Robust Disturbance Decoupling with Simultaneous Exact Model Matching via Static Measurement Output Feedback

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Abstract: The problem of robust exact model matching with simultaneous robust disturbance decoupling for linear systems with nonlinear uncertain structure (NLUS) and with measurable and nonmeasurable disturbances is solved via an independent of the uncertainties static measurement output feedback law. The necessary and sufficient conditions for the problem to have a solution, are established. The general analytical expressions of the feedback matrices are derived.

Key-Words: -Disturbance decoupling, model matching, robust control, linear systems, NLUS, feedback.

1 Introduction
Consider the linear system with (NLUS):
\[ \dot{x}(t) = A(q)x(t) + B(q)u(t) + D_1(q)z_1(t) + D_2(q)z_2(t), \]
\[ y_M(t) = M(q)x(t), y(t) = C(q)x(t) \]
where \( x \in \mathbb{R}^n \) denotes the state vector, \( u \in \mathbb{R}^m \) denotes the measurement output vector, \( y_M \in \mathbb{R}^p \) denotes the performance output vector, \( z_1 \in \mathbb{R}^{1 \times n} \) denotes the measurable disturbance vector and \( z_2 \in \mathbb{R}^{1 \times n} \) denotes the nonmeasurable disturbance vector. The elements of the matrices \( A(q), B(q), C(q), D_1(q), D_2(q) \) and \( M(q) \) belong to \( \mathcal{Q}(q) \) i.e. they are nonlinear function matrices depending upon the uncertainty vector \( q = \{q_1, \ldots, q_l\} \in \mathcal{Q} \), where \( \mathcal{Q} \) is the uncertainty domain and \( \mathcal{Q}(q) \) is the set of nonlinear functions of \( q \).

The uncertainty domain can be any set, while the values of the function of \( \mathcal{Q}(q) \) are considered to be real. The uncertainties \( q_1, \ldots, q_l \) do not depend upon the time. With regard to the nonlinear structure of \( A(q), B(q), D_1(q), D_2(q), M(q) \) and \( C(q) \) it is mentioned that, no limitations or specifications (continuity, boundness, smoothness, etc) are required.

The problem of robust disturbance rejection is a significant control design requirement having attracted considerable attention [1]-[4]. In [1] and [2] sufficient conditions for the problem to have a solution are derived. In [3] the necessary and sufficient conditions for the solution of the problem are derived for the case of left-invertible systems involving uncertain structure of the polynomial type.

In [4] the robust disturbance rejection of left invertible systems with uncertain structure, of the general nonlinear type have extensively been solved. Robust elimination of the influence of disturbances to the system outputs is a desirable goal which however does not guarantee satisfactory control of the outputs. To this end the design requirement of having a desired closed loop input-output map is simultaneous requested. The specification of having an ideal closed loop transfer function is the well known exact model matching problem [5-7].

Robust exact model matching with simultaneous robust disturbance rejection via static state feedback has been solved in [8]. In the present paper the problem of robust disturbance rejection with simultaneous robust model matching (RDRRMM) for linear systems with NLUS and with measurable and nonmeasurable disturbances is extensively solved.

The contribution of the present paper consist in establishing the following aspects: The necessary and sufficient conditions for the problem to have a solution via an independent of \( q \) static measurement output feedback. The general analytical expressions of the independent of \( q \) feedback.

2 Main Result
The problem will be studied, via an independent of \( q \) static measurement output feedback law. In the frequency domain system (1.1) takes the form
\[ sX(s) = A(q) + B(q)U(s) + D_1(q)Z_1(s) + D_2(q)Z_2(s) \]
\[
Y_M(s) = M(q)X(s), \quad Y(s) = C(q)X(s)
\] (2.1)

To system (2.1) apply the regular static measurement output feedback law
\[
U(s) = F_1Y_M(s) + F_2Z(s) + G\Omega(s)
\] (2.2)

where \(\Omega(s) \in \mathbb{C}^m\) is the external input vector and \(G\) is assumed to be invertible in order to insure linear independence of the \(m\) external inputs. The RDRMM problem for systems with measurable and nonmeasurable disturbances is formulated as in the definition:

