On the use of Coons' interpolation in CAD/CAE applications

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Abstract: - This paper presents the several possibilities of using the Coons' interpolation in both CAD and CAE applications. It is shown that not only curvilinear surfaces may be interpolated (as it is well known from the literature), but also finite element meshes may be developed in any arbitrarily-shaped domain. Moreover, it is shown how it is possible to built-up isoparametric macro-finite-elements with degrees of freedom appearing at the boundaries only, using global shape functions that are based on the same interpolation. The theory is sustained by one typical two-dimensional application of a U-notched elastic structural member.

Key-Words: - CAD/CAE, Coons' interpolation, Finite Element Method (FEM), Mesh generation, Macro-elements, Elastic Structures, Notched members, Stress concentration.

1 Introduction

The technique of bivariate «blending» function interpolation of S.A. Coons [1], was developed and applied to geometrical problems of computer-aided design and numerically controlled machining of free-form surfaces. Since then, the method has been used also in many engineering applications that require the description of three-dimensional surfaces in a form suitable for numerical analysis, such as in the finite element method and in mesh generation problems [17-19].

In two-dimensional problems, the domain is a patch, sometimes defined by its four surrounding sides, which may be meshed using any coordinate-mapping technique such as blending processes [8-10] and others [6,27].

With respect to the polynomial degree of the finite elements, after many years of using small-size isoparametric ones [6], the development of "large" elements with the purpose of reducing mesh generation work load, the total number of degrees of freedom, as well as the computational effort in both static and dynamic regimes, has kept researchers busy for a long time. Historically, it was Irons [7] who generalized the idea of arbitrarily noded elements, but also blending function methods based on the ideas put forward in [8] have been used to produce some interesting element families [9,10,11].

On the other hand, "large" elements were introduced by schemes [12-15] based on Trefftz's method [4]. These, as well as the Boundary Element Method (BEM) [16], require knowledge of the fundamental solution of the problem under consideration.

Coons's interpolation method has been generalised in a unique formula that describes C^0-, C^1-, C^2- etc. continuity of the first-, second- and third-derivative, respectively [17]. In the context of the FEM, Coons's interpolation is practically used for mesh generation in structured four-sided curvilinear patches [18-20].

El-Zafrany and Cookson [11] use Coons's idea for two-dimensional problems in conjunction with Lagrange and Hermite interpolation functions, allowing a small number of degrees of freedom per element. Also, Zhaobei and Zhiqiang [24] apply Coons's surface method to fit boundary conditions in some families of finite element of plates and shells. The use of large B-splines finite-elements based on Coon's interpolation theory, with degrees of freedom appearing only at the element boundaries has appeared in two-dimensional potential [25] and elasticity problems [26].

In this paper, the theory of Coons' is briefly presented for two-dimensional interpolation problems with C^0-continuity in curvilinear coordinates. Then, it is explained how it is possible to use this formula in order to generate a structured finite element mesh. A smoothening procedure that may significantly improve the quality of the mesh is discussed, too. In the sequence, the general theory of creating "large" finite elements (macro-elements) is presented and global cardinal (1-0-type) shape functions are illustrated. Finally, the theoretical aspects are applied to a structural member with two U-notches of rectangular section.

2 Problem Formulation

2.1 Coons' interpolation

Let us assume a patch that is defined in the (r,s)-coordinate system, with r and s being its normalised $(0 \le r, s \le 1)$ curvilinear co-ordinates. Furthermore, it is assumed that the coordinate vector $\mathbf{x}(r, s)$ of any point on this surface can be expressed in a closed, analytical form on the four "boundaries" $r = 0,1$ and $s = 0,1$ of the patch:

$$\mathbf{x}(0, s), \mathbf{x}(1, s), \mathbf{x}(r, 0), \mathbf{x}(r, 1) \tag{1}$$

Let now P_r be a "projector" interpolating $\mathbf{x}(r, s)$ along the boundaries $r = 0$ and $r = 1$. We will have it defined by the formula

$$P_r[\mathbf{x}] = E_0(r)\,\mathbf{x}(0, s) + E_1(r)\,\mathbf{x}(1, s) \tag{2}$$

