AN INTERVAL METHOD FOR GLOBAL INEQUALITY-CONSTRAINT OPTIMIZATION PROBLEMS

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Abstract - A simple interval method is suggested for globally solving optimization problems of the following type: minimize the objective function $\varphi_0(x)$ subject to the functional constraints $\varphi_i(x) \leq 0, i=1, \dots, r$ and the boundary constraints $x \in X^{(0)}$ where x is a n -dimensional vector and $X^{(0)}$ is a given initial search region (a box). All the functions involved are assumed continuously differentiable in $X^{(0)}$.

Known interval methods for solving the global optimization problem considered are iterative and are based on a specific interval linearization of all the functions associated with the problem at each iteration of the method. This linearization involves one real term and n interval terms. In contrast, the present method appeals to a new interval linearization of each function which is constructed in such a way that now only one additive term is an interval while the remaining n terms are real numbers.

The algorithm of the iterative method suggested includes the following basic procedures to be carried out at each iteration:

- (i) if the current subbox X is strictly feasible, then several tests for unconstrained minimization are applied and an upper bound on the global minimum is also found;
- (ii) if X is infeasible, X is discarded;
- (iii) if the first two cases do not occur, then a system of inequalities, including the objective function and some of the functional inequalities, is linearized and simultaneously solved in an attempt to reduce or completely discard X . The system involves at most n inequalities.

The use of the new interval linearization in the computational scheme of the present method leads to improved performance as compared to other known interval methods of the same class. CSCC'99 Proc.pp.3661-3664

Key-Words: global optimization, inequality-constraint problems, interval methods, interval linearization.

1 INTRODUCTION

Interval methods (methods applying interval analysis techniques) are capable of globally solving various optimization problems with mathematical certainty [1]-[5].

In this paper, a new interval method is suggested for global solution of the following inequality-constraint optimization problem:

Minimize

$$\varphi_0(x) \quad (1a)$$

subject to the constraints

$$\varphi_i(x) \leq 0, \quad i = 1, 2, \dots, r \quad (1b)$$

$$x \in X^{(0)} \subset R^n \quad (1c)$$

where x is a n -dimensional vector and $X^{(0)}$ is a given initial search region (a box). The functions in

(1a) and (1b) are assumed continuously differentiable (CD) in $X^{(0)}$.

Various methods have been suggested for solving (1) globally [1]-[5]. Let $f(x)$ denote any function in (1). Known interval methods for solving (1) are all based on the linearization of f in X as follows:

$$F(X) = f(x^c) + \sum_{i=1}^n G_i(X)(X_i - x_i^c) \quad (2)$$

where x_i^c are the components of the centre x^c of the box $X = (X_1, \dots, X_n)$ with components X_i and $G_i(X)$ is the interval extension of the derivative $g_i(x) = \partial f / \partial x_i$ or, better, the interval extension of the corresponding slope [3] in X (other more sophisticated extensions of the type (2) have also been suggested and used [1]-[5]). An elaborate algorithm for solving (1) by applying (2) is

presented in [3]. It is, however, too complicated for practical purposes. Indeed, it involves as an integral part a procedure for solving a nonlinear system derived on account of the so-called Fritz-John conditions [3]. Since (1c) is in fact the short notation of $2n$ inequalities:

$$\bar{x}_i - \underline{x}_i \leq 0, \quad i = 1, \dots, n \quad (3a)$$

$$\underline{x}_i - \bar{x}_i \leq 0, \quad i = 1, \dots, n \quad (3b)$$

where $\underline{x}_i$ and $\bar{x}_i$ are the end-points of X_i , the Fritz-John system will be a system of m equations in m unknowns with $m = 3n + r + 1$. To reduce the amount of computation required, an approach called "peeling of the boundary" which skips the inequalities (3) has been proposed in [5]. However, the simpler problem (1a), (1b) is to be solved 3^n times which is still prohibilively inefficient for larger n .

In this paper, a simple method for solving (1) is suggested. It is based on a new type of linear interval linearization of $f(x)$ in X in the form:

$$F(X) = \sum_{i=1}^n \alpha_i X_i + B, \quad x_i \in X_i \quad (4)$$

where α_i are real numbers and only B is an interval. The use of (4) in the computational scheme of the new method leads to improved performance as compared to other known interval methods. The present method is characterized by the fact that it involves the solution of a system of inequalities whose size does not exceed n .

