Dynamic control system to manage the maintenance of repairable parts

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Abstract: – Using dependability techniques ([3]-[4]) it has been possible to obtain a model of the lifecycle of a repairable device, for maintenance purposes.
The evolution of such approach is the application of control techniques to the periodicity of the check rates. Such rates are the cheapest parameters that can be used, in fact it is possible to conduct the checks also by means of sensors. The control law consists, then, in a suitable scheduling strategy of the control checks.

Key-Words: – Reliability, Fault analysis, Control systems, Stochastic systems, Decision support systems, Management systems, Semi-numerical algorithms.

1 Introduction

The repairable devices when damaged can be repaired and reused or stored in a warehouse waiting to be used. The time distributions of the events that interest such devices can be considered exponential ones with constant transition rates from one state to another ([5]). Given the randomness of the damage times, the usual logistic management strategy of such devices consists in doing periodic checks of their states ([3]), for example at the mean time to failure (MTTF), with eventual substitution if there is an upcoming failure (preventive maintenance) or after a damage has happened (corrective maintenance). After a certain number of reparations, when further investments are not convenient, the device is eliminated from the cycle. The main drawback of such management is the over dimensioning of the provision warehouse (passive management).
Instead, analysing the lifecycle of a similar device, it is possible to locate the control variables to be used in a feedback loop whose target is to modify the expected time to failure, to obtain, in such a way, the economic goal of the reduction of substitutions for maintenance. The increment of the MTTF thus obtained results in an abatement of the number of provisions of each device (active maintenance).
After a careful analysis the control variables to be used have been located in the check rates of correct functioning. The choice of such rates is the less expensive. The increment of the number of checks to be done in the unit of time characterises an increment of expenses, but reduces the necessity of corrective maintenance, which is more expensive of the preventive one, thus obtaining in a long period an effective thrift. Moreover, organisational problems that could rise by increasing the reparation rates are thus avoided.
The meaning of check rate is that of inverse of mean time between checks ([5]-[6]), thing that permits to schedule a plan of the checks.

2 Assumptions

A model of the lifecycle of a repairable device can be obtained with the following hypothesis:

- only one state for maintenance and checks;
- there are no stored or functioning devices whose failures are not detectable (failure coverage);
- the transition probability from the state S_i, at time instant t, to S_j, at time instant $t+\Delta t$, is proportional to the time interval Δt, for Δt small with respect to the transition times;
- the proportionality constant, called transition (or failure, reparation, check, etc.) rate does not depend on t and Δt; in particular eventual reparations do not modify in time the failure rates;
- independent devices of the same kind have the same probabilistic behaviour.

3 Lifecycle of a repairable device

The lifecycle of a repairable device, can be modelled with the state diagram of Fig. 1. Such a diagram represents only the active life of the device, infant mortality and obsolescence due to ageing are not considered.

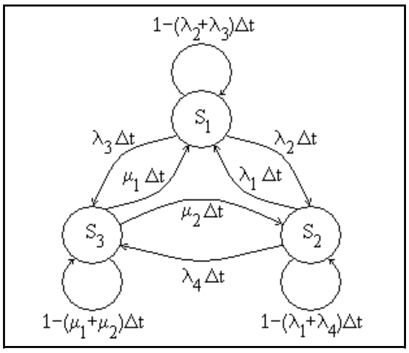

Fig. 1: Lifecycle of a repairable device

Symbols have the following meaning:

S_1: device not damaged stored in the warehouse,
S_2: correctly functioning device,
S_3: check or corrective maintenance intervention,

$x_1(t)$: probability of being in S_1 at time instant t,
$x_2(t)$: probability of being in S_2 at time instant t,
$x_3(t)$: probability of being in S_3 at time instant t,

$\lambda_i\Delta t$: transition probability in the interval $[t, t+\Delta t)$,

λ_1: rate of storage of a correctly functioning device,
λ_2: rate of going in use of a stored device,
$\lambda_3 = \nu_3+u_1$; ν_3: failure rate of a stored device,
u_1: check rate on a stored device
$\lambda_4 = \nu_4+u_2$; ν_4: failure rate of a functioning device,
u_2: check rate on a functioning device,
μ_1: rate of storage after a check or a storage intervention,
μ_2: rate of going in use after a check or a storage intervention.

From the diagram of Fig. 1 it is possible to obtain the following transition equations (Kolmogoroff equations):

$$\begin{aligned} x_1(t+\Delta t) &= [1-(\lambda_2+\nu_3)\Delta t]\cdot x_1(t)+\lambda_1\Delta t\cdot x_2(t)+ \\ &\quad +\mu_1\Delta t\cdot x_3(t)-u_1\Delta t\cdot x_1(t) \\ x_2(t+\Delta t) &= \lambda_2\Delta t\cdot x_1(t)+[1-(\lambda_1+\nu_4)\Delta t]\cdot x_2(t)+ \\ &\quad +\mu_2\Delta t\cdot x_3(t)-u_2\Delta t\cdot x_2(t) \\ x_3(t+\Delta t) &= \nu_3\Delta t\cdot x_1(t)+\nu_4\Delta t\cdot x_2(t)+ \\ &\quad +[1-(\mu_1+\mu_2)\Delta t]\cdot x_3(t)+u_1\Delta t\cdot x_1(t)+u_2\Delta t\cdot x_2(t) \end{aligned}$$

