problem is stated with subtraction performed analytically. One such a treatment is discussed below [1,3].

2 Mean and Deviation Potentials

This method was initially developed with specific reference to a class of permanent-magnet electric motors [1]. Were the conventional virtual work method to be used, there will be two boundary-value problems corresponding to two rotor positions as illustrated in Fig. 1.

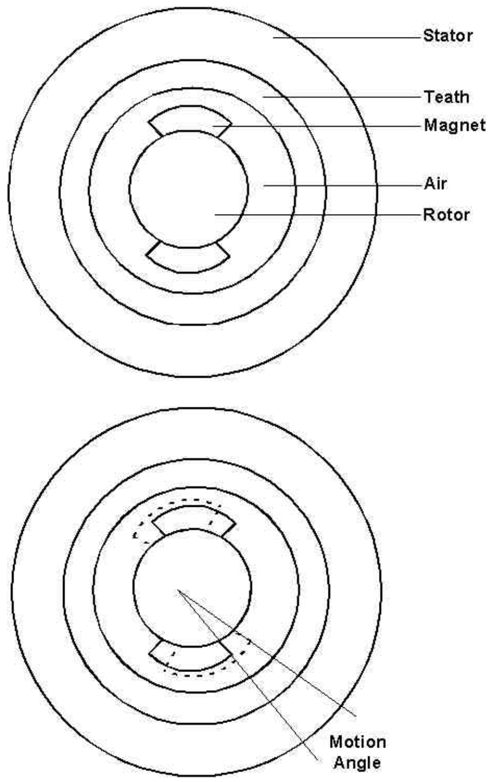

Figure 1: *Cogging torque boundary-value problems.*

It is seen that the two boundary-value problems differ only in the placement of the permanent magnetization. To facilitate development of the theory the remanence was replaced by equivalent current distributions J_1 and J_2 for rotor positions 1 and 2 respectively [1]. Furthermore, to allow a 2-dimensional analysis, the motor was assumed to be infinitely long and the permeability of the magnet material is assumed to be μ (*n*= *1/m*). This yields the following two boundary-value problems:

$$-\frac{\partial}{\partial x}\left(n\,\frac{\partial A_1}{\partial x}\right)-\frac{\partial}{\partial y}\left(n\,\frac{\partial A_1}{\partial y}\right)=J_1 \quad in\,\Omega \qquad (1)$$

and

$$-\frac{\partial}{\partial x}\left(n\,\frac{\partial A_2}{\partial x}\right)-\frac{\partial}{\partial y}\left(n\,\frac{\partial A_2}{\partial y}\right)=J_2 \quad in\,\Omega \qquad (2)$$

Both partial differential equations above are subject to similar boundary conditions. The problem region *W* (in the *x*-*y* plane) may be considered to have a boundary ¶*W* which may be partitioned into two portions ¶W_D and ¶W_N:

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$$

where

$A_1 = \overline{A}$ on ¶W_D

$\dfrac{\partial A_1}{\partial n} = \vec{A}'$ on ¶W_N

Rather than work with the two boundary-value problems in A_1 and A_2, two new problems are defined. The first is obtained by taking the difference of the two problems above. Define:

$$A_3 = \frac{A_1 - A_2}{2} \qquad \text{and} \qquad J_3 = \frac{J_1 - J_2}{2}$$

Then

$$-\frac{\partial}{\partial x}\left(n\,\frac{\partial A_3}{\partial x}\right)-\frac{\partial}{\partial y}\left(n\,\frac{\partial A_3}{\partial y}\right)=J_3 \qquad (3)$$

subject to the homogeneous boundary conditions:

