Euler/Quasi-wavelet method for the variable order fractional advection-diffusion equation with a nonlinear source term

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Abstract: New numerical techniques are presented for the solution of a class of the variable order fractional advection-diffusion equation with a nonlinear source term on one-dimensional finite domain. Quasi-wavelet and double quasi-wavelet are used for the spatial discretization. For the time stepping, Euler method is considered. We also tested the method proposed on several problems with very promising results.

Key–Words: Variable order fractional advection-diffusion equation, Quasi-wavelet method, Double quasi-wavelet method, Forward Euler scheme.

1 Introduction

Differential equations of fractional order have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [1, 2]. Many people have researched the numerical approximation for these kinds of equations [4, 6, 7, 12, 26, 27, 28, 29, 30].

For the variable-order setting, which has received a significant amount of attention in recent years. [15] presented the concept of variable-order fractional integration and differentiation. Evans and Jacob [21] derived Feller semigroups by using variable-order subordination. [14] found some diffusion processes may be better described using variable order exponents. [16] discussed the selection and meaning of variable-order operators for dynamic modeling. The feature of the variable-order fractional derivatives make the design of accurate and fast methods difficult. So the numerical analysis of variable-order fractional partial differential equations is relatively new. Zhuang et al. [17] considered the explicit and implicit Euler approximations for the variable order fractional advection-diffusion equation. [8] proposed the finite difference methods for the variable order equation. [34] developed numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation.

The objective of our present work is to introduce the quasi-wavelets scheme for the space variable-order fractional advection-diffusion equation. As we all know, quasi-wavelet is an important tool for solving some partial integro-differential equations and has highly accuracy [9, 17, 18, 20], because quasi-wavelet based numerical method is easy to implement and has distinctive local property that can achieve accurate results. In this paper, we used the quasi-wavelet based numerical method twice to approximate the function, so we call it double quasi-wavelet based numerical method, detail could be found in the following content. Comparing with the quasi-wavelet based numerical methods results in [9, 31, 32], double quasi-wavelet based numerical method is an effective method to extending the computation time. To the best of the authors’ knowledge, there are no research papers using the quasi-wavelet and double quasi-wavelet numerical algorithm for solving the variable-order fractional advection-diffusion equation.

The rest of this paper is organized as follows. In section 2 gives some preliminary knowledge of variable order fractional derivative. We give a detailed description of spatial-temporal discretizations for the space variable-order fractional partial differential equation in section 3. In section 4, we present some numerical examples. We have a conclusion at last.
2 Preliminary knowledge

Firstly, we introduce some useful variable-order fractional derivatives.

Definition 1 (Riemann-Liouville derivative) [3]

\[ a^D_x f(x) = \frac{1}{\Gamma(m - a(x))} \frac{d^m}{dx^m} \int_a^x (x - \eta)^{m-a(x)-1} f(\eta) d\eta, \]

\[ b^D_x f(x) = \frac{(-1)^m}{\Gamma(m - a(x))} \frac{d^m}{dx^m} \int_x^b (\eta - x)^{m-a(x)-1} f(\eta) d\eta, \]

where \( m - 1 < a(x, t) < m, m > 0 \).

Definition 2 (Caputo derivative) [3]

\[ a^C_x f(x) = \frac{1}{\Gamma(m - a(x))} \int_a^x (x - \eta)^{m-a(x)-1} f^{(m)}(\eta) d\eta, \]

\[ b^C_x f(x) = \frac{(-1)^m}{\Gamma(m - a(x))} \int_x^b (\eta - x)^{m-a(x)-1} f^{(m)}(\eta) d\eta, \]

where \( m - 1 < a(x, t) < m, m > 0 \).

Definition 3 (Riesz derivative) [33]

\[ \frac{\partial^a(x)}{\partial |x|^{a(x)}} f(x) = \frac{1}{2} \left[ D_x^a(x) f(x) - \phi_\beta(x) \frac{\partial^a(x)}{\partial |x|^{a(x)}} f(x) \right], \]

where \( m - 1 < a(x, t) < m, m > 0 \).