that, considering Δt infinitesimal, become:

$$\begin{aligned} dx_1(t)/dt &= -(\lambda_2+\nu_3)\cdot x_1(t)+\lambda_1\cdot x_2(t)+\mu_1\cdot x_3(t)-u_1\cdot x_1(t) \\ dx_2(t)/dt &= \lambda_2\cdot x_1(t)-(\lambda_1+\nu_4)\cdot x_2(t)+\mu_2\cdot x_3(t)-u_2\cdot x_2(t) \\ dx_3(t)/dt &= \nu_3\cdot x_1(t)+\nu_4\cdot x_2(t)-(\mu_1+\mu_2)\cdot x_3(t)+ \\ &\quad +u_1\cdot x_1(t)+u_2\cdot x_2(t) \end{aligned} \tag{1}$$

If u_1 and u_2 are decided a priori from the management system (for example according to legislation), the model can be used for analytic purposes. Instead, the main target here considered has been the design of a scheduling strategy for the maintenance checks.

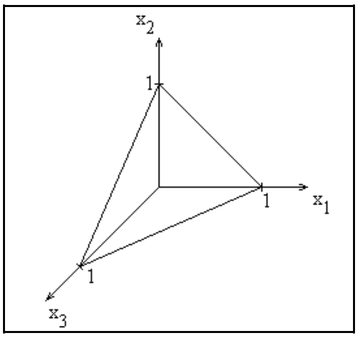

Fig. 2: Confinement domain of the motion of the system

Note that (1) is not a normal physical system, with physical inputs that are random variables, rather it is an implicit method to describe the probability distributions of certain temporal events. Then, instead of using the usual techniques of the optimal stochastic control, it is preferable to develop a method for modifying such probabilities, by making some events happen in a controlled way.
Since the system states are mutually exclusive, the motion of the system is confined in the triangle:

$$\begin{aligned} &x_1(t)+x_2(t)+x_3(t)=1, && \forall t\ge 0, \\ &0\le x_1(t)\le 1,\ \ 0\le x_2(t)\le 1,\ \ 0\le x_3(t)\le 1, && \forall t\ge 0, \end{aligned} \tag{2}$$

represented in Fig. 2. Note that, being system (1) bilinear, the goals to be obtained and the solution thus derived could not be trivial. Moreover, the physical meanings of the rates, dictate non-negativity and boundedness of the rates and of the control variables:

$$\begin{aligned} &0\le\lambda_1\le\lambda_{1,\max}, \quad 0\le\nu_3\le\nu_{3,\max}, \quad 0\le\mu_1\le\mu_{1,\max}, \\ &0\le\lambda_2\le\lambda_{2,\max}, \quad 0\le\nu_4\le\nu_{4,\max}, \quad 0\le\mu_2\le\mu_{2,\max}, \\ &0\le u_1(t)\le u_{1,\max}, \quad \forall t\ge 0, \\ &0\le u_2(t)\le u_{2,\max}, \quad \forall t\ge 0. \end{aligned} \tag{3}$$

A last consideration must be done on ν_3 and ν_4 both linked to the diagnostic capabilities of the devices:

- ν_3 is due to the functionality checks on stored devices before being put in operation, if such checks do not exist (i.e. for economic reasons), it can be assumed that ν_3=0,
- ν_4, instead, does not takes into account the fact that it is possible the existence of devices whose failures are not covered promptly by the diagnostic system.

Violation of the bounds (3) indicates that there are devices that need further checks or removal. A decisional strategy for removal must be based on two parameters:

- the fraction of the reparation time with respect to the total time (unavailability),
- the maintenance expenses with respect to a preventive budget and to the cost of a new device.

4 Analysis

The order of system (1) can be reduced by projecting the motion on the plane x_1x_2, where it is confined in the triangle:

$$\begin{aligned} &x_1(t)+x_2(t)\le 1, && \forall t\ge 0, \\ &0\le x_1(t)\le 1,\ \ 0\le x_2(t)\le 1, && \forall t\ge 0. \end{aligned} \tag{4}$$

The new equations of the motion are:

$$\begin{aligned} dx_1(t)/dt &= -(\lambda_2+\nu_3+\mu_1)\cdot x_1(t)+(\lambda_1-\mu_1)\cdot x_2(t)-u_1\cdot x_1(t)+\mu_1 \\ dx_2(t)/dt &= (\lambda_2-\mu_2)\cdot x_1(t)-(\lambda_1+\nu_4+\mu_2)\cdot x_2(t)-u_2\cdot x_2(t)+\mu_2 \end{aligned} \tag{5}$$

or:

$$dx(t)/dt = Ax(t)-U(t)x(t)+\mu = Ax(t)-X(t)u(t)+\mu \tag{6}$$

with

$$A=\begin{pmatrix}-d_1 & a\\ b & -d_2\end{pmatrix};\ x(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix};\ u(t)=\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}$$

$$\begin{aligned} d_1 &= \lambda_2+\nu_3+\mu_1; \quad a=\lambda_1-\mu_1; \\ d_2 &= \lambda_1+\nu_4+\mu_2; \quad b=\lambda_2-\mu_2; \end{aligned} \quad \mu=\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$$

$$U(t)=\begin{pmatrix}u_1(t) & 0\\ 0 & u_2(t)\end{pmatrix}; \qquad X(t)=\begin{pmatrix}x_1(t) & 0\\ 0 & x_2(t)\end{pmatrix}$$

(note that $U(t)x(t)=X(t)u(t)$).

