Sensitivity of Lyapunov Equations

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Abstract: A new approach for sensitivity analysis of general Lyapunov matrix equations is presented. Both local and non-local perturbation bounds are obtained.

Keywords: Perturbation analysis, Lyapunov matrix equations.

1. Introduction

In this paper we study the sensitivity of general Lyapunov matrix equations (LME). The continuous- and discrete-time Lyapunov equations, arising in linear systems theory, are particular cases of LME.

The following notations are used later on: $R^{m \times n}$ – the space of real $m \times n$ matrices; $I_n$ – the unit $n \times n$ matrix; $A^T = [a_{ij}]$ – the transpose of the matrix $A = [a_{ij}]$, $\text{vec}(A) \in R^{mn}$ – the column-wise vector representation of the matrix $A \in R^{m \times n}$; $\Pi_{n^2} \in R^{n^2 \times n^2}$ – the vec-permutation matrix, such that $\text{vec}(X^T) = \Pi_{n^2} \text{vec}(X)$ for all $X \in R^{n \times n}$; $A \otimes B = \begin{bmatrix} [a_{ij}]B \end{bmatrix}$ – the Kronecker product of the matrices $A$ and $B$; $\| \cdot \|_2$ – the spectral norm in $R^{m \times n}$; $\| \|_F$ – the Frobenius norm in $R^{m \times n}$. We write $A \preceq B$ if $a_{ij} \leq b_{ij}$ and $A \succeq 0$ if $A = A^T$ is a non-negative definite matrix. The notation ‘$=\,$’ stands for ‘equal by definition’.

2. Problem Statement

Consider the general LME

$$F(X, P) := A_0 + \sum_{i=1}^{r_1} (A_i XB_i^T + B_i X A_i^T) + \sum_{k=1}^{r_2} C_k X C_k^T = 0, \quad (1)$$

where $X \in R^{n \times n}$ is the unknown matrix. The function $F(\cdot, P) : R^{n \times n} \rightarrow R^{n \times n}$ is an affine symmetric matrix operator, depending on the parameter matrix $(1 + 2r_1 + r_2)$-tuple

$$P := (A_0; A_1, B_1, \ldots, A_{r_1}, B_{r_1}; C_1, \ldots, C_{r_2}),$$

where $A_0 = A_0^T$. With certain abuse of notation we shall eventually identify $P$ with the set of the matrix coefficients, and shall write $A_j \in P$, etc.

We assume that the partial Fréchet derivative $F_X$ of $F$ in $X$ is invertible. Then equation (1) has an unique solution $X$ for each $A_0$ and, in view of $A_0 = A_0^T$, we have $X = X^T$.

The perturbation problem for equation (1) is stated as follows. Let the matrices from $P$ be perturbed as

$$A_0 \mapsto A_0 + \delta A_0, \quad A_i \mapsto A_i + \delta A_i, \quad B_i \mapsto B_i + \delta B_i, \quad C_k \mapsto C_k + \delta C_k,$$

where $\delta A_0 = (\delta A_0)^T$. Denote by $P + \delta P$ the perturbed collection $P$, in which each matrix $Z \in P$ is replaced by $Z + \delta Z$. Then the perturbed equation is

$$F(Y, P + \delta P) = 0. \quad (2)$$

Since the operator $F_X$ is invertible, the perturbed equation (2) has an unique solution $Y = X + \delta X$, $Y =$
Y^T, in the neighbourhood of X if the perturbation \( \delta P \)
is sufficiently small.

Denote by

\[
\Delta := [\delta A_0, \delta A_1, \delta B_1, \ldots, \delta C_r, \ldots, \delta C_r]^\top \in \mathbb{R}^r
\]

the vector of absolute norm perturbations \( \delta Z := \| \delta Z \|_F \) in the data matrices, where \( r := 1 + 2r_1 + r_2 \).

The perturbation problem is to find a bound

\[
\delta X \leq f(\Delta), \quad \Delta \in \Omega \subset \mathbb{R}^r,
\]

for the perturbation \( \delta X := \| \delta X \|_F \), where \( \Omega \) is a given set and \( f \) is a continuous function, non-decreasing in each of its arguments and satisfying \( f(0) = 0 \). A first order local bound

\[
\delta X \leq f_1(\Delta) + O(\| \Delta \|^2), \quad \Delta \to 0,
\]

shall be first derived, which will then be incorporated in the non-local bound (3). The inclusion \( \Delta \in \Omega \) also guarantees that the perturbed equation (2) has an unique solution \( Y = X + \delta X \).