Based on the aforementioned knowledge, in this paper, we consider the following variable-order advection-diffusion equation with a nonlinear source term

\[ \frac{\partial u(x, t)}{\partial t} = k(x, t) R_{a(x)} u(x, t) - \nu(x, t) \frac{\partial u(x, t)}{\partial x} + f(u, x, t), (x, t) \in \Omega = [a, b] \times [0, T], \]

with initial and boundary conditions are

\[ u(x, 0) = \phi(x), \quad x \in [a, b] \quad (2) \]

and

\[ u(a, t) = u(b, t) = 0, \quad t \in [0, T], \quad (3) \]

where \( 1 < \alpha < \alpha(x, t) \leq \bar{\alpha} \leq 2 \). \( \nu(x, t)(1 \leq \nu(x, t) \leq \bar{\nu}) \) denotes the average fluid velocity. \( f(u, x, t) \) is the nonlinear source and satisfies the Lipschitz condition

\[ |f(u_1, x, t) - f(u_2, x, t)| \leq L |u_1 - u_2|. \]

\[ 0 \leq k(x, t) \leq \bar{k} R_{a(x)} u(x, t) \]

is the variable-order fractional derivative defined by

\[ R_{a(x)} u(x, t) = c_+(x, t) D_x^a(x) u(x, t) + c_-(x, t) D_b^a(x) u(x, t), \quad (4) \]

where \( 0 < c_+(x, t) \leq c_1, 0 < c_-(x, t) \leq c_2 \). \( c_+(x, t) = 1, c_-(x, t) = 0 \) and \( c_+(x, t) = 0, c_-(x, t) = 1 \) represent the left-hand and right-hand Riemann-Liouville derivative respectively.

When \( c_+(x, t) = c_-(x, t) = -\frac{1}{2} \cos \frac{a(x)}{2} \),

\[ R_{a(x)} u(x, t) = -\left( \Delta \right)^{\frac{a(x)}{2}} u(x, t), \]

\[ = -\frac{1}{2 \cos \frac{a(x)}{2}} \left[ D_x^a(x) u(x, t) + D_b^a(x) u(x, t) \right], \]

\[ = \frac{\partial^a(x)}{\partial |x|^{a(x)}} u(x, t), \]

denotes the Riesz variable fractional derivative [33].

3 Quasi-wavelet based numerical method

In this section, we give a brief introduction to quasi-wavelet theory, detail could be found in [5, 19, 20, 23, 25]. We first introduce one of the most important examples of the delta sequence kernel of Dirichlet type, such that

\[ \delta_\beta(x) = \frac{1}{\pi} \int_0^\beta \cos(xy)dy = \frac{\sin(\beta x)}{\pi x}, \quad (5) \]

it called the Shannon’s delta sequence kernel, which is important and widely used for many research fields. Shannon’s delta sequence kernel produces a basis for the Paley-Wiener reproducing kernel Hilbert space \( B^2_\beta \), which is a subspace of Hilbert space \( L^2 \), so for any \( f(x) \in B^2_\beta \), it can be uniquely reproduced by

\[ f(x) = \int_{-\infty}^{+\infty} \delta_\beta(x - y) f(y)dy = \int_{-\infty}^{+\infty} \frac{\sin(\beta(x - y))}{\pi(x - y)} f(y)dy. \]
In computation, we choose the Nyquist frequency \( \beta = \frac{\pi}{\Delta} \), which provides the highest computational efficiency both on and off grid (\( \Delta \) is the arbitrary spatial grid size). With (5) in (6), we obtain [22, 24]

\[
f(x) = \sum_{k=-\infty}^{+\infty} \int_{x_k}^{x_{k+1}} \delta_\beta(x - x_k)f(x_k)
= \sum_{k=-\infty}^{+\infty} \frac{\sin(\pi(x - x_k)/\Delta)}{x(x - x_k)/\Delta} f(x_k), \tag{7}
\]

that is the Shannon's sampling theorem. \( x_k \) is an appropriate set of discrete points. However, in practical computations, the truncation error of Shannon's sampling formula is substantial, we choose the regularized Shannon's delta kernel

\[
\delta_{\Delta, \sigma}(x) = \frac{\sin(\pi x/\Delta)}{\pi x/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{8}
\]