The following bounds imply physical realizability: $d_1-b>0$, $d_2-a>0$, $d_2+b>0$, $d_1+a>0$, $\mu_1\geq 0$, $\mu_2\geq 0$, $a+\mu_1\geq 0$, $b+\mu_2\geq 0$, $d_1-b-(\mu_1+\mu_2)\geq 0$, $d_2-a-(\mu_1+\mu_2)\geq 0$, (they are always fulfilled excluding the trivial cases $\lambda_1=\lambda_2=\nu_3=0$, $\mu_1=\mu_2=\nu_3=0$, $\lambda_1=\lambda_2=\nu_4=0$, $\mu_1=\mu_2=\nu_4=0$, without physical significance).

Let $u(t)=0$, then the evolution of system (6) is confined in triangle (4) and, respectively, the evolution of (1) in triangle (2), if and only if, besides the physical realizability bounds, the eigenvalues of the matrix A have strictly negative real part (asymptotic stability of matrix A).

Such eigenvalues are the roots of the characteristic polynomial $p(s)=s^2+(d_1+d_2)s+(d_1d_2-ab)$, and lay on the negative side of the complex plane if $(d_1d_2-ab)>0$, i.e. if $(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\nu_3(\lambda_1+\mu_2)+\nu_4(\lambda_2+\mu_1)+\nu_3\nu_4>0$, thing guaranteed by positivity of parameters (in fact such quantity reduces to zero only in the trivial cases, $\lambda_1=\lambda_2=\nu_3=\nu_4=0$, or $\mu_1=\mu_2=\nu_3=\nu_4=0$, without physical significance).

If the inputs (check rates) are constant in time, $u_1(t)=\bar{u}_1\geq 0$, $u_2(t)=\bar{u}_2\geq 0$, then it is possible to put $U(t)=\bar{U}$, $A_1=A+\bar{U}$ and, given the positivity of such inputs, also the matrix A_1 is asymptotically stable.

In such a case, the time evolution of system (6) is described by the following equation:

$$x(t)=e^{A_1t}x(0)+\int_0^t e^{A_1(t-\tau)}d\tau\cdot\mu$$

That thanks to asymptotic stability of A_1, reduces to:

$$x(t)=e^{A_1t}[x(0)+A_1^{-1}\mu]-A_1^{-1}\mu\ . \qquad (7)$$

For $t\to\infty$ the steady state solution is:

$$x(\infty)=\begin{pmatrix}x_1(\infty)\\ x_2(\infty)\end{pmatrix}=-A_1^{-1}\mu=$$

$$=\begin{pmatrix}\dfrac{(\lambda_1+\bar{\lambda}_4)\mu_1+\lambda_1\mu_2}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3(\lambda_1+\mu_2)+\bar{\lambda}_4(\lambda_2+\mu_1)+\bar{\lambda}_3\bar{\lambda}_4}\\ \dfrac{\lambda_2\mu_1+(\lambda_2+\bar{\lambda}_3)\mu_2}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3(\lambda_1+\mu_2)+\bar{\lambda}_4(\lambda_2+\mu_1)+\bar{\lambda}_3\bar{\lambda}_4}\end{pmatrix}$$

$$x_3(\infty)=1-x_1(\infty)-x_2(\infty)=$$

$$=\frac{\lambda_1\bar{\lambda}_3+\bar{\lambda}_3\bar{\lambda}_4+\lambda_2\bar{\lambda}_4}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3(\lambda_1+\mu_2)+\bar{\lambda}_4(\lambda_2+\mu_1)+\bar{\lambda}_3\bar{\lambda}_4}$$

where $\bar{\lambda}_3=\nu_3+\bar{u}_1$, e $\bar{\lambda}_4=\nu_4+\bar{u}_2$.

It is possible to define reliability, $R(t)$, of the device as probability of not being in S_3,

$R(t):=1-x_3(t)=C\,x(t)$, with $C=(1\quad 1)$.