Similarly, a projector $P_s[\mathbf{x}]$ interpolates $\mathbf{x}(r, s)$ along the boundaries $s = 0$ and $s = 1$,

$$P_s[\mathbf{x}] = E_0(s)\,\mathbf{x}(r,0) + E_1(s)\,\mathbf{x}(r,1) \tag{3}$$

Furthermore, the "product projector",

$$P_r\ P_s[\mathbf{x}] = \sum_{i=0}^{1}\sum_{j=0}^{1} E_i(r)E_j(s)\,\mathbf{x}(r_i,\ s_j) \tag{4}$$

interpolates $\mathbf{x}(r, s)$ at the four corners of the unit square $[0,1]\times[0,1]$. Finally, the "Boolean sum projector",

$$\mathbf{x}(r,s) = P_r[\mathbf{x}] + P_s[\mathbf{x}] - P_r P_s[\mathbf{x}] \tag{5}$$

interpolates $\mathbf{x}(r, s)$ over the entire domain $[0,1]\times[0,1]$ (see Fig.1: $\mathbf{u}$ has replaced $\mathbf{x}$, as is required in Section 2.3).

By the above interpolation technique, it is essentially intended to approximate the unknown function $\mathbf{x}(r,s)$ using on the one hand information given at the element's boundaries and on the other hand certain auxiliary scalar functions $E_j(h)$, where h is either r or s and j=0,1. These functions are termed "blending" functions and are taken to satisfy the cardinality conditions:

$$E_j(h_i) = d_{ij} \tag{6}$$

where d_{ij} is the Kronecker's delta.

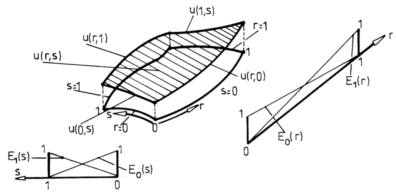


Fig.1: Boundary curves $u(0,s)$, $u(1,s)$, $u(r,0)$, $u(r,1)$ and 'blending functions' E_0 and E_1 of the macro-element.

For C^0-continuity discrete problems the blending function are linear and equal to:

$$E_0(h) = 1 - h,\ E_1(h) = h \tag{7}$$

2.2 Construction of a finite element mesh

The formula (5) can be applied an arbitrarily-shaped domain that may be considered as a four-sided two-dimensional patch. To do this, it is only necessary to know the coordinates along the four sides (eq.(1)). Then, using (5) a one-to-one mapping is established between the real $\mathbf{x}(x,y)$ and the normalized $\mathbf{r}(r,s)$ coordinates, the last belonging to a square (ABCD) of unit side (master or reference element, e.g. Fig.4b). If the number of the boundary nodes on the opposite sides (AB,CD) and (BC,DA) is the same, then r=const. and s=const. represent lines perpendicular to the lines AB and BC, respectively.

Generally, the so-produced mesh is irregular. However, due to its structured nature it is possible to carry out a smoothening as follows. By considering that eight other nodes surrounding each node in the interior of the patch, one can update these coordinates by the formula

$$\mathbf{x}^{new} = (1/8)\sum_{j=1}^{8}\mathbf{x}_j \tag{8}$$

This procedure is of low cost and leads to smooth meshes that may be used for the FEM analysis where conventional isoparametric or triangular elements are considered.

2.3 Construction of the global shape functions

As it has been previously presented [25,26], the above-mentioned patch may be discretized at its boundaries only, and constitute a macro-element.

According to this approach, the displacement vector $\boldsymbol{u}(x,y)$ inside the macro-element that covers the whole domain or a sub-domain W is approximated, again, by (5), where **x** is now replaced by **u**:

$$\mathbf{u}(r,s) = P_r[\mathbf{u}] + P_s[\mathbf{u}] - P_r P_s[\mathbf{u}] \qquad (9)$$

or, equivalently

$$\begin{aligned}\mathbf{u}(r,s) &= E_0(r)\mathbf{u}(0,s) + E_1(r)\mathbf{u}(1,s) \\ &\quad + E_0(s)\mathbf{u}(r,0) + E_1(s)\mathbf{u}(r,1) \\ &\quad - \sum_{i=0}^{1}\sum_{j=0}^{1} E_i(r)E_j(s)\mathbf{u}(r_i,s_j) \\ &= \sum_{k=1}^{K} N_k(r,s)\mathbf{u}_k \end{aligned} \qquad (10)$$

Moreover, if now $B_j(n)$, where n is either r or s, denote cardinal splines of degree m i.e.

$$B_j(n_i)=\ddot{a}_{ij} \qquad (11)$$

then the functions $\boldsymbol{u}(0, s)$, $\boldsymbol{u}(1, s)$, $\boldsymbol{u}(r, 0)$ and $\boldsymbol{u}(r, 1)$ may be written in the following form:

$$\begin{aligned} \mathbf{u}(0,s) &= \sum_{j=1}^{q} B_j(s)\mathbf{u}(0,s_j) \quad & \mathbf{u}(1,s) &= \sum_{j=1}^{q} B_j(s)\mathbf{u}(1,s_j) \\ \mathbf{u}(r,0) &= \sum_{j=1}^{q} B_j(r)\mathbf{u}(r_j,0) \quad & \mathbf{u}(r,1) &= \sum_{j=1}^{q} B_j(r)\mathbf{u}(r_j,1) \end{aligned} \qquad (12)$$