\( \Delta \) is the grid spacing, \( \sigma \) is a regularization parameter, which determines the width of the Gaussian envelop and is often varied in association with the grid spacing \( \Delta \), i.e. \( \sigma = r \Delta \). The function \( \delta_{\Delta, \sigma}(x) \) is no longer the exact orthogonal wavelet scaling function, so we call it the quasi scaling function, which has the global accuracy and local flexibility for linear and nonlinear system. Since Gaussian regularizer has rapidly decay properties to reduce the truncation error, in practice, we only need to select 2W sampling points near \( x \) which could reach the calculation precision.

Then we construct quasi-wavelet based numerical scheme in the following content, using the aforementioned quasi-scaling function, an arbitrary continuous function \( f(x) \in L_2 \) and it's nth-order derivative are approximated by

\[
f^{(n)}(x) \approx \sum_{k=-W}^{+W} \delta_{\Delta, \sigma}^{(n)}(x - x_k)u(x_k), \quad (n \geq 0), \tag{9}
\]

where \( \delta_{\Delta, \sigma}^{(n)} \) could be presented as

\[
\delta_{\Delta, \sigma}(x) = \begin{cases} 
\frac{\sin(\pi x/\Delta)}{\pi x/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right), & (x \neq 0) \\
1, & (x = 0).
\end{cases} \tag{10}
\]

2W + 1 is the computational bandwidth around \( x \). The values about \( W, \Delta, \sigma \) are determined by the accuracy requirement, mathematical estimation of approximation errors about \( W, \Delta, \sigma \) could be found in [13].

For the following calculation needs, we give the \( \delta_{\Delta, \sigma}^{(1)}(x) \) and \( \delta_{\Delta, \sigma}^{(2)}(x) \) as follows [28]:

\[
\delta_{\Delta, \sigma}^{(1)}(x) = \begin{cases} 
-\frac{\cos(\pi x/\Delta)}{\pi x/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) + \frac{\sin(\pi x/\Delta)}{\pi x/\Delta} \left(1 - \frac{x^2}{2\sigma^2}\right), & (x \neq 0) \\
0, & (x = 0)
\end{cases} \tag{11}
\]

\[
\delta_{\Delta, \sigma}^{(2)}(x) = \begin{cases} 
-\frac{\sin(\pi x/\Delta)}{\pi x/\Delta} \exp\left(-\frac{x^2}{2\sigma^2}\right) - 2\cos(\pi x/\Delta) \left(1 - \frac{x^2}{2\sigma^2}\right) + \frac{\sin(\pi x/\Delta)}{\pi x/\Delta} \left(3 - \frac{x^2}{\sigma^2}\right), & (x \neq 0) \\
0, & (x = 0).
\end{cases} \tag{12}
\]

4 Proposed algorithms

In this section, we propose the quasi wavelet based numerical algorithms for solving the Eq(1). We discretize Eq(1) by using the forward Euler scheme for time and the quasi-wavelet scheme for space.

4.1 Discretization in time: a forward Euler scheme

Let \( t_k = k\tau; (k = 0, 1, \ldots) \), \( \tau \) means the time grid size, \( u^k(x) \) is an approximation to \( u(t_k, x) \) and \( f^k(x) = f(t_k, x) \).