Such a definition, due to simplicity of the model, is not strictly undeviating, being the permanence in S_3 due not only to failures but also to checks. Anyway, for $t\to\infty$:

$$R(\infty)=-CA_1^{-1}\mu=$$

$$=\frac{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3\mu_2+\bar{\lambda}_4\mu_1}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3(\lambda_1+\mu_2)+\bar{\lambda}_4(\lambda_2+\mu_1)+\bar{\lambda}_3\bar{\lambda}_4}$$

For what concerns the MTTF, considering that $R'(t):=R(t)-R(\infty)$ goes to zero for $t\to\infty$, it is possible to define the following probability density:

$$f(t):=-\frac{dR'(t)}{dt}=-\frac{dR(t)}{dt},$$

Which implies the following definition for the mean time to failure or check:

$$MTTF:=\int_0^\infty t\,f(t)dt=\int_0^\infty R'(t)dt=-CA_1^{-1}[x(0)-x(\infty)]= \qquad (8)$$

$$=\frac{(\lambda_1+\lambda_2+\bar{\lambda}_4)(x_1(0)-x_1(\infty))+(\lambda_1+\lambda_2+\bar{\lambda}_3)(x_2(0)-x_2(\infty))}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\bar{\lambda}_3(\lambda_1+\mu_2)+\bar{\lambda}_4(\lambda_2+\mu_1)+\bar{\lambda}_3\bar{\lambda}_4}$$

5 Design

To avoid unboundedness of the control law when x_1 and/or x_2 goes to zero, triangle (4) has been splitted in four sets, indicated in Fig. 3, with ①,②,③,④.

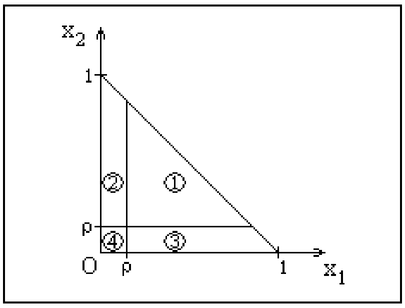

Fig. 3: Sets of validity of the control laws

In the set ①, where $\rho\leq x_1\leq 1$, $\rho\leq x_2\leq 1$ and $x_1+x_2\leq 1$, it can be used the following control law ([1]):

$$u(t)=\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}=[X(t)]^{-1}\left[(A+K)x(t)+\mu-K\bar{x}\right], \qquad (9)$$

where

$$K=\begin{pmatrix}k_1 & k_2\\ k_3 & k_4\end{pmatrix},\ \bar{x}=\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}$$

with $\bar{x}$ chosen inside the set ①, $-K$ asymptotically stable (i.e. such that $k_1>0$, $k_4>0$ e $k_1k_4-k_2k_3>0$), to assure stability of the feedback loop, and $K,\bar{x}$ such that $k_1-d_1\geq 0$, $k_4-d_2\geq 0$, $k_2+a\geq 0$, $k_3+b\geq 0$, $\mu_1-k_1\bar{x}_1-k_2\bar{x}_2\geq 0$, $\mu_2-k_3\bar{x}_1-k_4\bar{x}_2\geq 0$, to assure positivity, for all $t\geq 0$, of $u_1(t)\geq 0$ and $u_2(t)\geq 0$. Moreover, to be sure that the evolution of the feedback loop never leave triangle (4), the following physical realizability bounds must be satisfied: $k_1-k_2>0$, $k_4-k_3>0$, $k_4+k_2>0$, $k_1+k_3>0$, $(1-\bar{x}_1)(k_1+k_3)-(k_2+k_4)\bar{x}_2\geq 0$, $k_1\bar{x}_1-k_2(1-\bar{x}_2)\geq 0$, $(1-\bar{x}_2)(k_2+k_4)-(k_1+k_3)\bar{x}_1\geq 0$, $k_4\bar{x}_2-k_3(1-\bar{x}_1)\geq 0$, $k_1\bar{x}_1+k_2\bar{x}_2\geq 0$, $k_3\bar{x}_1+k_4\bar{x}_2\geq 0$.

Aim of (9) is that of modifying the dynamics of the system to obtain the model:

$dx_{m1}/dt=A_{m1}x_{m1}+\mu_{m1}$, with $A_{m1}=-K$, $\mu_{m1}=-A_{m1}\bar{x}=K\bar{x}$,

and new steady state solution:

$$x(\infty)=\begin{pmatrix}x_1(\infty)\\ x_2(\infty)\end{pmatrix}=-[A_{m1}]^{-1}\mu_{m1}=\bar{x}=\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}:=\bar{x}_{m1}$$

$$x_3(\infty)=1-\bar{x}_1-\bar{x}_2,$$

in such a way that the new steady state reliability is:

$$R_1(\infty)=1-x_3(\infty)=\bar{x}_1+\bar{x}_2\ . \qquad (10)$$

In the set ②, $0\leq x_1<\rho$, $\rho\leq x_2\leq 1$ and $x_1+x_2\leq 1$, the first component of (9) can become unbounded. To avoid this problem such a component is fixed to a constant value, leaving unchanged the second component of (9):

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu_1 + a\bar{x}_2}{\bar{x}_1} - d_1 \\ \dfrac{(k_3+b)x_1 + (k_4 - d_2)x_2 + (\mu_2 - k_3\bar{x}_1 - k_4\bar{x}_2)}{x_2} \end{pmatrix} \qquad (11)$$

with the same $\bar{x}$ as before (inside the set ①), and with the further condition $(\mu_1 + a\bar{x}_2)k_4 + a\bar{x}_1 k_3 > 0$ for the asymptotic stability of the loop (the physical realizability bounds do not add new conditions and the condition $\mu_1 - d_1\bar{x}_1 + a\bar{x}_2 \geq 0$ for the positivity of u_1 is already satisfied if the conditions on k_1 and k_2 are true). The dynamics of the system is modified by (11) in:

$dx_{m2}/dt = A_{m2}x_{m2} + \mu_{m2}$, with

$$A_{m2} = \begin{pmatrix} -\dfrac{\mu_1 + a\bar{x}_2}{\bar{x}_1} & a \\ -k_3 & -k_4 \end{pmatrix}, \quad \mu_{m2} = \begin{pmatrix} \mu_1 \\ k_3\bar{x}_1 + k_4\bar{x}_2 \end{pmatrix},$$

and the same steady state solution as before:

$x(\infty) = -[A_{m2}]^{-1}\mu_{m2} = \bar{x} := \bar{x}_{m2}$.