$A_3 = 0$ on ¶W_D

and,

$\dfrac{\partial A_3}{\partial n} = 0$ on ¶W_N

The second problem is obtained by averaging the two original problems, setting:

$$A_4 = \frac{A_1 + A_2}{2} \qquad \text{and} \qquad J_4 = \frac{J_1 + J_2}{2}$$

Then

$$-\frac{\partial}{\partial x}\left(n\,\frac{\partial A_4}{\partial x}\right)-\frac{\partial}{\partial y}\left(n\,\frac{\partial A_4}{\partial y}\right)=J_4 \qquad (4)$$

where the boundary conditions are similar to those of the original problems,

$A_4 = \overline{A}$ on ¶W_D

$\dfrac{\partial A_4}{\partial n} = \vec{A}'$ on ¶W_N

2.1 Virtual work

Let the total energy for each of the original rotor position be evaluated and subtracted, so that:

$$dW = \frac{1}{2}\int_\Omega A_1 J_1 d\Omega - \frac{1}{2}\int_\Omega A_2 J_2 d\Omega.$$

Rewriting in terms of A_3 and A_4, on collecting terms we get:

$$dW = \int_\Omega A_3 J_4 d\Omega + \int_\Omega A_4 J_3 d\Omega.$$

In other words the energy difference is evaluated by solving the *averaged* and the *deviation* boundary value problems then combining the potential of each with the current density of the other. It is noted that if the two rotor positions are in fact the same position, A_3 and J_3 both vanish. Then dW vanishes also – which is just as it should be.

2.2 Experimental evaluation

The mean and deviation potential method was experimentally evaluated elsewhere for the specific case of cogging torque in small permanent-magnet motors [1,4]. Instead of working with the two original boundary value problems, whose solutions are very similar, and then attempting to determine their small difference by subtraction, the energy difference is evaluated by solving the boundary value problems for the mean and deviation potentials. The latter two problems are quite different from each other, their solutions are nearly orthogonal to each other, as is illustrated in Fig. 2. [1].

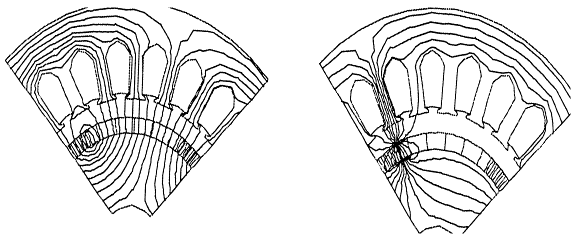

Figure 2: *Flux plots for mean (right) and deviation potential problems [1].*

Cogging torque characteristics obtained from the mean and deviation potentials method, and computed on a coarse mesh (1546 elements, 808 nodes) and on a fine mesh (1834 elements, 952 nodes) are presented in Fig. 3, together with measurements. The solid curve represents a least-square quintic polynomial fit to the measured values. It is seen that the error here is very small. Lowther and Forghani [5] suggest in their comparison study that this error level should be considered excellent for a two-dimensional analysis. Ferreira and Vaidya [6] corroborate this view, particularly for machines with large air gaps. The relative error here is not very far from the accuracy to be expected from the experimental results.

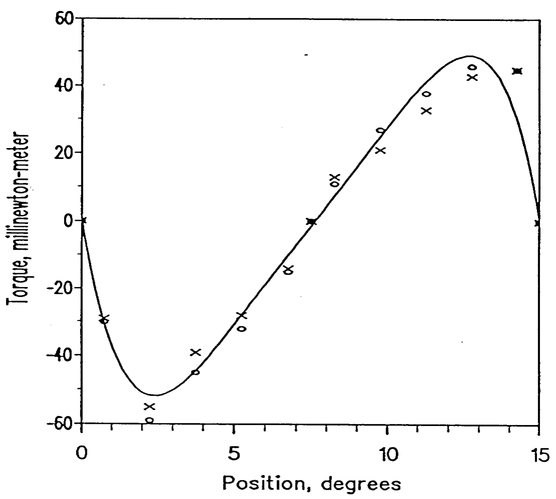

Figure 3: *Cogging torque computed by mean and deviation potential method [1].*

3 Error Analysis

While the method has been verified experimentally elsewhere, it is the purpose of this paper to present a mathematical formulation of its numerical errors. This of particular importance as it provides a general evaluation of the mathematical formulation.