3. Local perturbation analysis

Consider first the conditioning of the LME (1). Since \( F(X, P) = 0 \), the perturbed equation (2) may be written as

\[
F(X + \delta X, P + \delta P) := F_X(\delta X) + \sum_{Z \in P} F_Z(\delta Z) + G(\delta X, \delta P) = 0,
\]

where

\[
F_Z(\cdot) = F_Z(X, P)(\cdot), \quad Z \in P,
\]

are the partial Fréchet derivatives of \( F(X, \cdot) \) in the corresponding matrix arguments \( Z \), computed at the point \( (X, P) \). Note that \( F_X(\cdot) \) is a linear symmetric operator and \( F_{A_i}(\cdot) \) is the identity operator. The matrix \( G(\delta X, \delta P) \) contains second and higher order terms in \( \delta X, \delta P \).

A straightforward calculation leads to

\[
F_X(Z) = \sum_{i=1}^{r_1} (A_i Z B_i^T + B_i Z A_i^T) + \sum_{k=1}^{r_2} C_k Z C_k^T,
\]

\[
F_{A_i}(Z) = Z,
\]

\[
F_{B_i}(Z) = Z B_i^T + B_i Z^T,
\]

\[
F_{C_k}(Z) = Z C_k^T + C_k Z^T.
\]

Since the operator \( F_X(\cdot) \) is invertible we get

\[
\delta X = \Phi(\delta X, \delta P) := -\sum_{Z \in P} F_Z^{-1} \circ F_Z(\delta Z) - F_X^{-1}(G(\delta X, \delta P)).
\]

The relation (4) gives

\[
\delta X \leq \sum_{Z \in P} K_Z \delta Z + O(\| \Delta \|^2), \quad \Delta \to 0,
\]

where the quantities

\[
K_Z := \| F_X^{-1} \circ F_Z \|, \quad Z \in P,
\]

are the absolute individual condition numbers of LME (1). Here \( \| F \| \) is the norm of the operator \( F \), induced by the F-norm. If \( X \neq 0 \) then an estimate in terms of relative perturbations is

\[
\delta_X := \frac{\| \delta X \|_F}{\| X \|_F} \leq \sum_{Z \in P} k_z \delta_z + O(\| \Delta \|^2), \quad \Delta \to 0,
\]

where the scalars

\[
k_z := K_Z \frac{\| Z \|_F}{\| X \|_F}, \quad Z \in P,
\]

are the relative individual condition numbers with respect to perturbations in the matrix coefficients \( Z \in P \).

The calculation of the condition numbers \( K_Z \) is straightforward. Denote by \( M_X, M_{A_i}, \ldots, M_{C_k} \) the matrix representations of the operators \( F_X(\cdot), F_{A_i}(\cdot), \ldots, F_{C_k}(\cdot) \):

\[
M_X = \sum_{i=1}^{r_1} (A_i \otimes B_i + B_i \otimes A_i) + \sum_{k=1}^{r_2} C_k \otimes C_k,
\]

\[
M_{A_i} = I_{n_i},
\]

\[
M_{B_i} = (B_i X) \otimes I_{n_i} + (I_{n_i} \otimes (B_i X)) P_{i, n_i},
\]

\[
M_{C_k} = (I_{n_i} \otimes P_{i, n_i}) (C_k X) \otimes I_{n_i}.
\]

Then

\[
K_{A_i} = I := \| M_X^{-1} \|_2, \quad K_Z = \| M_X^{-1} M_Z \|_2, \quad Z \in P \setminus \{A_0\}.
\]