For analogy, in the set ③, $\rho \leq x_1 \leq 1$, $0 \leq x_2 < \rho$ and $x_1 + x_2 \leq 1$, where the second component of (9) can be unbounded, the control law is:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{(k_1 - d_1)x_1 + (k_3 + a)x_2 + (\mu_1 - k_1\bar{x}_1 - k_2\bar{x}_2)}{x_1} \\ \dfrac{\mu_2 + b\bar{x}_1}{\bar{x}_2} - d_2 \end{pmatrix} \qquad (12)$$

with the same $\bar{x}$ as before (inside the set ①), and with the further condition $(\mu_2 + b\bar{x}_1)k_1 + b\bar{x}_2 k_2 > 0$ for the asymptotic stability of the loop (the physical realizability bounds do not add new conditions and the condition $\mu_2 + b\bar{x}_1 - d_2\bar{x}_2 \geq 0$ for the positivity of u_2 is already satisfied if the conditions on k_3 and k_4 are true). The dynamics of the system is modified by (12) in:

$dx_{m3}/dt = A_{m3}x_{m3} + \mu_{m3}$, with

$$A_{m3} = \begin{pmatrix} -k_1 & -k_2 \\ b & -\dfrac{\mu_2 + b\bar{x}_1}{\bar{x}_2} \end{pmatrix}, \quad \mu_{m3} = \begin{pmatrix} k_1\bar{x}_1 + k_2\bar{x}_2 \\ \mu_2 \end{pmatrix},$$

and the same steady state solution as before:

$x(\infty) = -[A_{m3}]^{-1}\mu_{m3} = \bar{x} := \bar{x}_{m3}$.

In the set ④, $0 \leq x_1 < \rho$, $0 \leq x_2 < \rho$, the control law is:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu_1 + a\bar{x}_2}{\bar{x}_1} - d_1 \\ \dfrac{\mu_2 + b\bar{x}_1}{\bar{x}_2} - d_2 \end{pmatrix} \qquad (13)$$

where $\bar{x}$ is always the same, (the physical realizability bounds do not add new conditions and the condition $(\mu_2 + b\bar{x}_1)(\mu_1 + a\bar{x}_2) - ab\bar{x}_1\bar{x}_2 > 0$ for the asymptotic stability is already satisfied for any $\bar{x}_1$ and $\bar{x}_2$). The dynamics of the system is modified by (13) in:

$dx_{m4}/dt = A_{m4}x_{m4} + \mu_{m4}$, with

$$A_{m4} = \begin{pmatrix} -\dfrac{\mu_1 + a\bar{x}_2}{\bar{x}_1} & a \\ b & -\dfrac{\mu_2 + b\bar{x}_1}{\bar{x}_2} \end{pmatrix}, \quad \mu_{m4} = \mu,$$

Whose steady state solution is:

$x(\infty) = -[A_{m4}]^{-1}\mu_{m4} = \bar{x} := \bar{x}_{m4}$.

In the triangle (4) and in the respective sets of definition of the laws there are the following bounds:

$$u_{1,\min} = \min\left\{\left[\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1} - d_1\right], [(k_1 - d_1) + (\mu_1 - k_1\bar{x}_1 - k_2\bar{x}_2)]\right\}$$

$$u_{1,\max} = \max\left\{\left[\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1} - d_1\right], \left[(k_1 - d_1) + \frac{(k_2 + a)(1-\rho) + (\mu_1 - k_1\bar{x}_1 - k_2\bar{x}_2)}{\rho}\right]\right\}$$

$$u_{2,\min} = \min\left\{\left[\frac{\mu_2 + b\bar{x}_1}{\bar{x}_2} - d_2\right], [(k_4 - d_2) + (\mu_2 - k_3\bar{x}_1 - k_4\bar{x}_2)]\right\}$$

$$u_{2,\max} = \max\left\{\left[\frac{\mu_2 + b\bar{x}_1}{\bar{x}_2} - d_2\right], \left[(k_4 - d_2) + \frac{(k_3 + b)(1-\rho) + (\mu_2 - k_3\bar{x}_1 - k_4\bar{x}_2)}{\rho}\right]\right\}$$

Thanks to the physical realizability bounds and to the conditions for the asymptotic stability of the various models, it is possible to prove that the motion of the controlled system [with the law (9), (11), (12), (13)] tends to $\bar{x}$, starting from any initial condition inside the triangle (4). Moreover, the trajectories starting inside the set ① do not leave such set, those starting inside the set ② leave from such set to enter in ①, those starting inside the set ③ leave from such set to enter in ①, those starting inside the set ④ leave from such set to enter in ①, ②, ③.