We use the forward Euler method to approximate the first term in Eq.(1)

\[
u_t(x, t_k) \approx \frac{u^{k+1}(x) - u^k(x)}{\tau}, x \in [a, b], 0 \leq k. \tag{13}
\]

Substituting Eq.(13) into Eq.(1), we attain the temporal semi-discrete form for \( k > 0, x \in [a, b] \)

\[
u^{k+1}(x) \approx u^{k+1}(x) + \tau(k^k(x)R_{\alpha^k(x)}u^k(x) - \nu^k(x))\frac{\partial u^k(x)}{x} + f^k(x), \tag{14}
\]

for \( k = 0 \), we have

\[
u^1(x) \approx u^0(x) + \tau(0^0(x)R_{\alpha^0(x)}u^0(x) - \nu^0(x))\frac{\partial u^0(x)}{x} + f^0(x), x \in [a, b]. \tag{15}
\]
4.2 Discretization in space: quasi-wavelet based numerical method

We discretize the spatial-derivative of problem (1) by the described quasi-wavelet method in Section 3. Let \( x_i = ih, i = 0, 1, \ldots, N \), and \( h = (b - a)/N \) be the spatial step. \( u^k \) is an approximation to \( u(x_i, t_k) \) and \( f^k_j = f(x_j, t_k) \). The quasi-wavelet based numerical method indicates that only \( 2W \) grid points near the point \( x \) are needed to approximate the derivative function. For example, \( u^{(n)}(x_i) \), the \( n \)-th-order derivative of a function \( u(x) \) at the \( x_i \) point, for \( n \geq 0, i = 1, \cdots, N - 1 \), is approximated by

\[
u^{(n)}(x_i) \approx \sum_{s=-W}^{i+W} \delta^{(n)}_{h,\sigma}(x_i - x_s)u_s. \tag{16}\]

Using Eq.(16), we attain full-discrete form of \( v^k(x) \frac{\partial u^k(x)}{\partial x} \), for all \( k = 0, \cdots, i = 1, \cdots, N - 1 \),

\[

v^k(x_i)u^k(x_i) \approx \sum_{s=-W}^{i+W} \delta_{h,\sigma}^{(1)}(x_i - x_s)u^k_s
=

v^k \sum_{j=-W}^{W} \delta_{h,\sigma}^{(1)}(-jh)u^k_{i+j}. \tag{17}
\]

Combing Definition 2 with Eq.(4), for all \( k \geq 0 \), we have

\[
R_{a(x)}u^k(x) = c^k_+(x) D_x^a(x)u^k(x) + c^k_-(x) D_b^a(x)u^k(x)
=

\frac{c^k_+(x)}{\Gamma(2 - a^k(x))} \int_a^x (x - \eta)^{-a^k(x)} \eta \frac{d^2}{d\eta^2} u^k(\eta) d\eta
\]

Thus, by using Eq.(16) and considering the value of Eq.(19) at point \( x_i \), let \( s = i+j \), for all \( k \geq 0, i = 1, \cdots, N - 1 \), we have

\[
I_1 \approx \frac{c^k_+(x)}{\Gamma(2 - a^k(x))} \sum_{j=-W}^{W} \delta^{(2)}_{h,\sigma}(-jh) \int_a^x (x_i - \eta)^{-a^k_i} u(\eta) d\eta
=

\frac{c^k_+(x)}{\Gamma(2 - a^k(x))} \sum_{j=-W}^{W} \delta^{(2)}_{h,\sigma}(-jh) \sum_{l=0}^{i+j-1} \int_{x_i}^{x_{i+j}} (x_{i+j} - \eta)^{-a^k_{i+j}} u(\eta) d\eta. \tag{21}
\]

Let \( u(\eta) = \frac{u^k_i + u^k_{i+1}}{2} \) in Eq.(21), we attain

\[
I_1 \approx \frac{c^k_+(x)}{\Gamma(2 - a^k(x))} \sum_{j=-W}^{W} \delta^{(2)}_{h,\sigma}(-jh) \sum_{l=0}^{i+j-1} \int_{x_i}^{x_{i+j}} (x_{i+j} - \eta)^{-a^k_{i+j}} d\eta. \tag{22}
\]