The boundary-value problems (described by equations 1 to 4 above) in each of the potentials A_i (i= 1, ..4) may be described in terms of a linear operator $\boldsymbol{D}$ so that:

$$\boldsymbol{D}A_i = -J_i$$

means

$$\mathbf{1_z}\cdot curl\ n\ curl\ (\mathbf{1_z}A_i) = J_i \qquad \text{in } W,$$

$$A_i = \overline{A}_i \qquad \text{on ¶}W_D$$

$$\frac{\partial A_i}{\partial n} = \vec{A}_i' \qquad \text{on ¶}W_N$$

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$$

All four boundary-value problems are described by the same operator **D**, so all four will have the same discrete representation in matrix form. These will appear as:

$$\Delta \mathbf{a_i} = \mathbf{j_i}$$

Where Δ is a matrix, $\mathbf{a_i}$ and $\mathbf{J_i}$ are column vectors. Δ is of course subject to a discretisation error, it does not represent and exact discrete model of the operator $\boldsymbol{D}$. Instead, it is an exact model of some other operator ($\boldsymbol{D}$ - $\boldsymbol{E}$), where $\boldsymbol{E}$ accounts for the difference:

$$(\boldsymbol{D} - \boldsymbol{E})\ (A_i - e_i) = J_i$$

Here, e_i represents the error in the solution. Expanding:

$$\boldsymbol{D}\,A_i - \boldsymbol{E}\,A_i - \boldsymbol{D}\,e_i + \boldsymbol{E}\,e_i = J_i$$

The first term on the left, however, is exactly the solution of the boundary-value problem in A_i, so

$$(\boldsymbol{D} - \boldsymbol{E})\ e_i = -\boldsymbol{E}\,A_i$$

$$e_i = -(\boldsymbol{D} - \boldsymbol{E})^{-1}\ \boldsymbol{E}\,A_i$$

Clearly, one cannot usefully estimate the error e_i from this result, for the error operator **E** cannot ordinarily be known. However, one can bound the error in some convenient norm:

$$\|e_i\| \leq \frac{\|E\|}{\|D-E\|}\|A_i\|$$

The error in the energy increment *dW*, say *eW*, can be expressed as:

$$eW = 2\int_{\Omega}\left(e_3 J_4 + e_4 J_3\right) d\Omega$$

and it can now be bounded. First,

$$eW \leq 2\int_{\Omega}\left(\|e_3\|\,\|J_4\| + \|e_4\|\,\|J_3\|\right) d\Omega$$

and then, making use of the bounds on the e_i terms,

$$eW \leq 2\,\frac{\|E\|}{\|D-E\|}\int_{\Omega}\left(\|A_3\|\,\|J_4\| + \|A_4\|\,\|J_3\|\right) d\Omega$$

This analysis does indicate that the error in torque can be expected to behave no worse than the error in energy itself – which is generally quite good enough for most practical purposes.

4 Conclusion

The mean and deviation potential approach to torque computation yields good results, even for the very difficult problem of cogging torques where neither the Maxwell stress nor the virtual work method appear to be sufficiently robust to be useful. This paper shows that the method is numerically stable and demonstrates that the errors in torque are no worse than the errors in the computed energies.

References:

[1] E.S. Hamdi, A.F. Licariao-Nogueira and P.P. Silvester, Torque computation by mean and difference potentials, *IEE Proceeding-A*, Vol.140, No.2, March 1993, pp. 151-154.

[2] E.S. Hamdi, *Design of small electrical machines*, John Wiley & Sons, UK, 1994, 1996 & 1998.

[3] E.S. Hamdi, E.M. Freeman, and P.P. Silvester, On the stored energy computation in permanent-magnet motors, *Electromotion*, Vol.3, No.2, 1996, pp. 39-42.

[4] E.S. Hamdi, M.L. Isaac and P.P. Silvester, Difference-potential torque computation with magnet interpolar gaps, *Electrical Machines and Power Systems*, Vol.25, No.3, 1997, pp.263-269.

[5] D.A. Lowther and B. Forghani, Force calculation with three-dimentional scalar potential based finite-element programs, *J. Applied Physics*, Vol.67, No.9, Pt. 2B, 1990, p. 5833.

[6] C. Ferrerira and J. Vaidya, Torque analysis of permanent magnet coupling using 2D and 3D finite elements method, *IEEE Trans.*, MAG-25, 1989, pp. 3080-3082.