Local estimates, based on individual condition numbers, may eventually produce pessimistic results. At the same time it is possible to derive local, first order homogenous estimates, which are better in general. The operator equation (4) for the perturbation \( \delta X \) may be written in a vector form as

\[
\text{vec}(\delta X) = \sum_{Z \in P} N_Z \text{vec}(\delta Z) - M_X^{-1} \text{vec}(G(\delta X, \delta P)),
\]

where

\[
N_Z := -M_X^{-1} M_Z, \quad Z \in P.
\]
The condition number based estimate is a corollary of (5):

\[ \delta_X = ||\delta X||_F = ||\text{vec}(\delta X)||_2 \]
\[ \leq \text{est}_1(\Delta, N) + O(||\Delta||^2) \]
\[ := \sum_{Z \in P} ||N_Z||_2 \delta Z + O(||\Delta||^2) \]
\[ = \sum_{Z \in P} K_Z \delta Z + O(||\Delta||^2), \Delta \to 0, \]

since \( ||\text{vec}(\delta Z)||_2 \leq \delta Z \), where

\[ N := \{N_1, N_2, \ldots, N_r\} \]
\[ := \{N_{Z_1}, N_{Z_2}, \ldots, N_{Z_r}\} \in \mathbb{R}^{2 \times r n^2}. \]

Relation (5) also gives

\[ \delta_X \leq \text{est}_2(\Delta, N) + O(||\Delta||^2) \]
\[ := ||N||_2 ||\Delta||_2 + O(||\Delta||^2), \Delta \to 0. \]

The bounds \( \text{est}_1(\Delta, N) \) and \( \text{est}_2(\Delta, N) \) are alternative, i.e., which one is less depends on the particular value of \( \Delta \). There is also a third bound, which is always less than or equal to \( \text{est}_1(\Delta, N) \). We have

\[ \delta_X \leq \text{est}_3(\Delta, N) := \sqrt{\Delta^T S(N) \Delta} + O(||\Delta||^2), \Delta \to 0, \]

where \( S(N) \) is the \( r \times r \) matrix with elements \( s_{ij}(N) = ||N_i^T N_j||_2 \). Since

\[ ||N_i^T N_j||_2 \leq ||N_i||_2 ||N_j||_2 \]

we get \( \text{est}_3(\Delta, N) \leq \text{est}_1(\Delta, N) \). Hence we have the overall estimate

\[ \delta_X \leq \text{est}(\Delta, N) + O(||\Delta||^2), \Delta \to 0, \quad (6) \]

where

\[ \text{est}(\Delta, N) := \min\{\text{est}_2(\Delta, N), \text{est}_3(\Delta, N)\}. \quad (7) \]

The local bound \( \text{est} \) in (6), (7) is a non-linear, first order homogeneous and piece-wise real analytic function in \( \Delta \).

4. Non-local Perturbation Analysis

Local bounds are usually used neglecting terms of order \( O(||\Delta||^2) \), i.e., they are valid only asymptotically, for \( \Delta \to 0 \). Unfortunately, it is usually impossible to say, having a small but a finite perturbation \( \Delta \), whether the neglected terms are indeed negligible. We can not even claim that the magnitude of the neglected terms is less, or of the order of magnitude of the local bound. Even for simple linear equations the local bound may always underestimate the actual perturbation in the solution for a class perturbations in the data. Moreover, for some critical values of the perturbations in the coefficient matrices the solution may not exist (or may go to infinity when these critical values are approached). This will be the case when the perturbed linear operator \( F(\ast, P + \delta P) \) is not invertible, or is near to the set of non-invertible operators. Nevertheless, even in such cases the local estimates will still produce a ‘bound’ for a very large or even for a non-existing solution, which is a serious drawback.

The disadvantages of the local estimates may be overcome using the techniques of non-linear perturbation analysis. As a result we get a domain \( \Omega \subset \mathbb{R}_+ \) and a non-linear function \( f : \Omega \to \mathbb{R}_+ \) such that \( \delta_X \leq f(\Delta) \) for all \( \Delta \in \Omega \). In connection with this we would like to emphasis two important issues. First, the inclusion \( \Delta \in \Omega \) guarantees that the perturbed equation has a solution (this is an independent and essential point). And second, the estimate \( \delta_X \leq f(\Delta) \) is rigorous, i.e. it is true for perturbations with \( \Delta \in \Omega \), unlike the local bounds. However, in some cases the non-local bounds may not exist or may be pessimistic.

Let the collection \( \hat{P} \) be perturbed to \( P + \delta P \) and \( \tilde{Y} = X + \delta X \) be the solution of the perturbed equation (2). In what follows we shall mark only the dependence on the perturbations \( \delta X \) and \( \delta P \), recalling that \( X \) is a fixed solution of (1).