**Definition 3.1.** The RDRMM problem for linear systems with NLUS and with measurable and nonmeasurable disturbances of the form (2.1), via a feedback law of the form (2.2), consists in finding independent of \(q\) matrices \(F_1, F_2\) and \(G\) such that

\[
C(q)[sI - A(q) - B(q)F_1M(q)]^{-1} \times \\
[B(q)G \mid D_1(q) \mid D_2(q) ] = \\
H_M(s)\left[ \Gamma - \Phi M(q)(sI - A(q))^{-1}B(q) \right] \\
- \Phi M(q)(sI - A(q))^{-1}D_1(q) - \Theta] \\
- \Phi M(q)(sI - A(q))^{-1}D_2(q) \right) (2.4)
\]

where

\[
\Gamma = G^{-1}, \Phi = G^{-1}F_1, \Theta = G^{-1}F_2 
\] (2.5)

According to (2.4 and 5), \(\Gamma\) must be invertible, i.e.

\[
\det \Gamma \neq 0 
\] (2.6)

Expansion of both sides of (2.4) in negative power series of \(s\) yields

\[
C(q)\Delta(q)s^{-1} + C(q)A(q)\Delta(q)s^{-2} + \cdots = \\
\left( H_M(s)^{-1}H_M s^{-2} + \cdots \right) \times \\
\times \left( J - \Phi M(q)\Delta(q)s^{-1} - \Phi M(q)A(q)\Delta(q)s^{-2} - \cdots \right) (2.7)
\]

where

\[
\Delta(q) = [B(q) \mid D_1(q) \mid D_2(q) ], \\
J = [\Gamma \mid - \Theta \mid 0 ] \\
H_M(s) = \sum_{k=0}^{\infty} H_{M,s^{-k}}
\]

Equating coefficients of like powers of \(s\) in (2.7) derive an infinite set of algebraic equations. Since it suffices to keep only the \(2n+1\) equations, the following nonhomogeneous system of equations is derived

\[
\eta \Pi(q) = \xi(q) 
\] (2.8)

where

\[
\eta = [\gamma_1, \ldots, \gamma_m ] \mid \theta_1, \ldots, \theta_m \mid \phi_1, \ldots, \phi_m \] (2.9)

\[
\xi(q) = [c_1(q)A(q)^\eta\Delta(q) \cdots c_p(q)A(q)^\eta\Delta(q) \cdots] (2.10)
\]

\[
\Pi(q) = [\Pi_1, \ldots, \Pi_{2n+1} ] (2.11)
\]

where \(\gamma_i, \phi_i, \theta_i\) are the \(i\)-th rows of \(\Gamma, \Phi, \Theta; \Pi(q) = \sum_{k=0}^{\infty} H_{M,s^{-k}}\).
\[ Q_0 = \begin{bmatrix} (q_0)_1 \\ \vdots \\ (q_0)_m \end{bmatrix}, \quad S_0 = \begin{bmatrix} (r_0)_1 \\ \vdots \\ (r_0)_m \end{bmatrix}, \quad S_0 = \begin{bmatrix} (\tilde{r}_0)_1 \\ \vdots \\ (\tilde{r}_0)_m \end{bmatrix} \]  
(2.17)

where \( \hat{K}_i, K_i, \) and \( \bar{K}_i \) are submatrices of dimensions \( v \times m, v \times \zeta_i, \) and \( v \times \mu, \) respectively, \( (q_0)_i \in \mathbb{R}^{1 \times m}, \)
\( (r_0)_i \in \mathbb{R}^{1 \times \zeta_i}, \) \( (\tilde{r}_0)_i \in \mathbb{R}^{1 \times \mu} \) and where \( (k_i), (\hat{k}_i) \) and \( (\bar{k}_i) \) are the \( j \)-th rows of \( K_i, \bar{K}_i \) and \( \bar{K}_i \) respectively. Using the above definitions as well as (2.13), the general analytical expressions of the independent of \( q \) matrices \( \Gamma, \Theta \) and \( \Phi \) are derived to be

\[ \Gamma = Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_q Q_q \]  
(2.18)

\[ \Theta = S_0 + \sum_{i=1}^{v} \lambda_i S_1, \quad \Phi = S_0 + \sum_{i=1}^{v} \lambda_i S_1 \]  
(2.19)