Then, for initial conditions inside ①, ②, ③ and ④, the MTTF can be calculated as follows. Let:

$$p_2(s) = s^2 + \left(\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1} + k_4\right)s + \left(\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1}k_4 + ak_3\right), \quad q_1(s) = s + \frac{\mu_1 + a\bar{x}_2}{\bar{x}_1}$$

$$p_3(s) = s^2 + \left(k_1 + \frac{\mu_2 + b\bar{x}_1}{\bar{x}_2}\right)s + \left(k_1\frac{\mu_2 + b\bar{x}_1}{\bar{x}_2} + k_2 b\right), \quad q_2(s) = s + \frac{\mu_2 + b\bar{x}_1}{\bar{x}_2}$$

$$p_4(s) = s^2 + \left(\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1} + \frac{\mu_2 + b\bar{x}_1}{\bar{x}_2}\right)s + \left(\frac{\mu_1 + a\bar{x}_2}{\bar{x}_1}\frac{\mu_2 + b\bar{x}_1}{\bar{x}_2} - ab\right),$$

$$\begin{pmatrix} f_{11}(t) & f_{12}(t) \\ f_{13}(t) & f_{14}(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \dfrac{s+d_2}{p(s)} & \dfrac{a}{p(s)} \\ \dfrac{b}{p(s)} & \dfrac{s+d_1}{p(s)} \end{pmatrix} \quad \begin{pmatrix} f_{21}(t) & f_{22}(t) \\ f_{23}(t) & f_{24}(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \dfrac{s+k_4}{p_2(s)} & \dfrac{a}{p_2(s)} \\ \dfrac{-k_3}{p_2(s)} & \dfrac{q_1(s)}{p_2(s)} \end{pmatrix}$$

$$\begin{pmatrix} f_{31}(t) & f_{32}(t) \\ f_{33}(t) & f_{34}(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \dfrac{q_2(s)}{p_3(s)} & \dfrac{-k_2}{p_3(s)} \\ \dfrac{b}{p_3(s)} & \dfrac{s+k_1}{p_3(s)} \end{pmatrix} \quad \begin{pmatrix} f_{41}(t) & f_{42}(t) \\ f_{43}(t) & f_{44}(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \dfrac{q_2(s)}{p_4(s)} & \dfrac{a}{p_4(s)} \\ \dfrac{b}{p_4(s)} & \dfrac{q_1(s)}{p_4(s)} \end{pmatrix}$$

$x_{22}(t,\xi,\eta) = f_{23}(t)[\xi - \bar{x}_1] + f_{24}(t)[\eta - \bar{x}_2] + \bar{x}_2$,

$x_{13}(t,\xi,\eta) = f_{31}(t)[\xi - \bar{x}_1] + f_{32}(t)[\eta - \bar{x}_2] + \bar{x}_1$,

$t_2(\xi,\eta)$, be the smallest solution of the equation in t_2,

$$f_{21}(t_2)[\xi - \bar{x}_1] + f_{22}(t_2)[\eta - \bar{x}_2] + \bar{x}_1 = \rho,$$

$t_3(\xi,\eta)$, be the smallest solution of the equation in t_3,

$$f_{33}(t_3)[\xi - \bar{x}_1] + f_{34}(t_3)[\eta - \bar{x}_2] + \bar{x}_2 = \rho,$$

$t_{41}(\xi,\eta)$, be the smallest solution of the equation in t_{41},

$$f_{41}(t_{41})[\xi - \bar{x}_1] + f_{42}(t_{41})[\eta - \bar{x}_2] + \bar{x}_1 = \rho,$$

$t_{42}(\xi,\eta)$, be the smallest solution of the equation in t_{42},

$$f_{43}(t_{42})[\xi - \bar{x}_1] + f_{44}(t_{42})[\eta - \bar{x}_2] + \bar{x}_2 = \rho,$$

$$T_1(\xi,\eta) = \frac{(k_4 - k_3)(\xi - \bar{x}_1) + (k_1 - k_2)(\eta - \bar{x}_2)}{k_1k_4 - k_2k_3},$$

$T_2(\xi,\eta) = t_2(\xi,\eta) + T_1(\rho, x_{22}(t_2(\xi,\eta),\xi,\eta))$,

$T_3(\xi,\eta) = t_3(\xi,\eta) + T_1(x_{13}(t_3(\xi,\eta),\xi,\eta),\rho)$,

$T_{41}(\xi,\eta) = t_{41}(\xi,\eta) + T_1(\rho, x_{22}(t_{41}(\xi,\eta),\xi,\eta))$,

$T_{42}(\xi,\eta) = t_{42}(\xi,\eta) + T_1(x_{13}(t_{42}(\xi,\eta),\xi,\eta),\rho)$.

If the initial condition is inside the set ①, then:

$$MTTF_1[x_1(0),x_2(0)] := \int_0^{\infty} R_1'(t)dt = CK^{-1}[x(0) - \bar{x}] = T_1[x_1(0),x_2(0)]$$

If the initial condition is inside the set ②, then:

$MTTF_2[x_1(0),x_2(0)] = T_2(x_1(0),x_2(0))$.