The perturbed equation (2) may be rewritten in the form

\[ \delta X = \Phi(\delta X, \delta P) := \Phi_0(\delta P) + \Phi_1(\delta X, \delta P), \quad (8) \]

where

\[ \Phi_0(\delta P) := -F_X^{-1}(G_0(\delta P)), \]
\[ \Phi_1(\delta X, \delta P) := -F_X^{-1}(G_1(\delta X, \delta P)) \]

and

\[ G_0(\delta P) = \delta A_0 + R_1(X, \delta P) + R_2(X, \delta P), \]
\[ G_1(\delta X, \delta P) = R_1(\delta X, \delta P) + R_2(\delta X, \delta P). \]

Here \( R_k(\ast, \delta P) \) are linear operators of asymptotic order \( k \) relative to \( \delta P, \delta P \to 0 \), determined from

\[ R_1(Z, \delta P) := \sum_{i=1}^m \left( \delta A_i Z B_i^T + A_i Z \delta B_i^T + \delta B_i Z A_i^T + B_i Z \delta A_i^T \right) + \sum_{k=1}^n \left( \delta C_k Z C_k + C_k Z \delta C_k \right), \]
\[ R_2(Z, \delta P) := \sum_{i=1}^{r_1} (\delta A_i Z \delta B_i^T + \delta B_i Z \delta A_i^T) + \sum_{k=1}^{r_2} \delta C_k Z \delta C_k^T. \]

Suppose that \( \|Z\|_F \leq \rho \). Then we have
\[
\|\Phi_1(\delta P)\|_F \leq a_1(\Delta), \\
\|\Phi_1(Z, \delta P)\|_F \leq a_1(\Delta)\rho,
\]
where
\[
a_0(\Delta) := a_{01}(\Delta) + a_{02}(\Delta), \quad (9) \\
a_1(\Delta) := a_{11}(\Delta) + a_{12}(\Delta). \]

The quantities \( a_{1k}(\Delta) \) are of asymptotic order \( O(|\Delta|^k) \) for \( \Delta \to 0 \) and are determined as follows.

The case \( i = 0 \):
\[
a_{01}(\Delta) := \text{est}(\Delta, N), \quad (10) \\
a_{02}(\Delta) := l\|X\|_2 \left( 2 \sum_{i=1}^{r_1} \delta A_i \delta B_i + \sum_{k=1}^{r_2} \delta C_k^2 \right). \]

The case \( i = 1 \):
\[
a_{11}(\Delta) := \sum_{i=1}^{r_1} \| M^{-1}_X (I_{n^2} + \Pi_{n^2}) (B_i \otimes I_{n}) \|_2 \delta A_i, \quad (11) \\
+ \sum_{i=1}^{r_1} \| M^{-1}_X (I_{n^2} + \Pi_{n^2}) (A_i \otimes I_{n}) \|_2 \delta B_i \\
+ \sum_{k=1}^{r_2} \| M^{-1}_X (I_{n^2} + \Pi_{n^2}) (C_k \otimes I_{n}) \|_2 \delta C_k, \\
a_{12}(\Delta) := l \left( 2 \sum_{i=1}^{r_1} \delta A_i \delta B_i + \sum_{k=1}^{r_2} \delta C_k^2 \right). \]

Let \( \|Z\|_F, \|\bar{Z}\|_F \leq \rho \). The Lyapunov majorant for equation (8) is a function \( (\rho, \Delta) \mapsto h(\rho, \Delta) \), defined on a subset of \( \mathbb{R}_+ \times \mathbb{R}_+^2 \) and satisfying the conditions
\[
\|\Phi(Z, \delta P)\|_F \leq h(\rho, \Delta)
\]
and
\[
\|\Phi(Z, \delta P) - \Phi(\bar{Z}, \delta P)\|_F \leq K_h(\rho, \Delta) \|Z - \bar{Z}\|_F.
\]

According to the above considerations, the Lyapunov majorant here is linear and is determined from
\[
h(\rho, \Delta) = a_0(\Delta) + a_1(\Delta)\rho.
\]

In this case the fundamental equation \( h(\rho, \Delta) = \rho \) for determining the non-local bound \( \rho = \rho(\Delta) \) for \( \delta X \) gives
\[
\delta X \leq f(\delta) := \frac{a_0(\Delta)}{1 - a_1(\Delta)}, \quad \Delta \in \Omega, \quad (12)
\]
where
\[
\Omega := \{ \Delta \geq 0 : a_1(\Delta) < 1 \} \subset \mathbb{R}_+. \quad (13)
\]