Using property of Gamma function that \( (2 - a^k_i)\Gamma(2 - a^k_i) = \Gamma(3 - a^k_i) \), and rearranging terms in Eq.(22), for \( k \geq 0, i = 1, \cdots, N - 1 \), we can obtain full-discrete form of \( I_1 \)

\[
I_1 \approx \frac{c^k_+(x) h^{2-a^k_i}}{2\Gamma(3 - a^k_i)} \sum_{j=-W}^{W} \delta^{(2)}_{h,\sigma}(-jh) \sum_{l=0}^{i+j-1} \left[ u^k_i \sum_{l=0}^{i+j-1} (u^k_i + u^k_{i+1}) \sum_{l=0}^{i+j-1} \int_{x_i}^{x_{i+j}} (x_{i+j} - \eta)^{-a^k_{i+j}} d\eta \right]. \tag{23}
\]

Similarly, for \( k \geq 0, i = 1, \cdots, N - 1 \), the full-discrete form of \( I_2 \) can be rearranged as

\[
I_2 \approx \frac{c^k_+(x) h^{2-a^k_i}}{2\Gamma(3 - a^k_i)} \sum_{j=-W}^{W} \delta^{(2)}_{h,\sigma}(-jh) \sum_{p=0}^{N-1} \sum_{l=0}^{i+j-1} \left[ u^k_i \sum_{l=0}^{i+j-1} (u^k_i + u^k_{i+1}) \sum_{l=0}^{i+j-1} \int_{x_i}^{x_{i+j}} (x_{i+j} - \eta)^{-a^k_{i+j}} d\eta \right]. \tag{24}
\]
Thus, with the full-discrete form of $I_1$ and $I_2$ in Eq.(18), we obtain
\[
R_{a^k}(x)u^k(x) - \frac{c_{k,i}^2h^{2-a_i^k}}{2\Gamma(3-a_i^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \sum_{i=0}^{i+j-1} \left( u_i^k((i+j-1)^{2-a_i^k} - (i+j-l-1)^{2-a_i^k}) + u_{i+1}^k((i+j-l)^{2-a_i^k} - (i+j-l-1)^{2-a_i^k}) \right) \\
+ \frac{c_{-i}^2h^{2-a_{-i}^k}}{2\Gamma(3-a_{-i}^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \sum_{p=i+j}^{N-1} \left( u_p^k((p+1-i-j)^{2-a_i^k} - (p-1-i-j)^{2-a_i^k}) + u_{p+1}^k((p+1-i-j)^{2-a_i^k} - (p-1-i-j)^{2-a_i^k}) \right). \tag{25}
\]

By substituting Eq.(17), and Eq.(25) in Eq.(14), for $k \geq 0, i = 1, \cdots, N - 1$, we obtain the full-discrete form of Eq.(14) as follows
\[
u_{i+1}^k \approx u_i^k + \tau \left\{ f_{i}^k - v_{i}^k \sum_{j=-W}^{W} \delta_{h,\sigma}^{(1)}(-jh)u_{i+j}^k \right\} \\
+ \frac{k_{i}^2c_{i}^2h^{2-a_i^k}}{2\Gamma(3-a_i^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \sum_{i=0}^{i+j-1} \left( u_i^k((i+j-1)^{2-a_i^k} - (i+j-l-1)^{2-a_i^k}) + u_{i+1}^k((i+j-l)^{2-a_i^k} - (i+j-l-1)^{2-a_i^k}) \right) \\
+ \frac{k_{-i}^2c_{-i}^2h^{2-a_{-i}^k}}{2\Gamma(3-a_{-i}^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \sum_{p=i+j}^{N-1} \left( u_p^k((p+1-i-j)^{2-a_i^k} - (p-1-i-j)^{2-a_i^k}) + u_{p+1}^k((p+1-i-j)^{2-a_i^k} - (p-1-i-j)^{2-a_i^k}) \right). \tag{26}
\]

the boundary condition (3), $u(a, t) = u(b, t) = 0$, is easily discretized as
\[
u_0^k = u_N^k = 0, \quad k = 0, 1, \cdots. \tag{27}
\]

Because some values of $u_i$ are usually undefined outside the spatial interval $[a, b]$, and the boundary condition is assumed Dirichlet type, we set
\[
u_i^k = \begin{cases} 
    u_0^k, & i < 0. \\
    u_N^k, & i > N. 
\end{cases} \tag{28}
\]

If some interested readers want to learn more detail about method for mathematical treatments for boundary conditions, you can refer to [19].