If the initial condition is inside the set ③ then:

$MTTF_3[x_1(0),x_2(0)] = T_3(x_1(0),x_2(0))$.

If the initial condition is inside the set ④ then:

$$MTTF_4[x_1(0),x_2(0)] = \begin{cases} T_{41}(x_1(0),x_2(0)) & if \quad \bar{t}_{41} < \bar{t}_{42} \\ \bar{t}_4 + T_1(\rho,\rho) & if \quad \bar{t}_{41} = \bar{t}_{42} = \bar{t}_4 \\ T_{42}(x_1(0),x_2(0)) & if \quad \bar{t}_{41} > \bar{t}_{42} \end{cases}$$

where $\bar{t}_{41} = t_{41}(x_1(0),x_2(0))$ e $\bar{t}_{42} = t_{42}(x_1(0),x_2(0))$.

6 Controller realizability

During the design of the control law (9), (11), (12), (13), three groups of conditions among the controller parameters K and $\bar{x}$ and the system parameters A and μ have been found (asymptotic stability, physical realizability[1] and positivity of inputs), it is licit to ask if such a set of bounds can be compatible.

Not all points $\bar{x}$ inside triangle (4) can be chosen: the valid ones are those of the set

$$\mu_1 - d_1\bar{x}_1 + a\bar{x}_2 \geq 0, \quad \mu_2 + b\bar{x}_1 - d_2\bar{x}_2 \geq 0. \tag{14}$$

Moreover, the bounds that the components of K must satisfy are, in the order:

$$-a \leq k_2 \leq \min\left\{\mu_1, \frac{\mu_1 - d_1\bar{x}_1}{\bar{x}_2}\right\}, \quad -b \leq k_3 \leq \min\left\{\mu_2, \frac{\mu_2 - d_2\bar{x}_2}{\bar{x}_1}\right\}, \tag{15}$$

$$\max\left\{d_1, \frac{-b\bar{x}_2k_2}{\mu_2 + b\bar{x}_1}, \frac{(1-\bar{x}_2)}{\bar{x}_1}k_2, -\frac{\bar{x}_2}{\bar{x}_1}k_2\right\} \leq k_1 \leq \frac{\mu_1 - k_2\bar{x}_2}{\bar{x}_1}, \quad \max\left\{d_2, \frac{-a\bar{x}_1k_3}{\mu_1 + a\bar{x}_2}, \frac{(1-\bar{x}_1)}{\bar{x}_2}k_3, -\frac{\bar{x}_1}{\bar{x}_2}k_3\right\} \leq k_4 \leq \frac{\mu_2 - k_3\bar{x}_1}{\bar{x}_2}, \tag{16}$$

$$k_1k_4 - k_2k_3 > 0. \tag{17}$$

It can be proved that if $\bar{x}$ satisfies (14), then there exist values k_2 and k_3 satisfying (15) and, consequently, values k_1 and k_4 satisfying (16). Moreover, it is possible to give examples of more restrictive conditions such that (17) is automatically true.

<u>Ex. 1</u>: if (15) are true and

$$d_1d_2 - k_2k_3 > 0. \tag{18}$$

It follows that (16) imply (17).

<u>Ex. 2</u>: if (15) are true and

$$\frac{k_2\bar{x}_2}{\mu_1} + \frac{k_3\bar{x}_1}{\mu_2} < 1 \tag{19}$$

and

$$k_1 = \frac{\mu_1 - k_2\bar{x}_2}{\bar{x}_1}, \quad k_4 = \frac{\mu_2 - k_3\bar{x}_1}{\bar{x}_2}, \tag{20}$$

then (17) is automatically true.

<u>Ex. 3</u>: if $k_2 = -a$, $k_3 = -b$ and k_1, k_4 satisfy (16), then (17) is automatically true.

[1] Physical realizability is necessary to assure that the reference model be a correct physical model, for example for the set ① $\mu_{d1}=\mu_{m11}$, $\mu_{d2}=\mu_{m12}$, $\lambda_{d1}=\mu_{m11}-k_2$, $\lambda_{d2}=\mu_{m12}-k_3$, $\nu_{d3}=k_1+k_3-(\mu_{m11}+\mu_{m12})$, $\nu_{d4}=k_4+k_2-(\mu_{m11}+\mu_{m12})$.

7 Probabilities estimation

By applying the model to many identical devices, an initial estimation of the probability $\tilde{x}_i(t)$, that the generic device at the time instant t, be in the state S_i, can be obtained with the estimator ([2]):

$$\tilde{x}_i(t) = \frac{n_i(t)}{n(t)}, \tag{21}$$

where $n_i(t)$ is the number of devices in the state S_i at time t, and $n(t)$ is the number of devices managed at the same time. Note that $\tilde{x}_i(t) = x_i(t) + \delta_i(t)$, where $\delta_i(t)$ is a bounded random variable describing the difference between the real value of $x_i(t)$ and its estimation. Now, rather than using standard techniques (i.e.: Kalman-Bucy filter [2]), the intrinsic stability of the model is used to obtain the estimator:

$$d\hat{x}(t)/dt = A\hat{x}(t) - \hat{X}(t)u(t) + \mu, \quad \hat{x}(0) = \tilde{x}(0) = x(0) + \delta(0), \tag{22}$$

Such that

$$\|\delta(t)\| \leq \alpha\|\delta(0)\|e^{-\beta\cdot t},$$

where $\delta = \hat{x} - x$ (note that such bound on the estimation error is not obtained using probabilistic but deterministic arguments due to the boundedness of the error δ and, in particular of its initial value δ(0)).