Finally, by substituting (12) in (10), one can finally obtain

$$\mathbf{u}(r,s) = \sum_{k=1}^{K} \mathbf{N}_k(r,s)\mathbf{u}_k(r) \qquad (13)$$

with $\mathbf{N}_k(r,s)$ denoting the macro-element shape function, $\mathbf{u}_k(r)$ nodal degrees of freedom appearingonly at the boundaries of the macro element.

The advantage of using this kind of interpolation is that undesirable oscillations between two arbitrary abscissae h_i and h_{i+1} (as in Lagrangian interpolation) are avoided. A typical global function is illustrated in Fig.2 and Fig.3 for a 24-node macro-element.

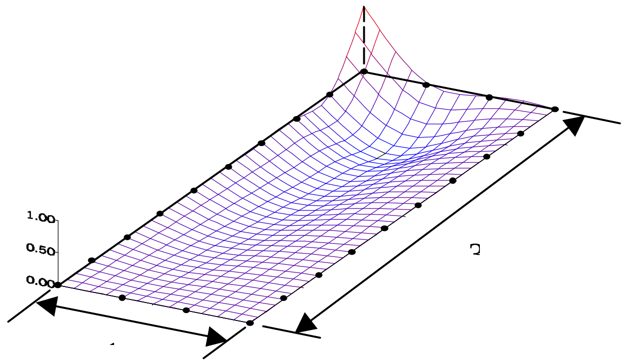


Fig.2: Shape function at a corner node of a 24-noded macro-element.

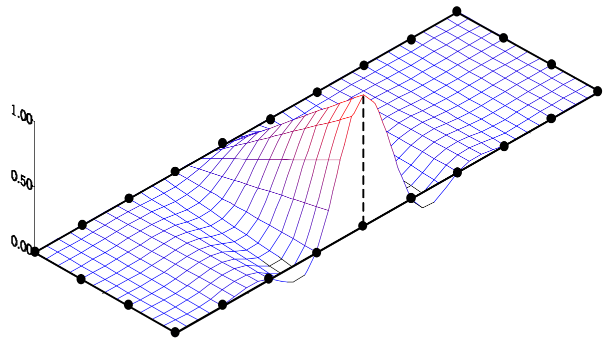
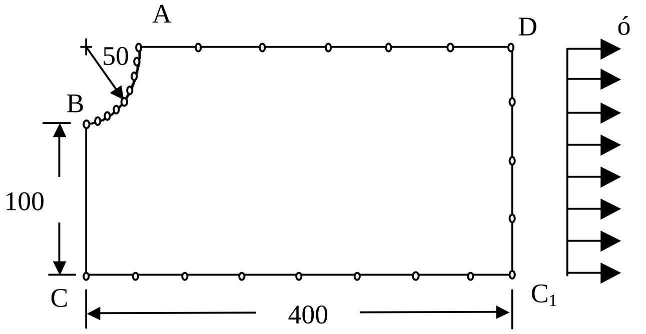


Fig.3: Shape function at an intermediate node of a 24-noded macro-element.

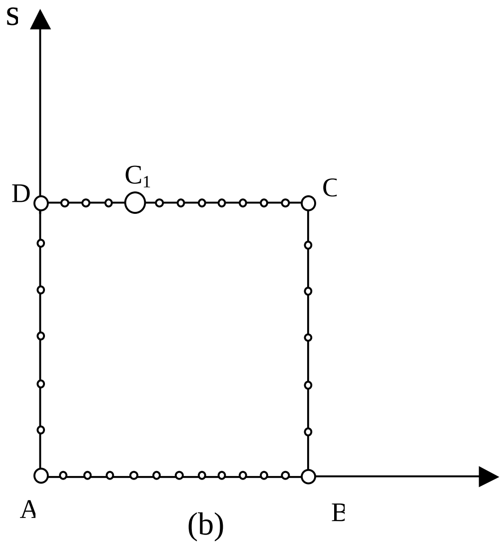
3 Numerical examples

In mechanical engineering, it is usually necessary to estimate the stress concentration factor k, which is defined as the ratio of the maximum stress in the stress raiser to the nominal stress, using the net cross section. In the past, photoelastic methods had been applied and a list of many such factors may be found in the literature [28].