Based upon the conditions (2.12) and (2.6) as well as the formula (2.18), the necessary and sufficient conditions for the problem to have a solution can be established. To this end, define

\[ T_j = \begin{cases} m & j = 0 \\ (T_j + 1) & j \geq 1 \end{cases}, \quad \rho_j = \begin{cases} m & j = 0 \\ (T_j + 1) & j \geq 1 \end{cases} \]  
(2.20)

\[ N_j(q) = \begin{bmatrix} Q_i \\ N_j(q - 1) \otimes \begin{bmatrix} 0 & 0 \\ I_{T_j+1} \end{bmatrix} \end{bmatrix} \]  
(2.21)

\[ P_j = \begin{cases} (q_0) & j = 0, \forall i > 0 \\ \bar{P} & j \geq 1, \forall i > 0 \end{cases} \]  
(2.22)

where \( \bar{P} = P_j \otimes \begin{bmatrix} 0 & 0 \\ I_{T_j+1} \end{bmatrix} + \sum_{i=1}^{v} \lambda_i (j - 1) \otimes \begin{bmatrix} I_{T_j+1} \\ 0 & 0 \end{bmatrix} \).

**Theorem 3.1.** The necessary and sufficient conditions for the solvability of the RDRRRM problem, via an independent of the uncertainties static measurement output feedback law, are

\[ \text{rank}_R \begin{bmatrix} \Pi(q) \\ \zeta(q) \end{bmatrix} = \text{rank}_R \begin{bmatrix} \Pi(q) \end{bmatrix} \]  
(2.23)

\[ \text{rank}_R [P_j] = \rho_j \]  
(2.24)

**Proof:** The condition (2.23) is identical to that in (2.12). If the condition (2.23) is satisfied the problem is reduced to that of finding a vector \( \gamma_i \) \((i = 1, \ldots, m)\) such that (2.6) is satisfied with \( \Gamma \) as in (2.18) or equivalently if (2.24) is satisfied. \( \blacksquare \)

**3.2 The feedback matrices**

In this subsection the general forms of the matrices \( G, F_1 \) and \( F_2 \) will be derived. The following theorem results directly from (2.5), (2.18) and (2.19).

**Theorem 3.2.** Assume that system (1.1) satisfies the conditions of Theorem 3.1., then the general analytical expressions of the independent of \( q \) feedback matrices \( G \) and \( F \), are

\[ G = (Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_q Q_q)^{-1} \]  
(2.25)

\[ F_1 = (Q_0 + \sum_{i=1}^{v} \lambda_i Q_i)^{-1}(S_0 + \sum_{i=1}^{v} \lambda_i S_i) \]  
(2.26a)

\[ F_2 = (Q_0 + \sum_{i=1}^{v} \lambda_i Q_i)^{-1}(S_0 + \sum_{i=1}^{v} \lambda_i S_i) \]  
(2.26b)

where

\[ \det(Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_q Q_q) \neq 0 \]  
(2.27)

The condition (2.27) upon the degrees of freedom of \( \lambda_i \) in the controller matrices \( G, F_1 \) and \( F_2 \) may be viewed as a forbidden hypersurface in the space \( \mathbb{R}^v \) of the arbitrary parameters \( \lambda_i \). Moving closer to this surface results in the norms of \( G, F_1 \) and \( F_2 \) approaching infinity. No matter how close to this surface we are, the problem has a solution, since we are not exactly on this surface. However, from a practical point of view, we should choose \( \lambda_i \) such that the norms of \( G, F_1 \) and \( F_2 \) have suitable values.

**4 Conclusions**

RDRRRM problem for systems with measurable and nonmeasurable disturbances for systems with nonlinear uncertain structure, via static state feedback, has extensively been solved. The necessary and sufficient conditions have been established. The general analytical expressions of the independent of uncertainties feedback matrices have been derived.

**References**