### 4.3 Discretization in space: double quasi-wavelet based numerical method

In this part, our purpose is to propose double quasi-wavelet based numerical scheme of Eq.(14) in space. As we all know, quasi-wavelet based has distinctive local property that can achieve accurate results, so we use quasi-wavelet based numerical method instead of $u(\eta) = \frac{u_i^k + u_{i+1}^k}{2}$ to approximate the $u(\eta)$ in (21). As the $v^k(x)\frac{u^k}{\partial x}$ term in Eq.(14) have the first-order derivative, that is say, quasi-wavelet discrete scheme is the same as double quasi-wavelet numerical scheme for this term. We only need to give the double quasi-wavelet based numerical format for the $R_{a^k}(x)$. From Eq.(21), for all $k \geq 0, i = 1, \cdots, N - 1$, we know
\[
I_1 \approx \frac{c_{k,i}^2h^{2-a_i^k}}{2\Gamma(3-a_i^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \int_a^{x_{i+j}} (x_{i+j} - \eta)^{1-a_i^k} u(\eta) d\eta. \tag{29}
\]

Using Eq.(9) to approximate $u(\eta)$ in Eq.(29), we have
\[
I_1 \approx \frac{c_{k,i}^2h^{2-a_i^k}}{2\Gamma(3-a_i^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \int_a^{x_{i+j}} (x_{i+j} - \eta)^{1-a_i^k} \sum_{q=-W_1}^{W_1} u_q^k \delta(\eta - x_q) d\eta. \tag{30}
\]

$u_q^k$ is irrelevant to $\eta$, thus Eq.(30) can be rewritten as
\[
I_1 \approx \frac{c_{k,i}^2h^{2-a_i^k}}{2\Gamma(3-a_i^k)} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-jh) \sum_{l=0}^{i+j-1} \sum_{q=-W_1}^{W_1} u_q^k \int_{x_l}^{x_{l+1}} (x_{i+j} - \eta)^{1-a_i^k} \delta(\eta - x_q) d\eta. \tag{31}
\]

In Eq.(31), integral term can be approximated by the Simpson numerical integral and left quadrangle numerical integral in first $i + j - 1$ intervals and the last interval, respectively, for all
we obtain the full-discrete form of Eq.(14)

\[ u_{i}^{k+1} \approx u_{i}^{k} + \tau \{ - \gamma_{i}^{k} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(1)}(-j)u_{i+j}^{k} + f_{i}^{k} \] 

\[ + \frac{h^{2-a_{i}}}{6\Gamma(2-a_{i})} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-j) \sum_{q=-W_{i}}^{W_{i}} u_{q}^{k}(x_{i+j} - x_{q}) \] 

Furthermore, Eq.(32) can be rewritten as

\[ I_{1} \approx \frac{h^{2-a_{i}}}{6\Gamma(2-a_{i})} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-j) \sum_{q=-W_{i}}^{W_{i}} u_{q}^{k}(x_{i+j} - x_{q}) \]

\[(i + j - 1)^{1-a_{i}} \delta_{h,\sigma}((-l - 1)q) + (i + j - l - 1)^{1-a_{i}} \delta_{h,\sigma}((l + 1 - q)h) \]

\[+ \frac{h^{2-a_{i}}}{\Gamma(2-a_{i})} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-j) \sum_{q=-W_{i}}^{W_{i}} u_{q}^{k} \delta_{h,\sigma}((i + j - 1 - q)h). \] (33)

Similarly, for \( I_{2} \), we attain

\[ I_{2} \approx \frac{h^{2-a_{i}}}{6\Gamma(2-a_{i})} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-j) \sum_{p=-1+j+1}^{N-1} \sum_{q=-W_{i}}^{W_{i}} u_{q}^{k} \]

\[(p - i - j)^{1-a_{i}} \delta_{h,\sigma}((p - q)h) + (p + 1 - i - j)^{1-a_{i}} \delta_{h,\sigma}((p - q + \frac{1}{2}h) + (p + 1 - i - j)^{1-a_{i}} \delta_{h,\sigma}((p - q)h) \]

\[+ \frac{h^{2-a_{i}}}{\Gamma(2-a_{i})} \sum_{j=-W}^{W} \delta_{h,\sigma}^{(2)}(-j) \sum_{q=-W_{i}}^{W_{i}} u_{q}^{k} \delta_{h,\sigma}((i + j + 1 - q)h). \] (34)