As a typical case, we choose the case of two U-notches in a member of rectangular section. The geometry of a relevant specimen is shown in Fig.4. In order to examine the efficiency of the mesh generation technique and the macro-element, a coarse mesh of only thirty-six boundary nodes is considered for one-fourth of the U-notch, as it is illustrated in Fig.4 (a). In the same figure, one can notice the correspondence between the real boundaries and the four sides of the reference square (ABCD), shown in Fig.4(b). It should be noticed that the real side CD is composed of two straight segments (CC1) and (C1D).



(a)



(b)

Fig.4: (a) Geometry and boundary discretization of a U-notched specimen (one fourth), and
(b) Reference square-element (*r*,*s*)

The application of Coons' formula (5) leads to a irregular finite element mesh of seventy-two elements and ninety-one nodes, as shown in Fig.5. However, after forty-three smoothening iterations according to (8), the mesh becomes significantly regular as shown in Fig.6.

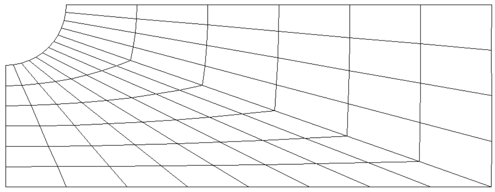


Fig.5: Initial finite element mesh

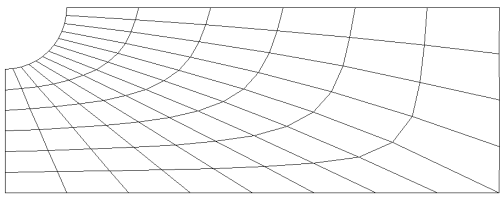


Fig.6: Final finite element mesh, after smoothening.

The accuracy of the macro-element is presented in Table 1. One can notice that the displacements of both the conventional mesh and the macro-element are close each other, again, for the same number of boundary nodes. It can noticed that in both cases the accuracy is adequate, compared with a large finite element model of 6203 nodes and 6041 four-noded elements using the FEM-code ALGOR (U.S.A.).

Table 1. Mean averaged value of the calculated displacements along the side (C1D) using macro-elements (Fig.4a) and conventional four-noded elements (Fig.6).

TYPE OF ELEMENTS	Displacement
Macro element (Fig.4a)	.437
Four-node elements (Fig.6)	.438
ALGOR	.440

With respect to the stress concentration factor, two cases are presented below. In Table 2, we present the stress ratio at the Gauss points that correspond to the center of the twelfth (last) element along the arc AB, adjacent to the point B, as well as the next three ones. The stress ratio is defined as the ratio of the current stress towards the direction of loading over the externally applied stress. Obviously, the maximum stress ratio coincides with the stress concentration factor *k*. One can notice that results are close each other.

Table 2. Calculated stress ratio at the center of the elements adjacent to the side (BC) using macro-elements (Fig.4a) and conventional four-noded elements (Fig.6).

TYPE OF ELEMENTS	Stress Ratio
Macro element (Fig.4a)	2.099 1.609 1.398 1.312
Four-node elements (Fig.6)	2.229 1.604 1.370 1.267

Finally, Table 3 presents the stress ratio at the boundary nodes. This is possible for the macro-elements, where strain and stress are continuous functions, while in conventional elements extrapolation is required.

Table 3. Calculated stress ratio at the center of the elements adjacent to the side (C1D) using macro-elements (Fig.4a), compared with ALGOR.

TYPE OF ELEMENTS	Stress Ratio
Macro element (Fig.4a)	2.918 2.956 1.758 1.509
ALGOR (stress concentration factor)	3.072

Obviously, the calculated stress concentrated factor through the macro-element approach (k=2.956) is very close to that, which was calculated with ALGOR with much more elements, as was previously mentioned.

4 Conclusion

It was shown that Coons' interpolation formula, which was firstly applied in automotive CAD/CAM applications, is a powerful tool for also CAE applications. After a suitable smoothening, the formula is capable of constructing regular structured meshes that are near to orthogonal ones. The domain may be arbitrarily-shaped and must be seen as a four-sided patch. It should become clear that each side of the patch is not necessarily a unique entity, arc or straight line, e.t.c., but it can be also a composite line made of several different segments. Moreover, it was shown that Coons' interpolation formula gives the capability to built-up global cardinal shape functions of (1-0)-type. The theory was successfully tested on a U-notched specimen, where displacements and the stress concentration factor were calculated.

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