By using the estimator (22), the control laws (9), (11), (12), (13), can be modified with the substitution to the real state x its estimation $\hat{x}$. Note that, for the conditions (14)–(17), the components of u are non-negative for any $\hat{x}$ inside triangle (4), and then also the motion of the closed loop remains inside such triangle for any $\hat{x}$, even though, for $\hat{x}$ chosen without a certain strategy (such as (22)) the convergence to $\bar{x}$ is not assured.

8 Control strategy

The following considerations are limited to the set ①, since, for initial conditions inside of it and any positive input, the evolution of system (6) is confined inside such a set.

Using $\hat{x}$ obtained from (22) into (9) one has

$$u(t) = [\hat{X}(t)]^{-1}\left[(A+K)\hat{x}(t) + \mu - K\bar{x}\right], \tag{9'}$$

that, applied to (6), gives

$$\|x(t) - \bar{x}\| \leq \|x(0) - \bar{x}\|e^{-\beta_k t} + \alpha\theta\|\varepsilon(0)\|\left[\frac{e^{-\beta_k t} - e^{-\beta t}}{\beta - \beta_k}\right] \tag{23}$$

where $\theta = \|A+K-U\|_{\max} = max\{\|A+K-U\| : \hat{x} \in ①\}$, u in the matrix U is given by (9′) and $\beta_k \neq \beta$ depends on K.

The second term of (23) decreases rapidly after a maximum at time

$$T = \frac{1}{\beta - \beta_k}\log\left(\frac{\beta}{\beta_k}\right) \approx m_1\Delta t$$

(for the effective implementation a small, but finite, time interval Δt must be used).

The strategy of the controller is that of applying (9′) for time intervals larger than m_2T (with m_1, m_2 and $m := m_1 \cdot m_2$, suitable integers), and then to control the course of the error $\|\tilde{x} - \hat{x}\|$. If this error do not remain bounded, the estimator (22) must be reinitialised with

$$\hat{x}(m\Delta t) = \tilde{x}(m\Delta t)\,. \qquad (24)$$

In fact, since the $x_i(t)$ are the probability distributions of some random variables t_i (giving the exit time instants from the state S_i), i.e. $x_i(t)=P(t_i \le t)$, it is possible to define the following random variables $\varepsilon_{ai} := \max_{t\in R^+}\{|\tilde{x}_i(t) - \hat{x}_i(t)|\}$ and $\varepsilon_i(t) := \max_{\tau \le t}\{|\tilde{x}_i(\tau) - \hat{x}_i(\tau)|\}$, such that $\varepsilon_i(t) \le \varepsilon_{ai}$, $\forall t \ge 0$. Then for a sufficiently large $n(t)$ (such that $\varepsilon_{ai} > 1/\sqrt{n(t)}$), one has $P(\varepsilon_i(t) \le c_i(t)) = \gamma_i(t)$, or $\tilde{x}_i(t) \in [\hat{x}_i(t) - c_i(t), \hat{x}_i(t) + c_i(t)]$ with confidence interval $\gamma_i(t)$, where ([2])

$$c_i(t) = \sqrt{-\frac{1}{2n(t)}\log\frac{1-\gamma_i(t)}{2}}\,.$$

If $n(t)$ and $\gamma_i(t)$ do not change with time, also $c_i(t)$ is constant. Then (if there is a reliable knowledge of the parameters A and μ), if after a time $m\Delta t$, one has that

$$\varepsilon_i(t) > c_i, \qquad (25)$$

the reinitialisation (24) should be done.
For the actual realisation of (9′), (22), note that an explicit expression of $u=u(t)$ can be obtained by integrating such equations. Then the check times t_{u_1} and t_{u_2}, with probability distributions derived from (9′), (22), are the minimal time instants for which:

$$\int_0^{t_{u1}(N)} u_1(\tau)\,d\tau = -\log(1-\delta_N),$$
$$\int_0^{t_{u2}(M)} u_2(\tau)\,d\tau = -\log(1-\varphi_M), \qquad (26)$$

where δ_N and φ_M are independent random numerical sequences with uniform distribution in [0,1] and N and M are two indexes.

Conclusion

One of the more important logistic problems regards the provision management of devices. The planning of the control checks is settled by safety standards in such a way to be correlated to the expected mean time to failure (MTTF).
Thanks to a probabilistic model ([4]-[6]) of the lifecycle of a repairable device, it is possible to implement a control strategy that modifies the MTTF, to reduce the rate of the corrective maintenance interventions.
This fact allows to find those devices which need more resources, by making control checks with a suitable rate.
The control law proposed modifies some probabilities by means of the random repetition of checks with established distribution.

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