2 NEW LINEAR INTERVAL

LINEARIZATION OF NONLINEAR FUNCTIONS

The transformation of a nonlinear function $f(x)$, $x \in X$, to the new linear interval form (4) can be done following the approach suggested in [6]-[9]. If $f(x)$ is a separable function, then α_i and B , which are in fact dependent on X , i.e. $\alpha_i = \alpha_i(X)$ and $B = B(X)$, can be determined by a procedure given in [6], [7]. This procedure is applicable both to CD components $f_i(x)$ of f and to continuous (C) components as well. This is a major advantage because many optimization problems involve C functions such as $\text{abs}(\cdot)$, etc. If $f(x)$ is not in separable form, the linearization (4) is obtained in the following manner. Based on the approach adopted in [8], [9], f is first transformed equivalently into a system of $q+1$ functions by introducing q new auxiliary variables. More specifically

$$f(x) = f_0(y) = f_0(x, u) \quad (6a)$$

where $y = (x, u)$ and $u = (u_1, \dots, u_q)$,

$$u_p = f_p(y) = f_p(x, u), \quad p = 1, \dots, q \quad (6b)$$

Each function $f_p(y)$ in (6) is in the so-called semiseparable form

$$f_p(y) = \sum_{i=1}^m f_{pi}(y_i) + \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m \alpha_{ij}^{(p)} y_i y_j, \quad p = 0, 1, \dots, q \quad (7)$$

The transformation of $f(x)$ into (6), (7) can be done automatically on a computer. Each function $f_p(y)$ is then transformed readily into (4). Indeed, the first separable form term in (7) is handled in the same way as in [5] while the products in (7) are transformed as shown in [8]. Thus, system (6) is replaced by the following linear interval system

$$F_o(X, U) = \sum_{i=1}^n \alpha_{oi} X_i + \sum_{i=n+1}^q \alpha_{oi} U_i + B_o \quad (8a)$$

$$U_p = \sum_{i=1}^n \alpha_{pi} X_i + \sum_{i=n+1}^q \alpha_{pi} U_i + B_p \quad (8b)$$

Now, (8b) is substituted into (8a) to finally obtain

$$F(X) = F_o(X) = \sum_{i=1}^n \alpha_i X_i + B \quad (9)$$

Example 1. Let

$$f(x) = (x_1 + x_2)x_3, \quad x_i \in X_i, \quad (10)$$

We introduce a new variable $u_1 = x_1 + x_2$. Hence (10) is transformed into the system

$$f(x) = f_o(y) = f_o(x, u_1) = u_1 x_3, \quad (11a)$$

$$x_3 \in X_3, \quad x_2 \in X_2$$

$$u_1 = x_1 + x_2, \quad x_1 \in X_1, \quad x_2 \in X_2 \quad (11b)$$

The first function $f_o = u_1 x$, is in semiseparable form and can be linearized as [8]:

$$F_o = x_3^c U_1 + u_1^c X_3 + B_o \quad (12)$$

where x_3^c and u_1^c are the centres of the corresponding intervals X_3 and U_1 . Function (11b) is in separable form and

$$U_1 = X_1 + X_2 \quad (13)$$

On substituting (13) into (12), we get

$$F(X) = F_o(X) + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + B_o \quad (14)$$

This simple example illustrates the process of transforming an arbitrary nonlinear function into the linear interval form (4). This process can be formalized and fully automated.

3 ALGORITHM OF THE METHOD

The algorithm of the present iterative method is based on the execution, at each iteration, of several basic procedures. The linear interval extensions used

throughout these procedures will be constructed in the new form (4).

Let $X \subseteq X^{(0)}$ be the current box to be analysed. The box X is strictly feasible (s.f.) [3] if all the inequalities

$$\begin{aligned} \Phi_i(X) &< 0, \quad \forall i = 1, \dots, r \\ x_i - \bar{x}_i &< 0, \quad \forall i = 1, \dots, n \\ -x_i + \underline{x}_i &< 0, \quad \forall i = 1, \dots, n \end{aligned} \quad (15)$$

are valid. If X is s.f., then the following 3 procedures are carried out (otherwise the iterative process continues with Procedure 4 to be presented below).

Procedure 1. Let $\bar{\varphi}_0$ denote an upper bound on the global minimum φ_0^* , obtained at a previous iteration (at the first iteration $\bar{\varphi}_0$ is set to $+\infty$). Now we compute $\varphi_0(x^c)$ where x^c is the centre of X and update $\bar{\varphi}_0$, i.e.

$$\bar{\varphi}_0 = \varphi_0(x^c), \quad \text{if } \varphi_0(x^c) < \bar{\varphi}_0, \quad (16)$$

otherwise $\bar{\varphi}_0$ remains unchanged.

Procedure 2. (Monotonocity test [1], [3]). If for some i

$$G_i(X) > 0 \text{ or } G_i(X) < 0 \quad (17)$$

where $G_i(X)$ is the interval extension of $\partial\varphi_0 / \partial x_i$ then X is discarded.

Procedure 3. (Nonconvexity test [3]). Let $H_{ii}(X)$ be an interval extension of $h_{ii}(x) = \partial^2 \varphi_0 / \partial x_i^2$. If

$$H_{ii}(X) < 0 \quad (18)$$

for some i , then X is discarded [3].

Procedure 4. (Extended infeasibility test).

Let

$$\varphi_0(x) := \varphi_0(x) - \bar{\varphi}_0 \quad (19)$$

If for some i from $0, 1, \dots, r$

$$\Phi_i(X) > 0 \quad (20)$$

the current box X is discarded

If one of the above three tests (17), (18) or (20) is satisfied, a new box X is retrieved from a queue Q of boxes to be processed and the iterative process continues with checking feasibility. Otherwise, we proceed to the following procedure.

Procedure 5. The set of intervals $\Phi_i(X) = [\bar{\Phi}_i, \underline{\Phi}_i]$, $i=0, 1, \dots, r$ is ordered as follows. We first compute the positive coefficients

$$s_i = \frac{\bar{\Phi}_i}{\bar{\Phi}_i - \underline{\Phi}_i} \geq 0, \quad (21)$$

only for those indices i for which $\bar{\Phi}_i \geq 0$. Let the number of these coefficients be r' . Then s_i are

ordered by magnitude. We choose the first p numbers where $p=n$ if $r' \geq n$ or $p=r'$ if $r' < n$. The corresponding $\varphi_i(x)$ are linearized by using (4) and the following system of p inequalities in n variables is set up

$$\sum_{j=1}^p \alpha_{kj} x_j + B_k \leq 0, \quad k = 1, \dots, p \quad (22)$$

Procedure 6. System (22) is now "solved" in the sense of [3]. To do this, first (22) is transformed (fully or partially) to a "triangular" form using a slightly modified Gauss-Seidel elimination. The modification involves the following two restrictions: (i) the pivot element can be only positive, (ii) in choosing the pivot only column interchanges are allowed. After the triangularization all the remaining columns are moved to the right of the inequality sign and the corresponding x_j are replaced by X_j . Thus, the following system is obtained

$$Tx \leq B' \quad (23)$$

where (for simplicity) T is assumed to be a fully triangularized real matrix and B' is an interval vector. System (23) is easily "solved" by a backward sweep starting from the last equation:

$$t_{pp} x_p \leq B'_p \quad (24)$$

The solution of (24) is the interval

$$Y_p \leq B'_p / t_{pp} \quad (25)$$

Now Y_p is intersected with X_p , i.e.

$$X'_p = Y_p \cap X_p \quad (26)$$

and the result is replaced in the $(p-1)$ th row of (24). Thus, a new solution Y_{p-1} is obtained and the computation process continues until $p=1$.

Remark. The remaining steps of the algorithm of the present method such as control of the accuracy, splitting the current box whenever necessary etc. are omitted in order not to clutter the presentation.

4 NUMERICAL EXAMPLE

To illustrate the new method the following example will be considered [3].

Example 2. The function to be minimized is $\varphi_0(x) = x_1^6 - 6.3x_1^4 + 12x_1^2 + 6x_1x_2 + 6x_2^2$ (27a) subject to the constraints

$$\varphi_1(x) = 1 - 16x_1^2 - 25x_2^2 \leq 0$$

$$\varphi_2(x) = 13x_1^3 - 145x_1 + 85x_2 - 400 \leq 0 \quad (27b)$$

$$\varphi_3(x) = x_1x_2 - 4 \leq 0$$

There are two global solutions:

$$x_1^* = \pm 0.06604, \quad x_2^* = 0.1929 \quad (28)$$

and the global minimum is

$$\varphi_0^* = 0.1990 \quad (29)$$

The problem has been solved for various starting boxes $X^{(0)}$. The results are encouraging. Thus, for $X_1^{(0)} = X_2^{(0)} = [-2, 4]$ the method required $N_i=1198$ iterations to find the global solutions (28), (29) and took $t=4.12$ sec. when run on a 100 MHz DX486 computer. As the starting box $X^{(0)}$ was enlarged 4 times, the run time was increased only 1.39 times.

5 CONCLUSION

A new interval method for globally solving inequality constrained optimization problems defined as in (1) has been suggested. The method has one major feature which distinguishes it favourably from other known interval methods for global optimization. It consists in the fact that the interval extensions $\Phi_i(X)$ of the functions $\varphi_i(x)$, $i=0, \dots, r$, associated with the optimization problem, are determined in a new way, using the linear interval form (4) suggested in the paper. The new form differs from the known form (2) in that the coefficients α_i before the variables x_i are real numbers and only the additive term B is an interval while in (2) all the coefficients are intervals and only the additive term is a real number. This explains the improved performance of the present method, especially in solving systems of inequalities that arise in Procedure 5 from Section 3.

The applicability of the method has been illustrated by a numerical example having three functional nonlinear constraints.

An improvement of the numerical performance of the present method seems to be possible by introducing the so-called constraint propagation (exploited already in [9] for solving system of equalities) into the computational scheme of Procedure 6 of the method where systems of inequalities (22) are to be solved.

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