Substituting Eq.(33), (34) into Eq.(18), then combing with Eq.(17), for \( 0 \leq k \), \( 0 \leq i \leq N - 1 \), we obtain

\[ k \geq 0, i = 1, \ldots, N - 1, \]

5 Numerical experiments

We provide some numerical examples in this section to demonstrate the effectiveness and accuracy of the proposed method. We set \( a(x, t) = 1.5 + 0.4\sin(0.5\pi t x) \) in the following numerical examples, for simplicity, uniform grid is considered in all calculations. Taking \( t_{n} = n\tau, n = 0, \ldots, M, x_{i} = hi, i = 0, \ldots, N, h = 1/N \), and we mainly consider the following errors

\[ L_{2} = \sqrt{\sum_{i=0}^{i=N} (u_{i}^{\text{compt}} - u_{i}^{\text{exact}})^{2}}. \]

\[ L_{\infty} = \max_{i=0}^{i=N} |u_{i}^{\text{compt}} - u_{i}^{\text{exact}}| \]

\[ L_{\infty} = \max_{i=0}^{i=N} |u_{i}^{\text{compt}} - u_{i}^{\text{exact}}| \]
5.1 The numerical solutions using quasi-wavelet approximation

In this part, we present the numerical results to illustrate the effectiveness of the quasi-wavelet method. We remark here that $\tau$ must be small enough, as it has been proved that the truncation error of the algorithm using the regularized Shannon's kernel decays exponentially when the spatial sampling point increases [26], that is to say, only if we use a small time increment, the stability condition of Euler difference scheme can be satisfied.

Example 1 We consider the following initial-boundary-value problem [33]

\[
\frac{\partial u}{\partial t} = -v(x, t) \frac{\partial u}{\partial x} + c_{+}(x, t) D_{x}^{a(x, t)} u \\
+ c_{-}(x, t) \frac{\partial D_{x}^{b(x, t)} u}{\partial x} + f(u, x, t),
\]

\[ (x, t) \in \Omega, \quad (36) \]

\[ u(x, 0) = x^{2}(1 - x)^{2}; \quad 0 \leq x \leq 1. \quad (37) \]

\[ u(0, t) = u(1, t) = 0; \quad 0 \leq t \leq 1. \quad (38) \]

Where

\[ v(x, t) = 6x^{2}(1 - x)^{3} e^{t}, \]

\[ c_{+}(x, t) = 0.5 \Gamma(5 - \alpha(x, t)) x^{2+\alpha(1 - x)} e^{t}, \]

\[ c_{-}(x, t) = 0.5 \Gamma(5 - \alpha(x, t)) x^{4(1 - x)^{2+\alpha}} e^{t}. \]

The analytical solution is $u(x, t) = e^{t} x^{2}(1 - x)^{2}$, and the associated forcing term is $f(u, x, t) = u + u^{2}(-24x^{2} + 2\alpha(x, t)(4 - \alpha(x, t)))$. We use parameters $r = 3.4, W = 35$ to attain the numerical results in table 1, we could conclude from the table 1 and 2 that

(1) As expected, quasi wavelet method is efficient for the space variable-order fractional advection-diffusion equation and has high accuracy from the results in table 1 and table 2.

(2) In table 1, both the time increment $\tau$ and the number of grid points $N$ and $M$ are varied to test the performance of the proposed method. The comparison has been done for two time steps $\tau = 0.00001$ and $\tau = 0.000001$ with two spatial discretization $N = 20$ and $N = 10$, and results indicate that, the proposed algorithm is efficient.

(3) In table 2, we provide the different parameters of $W$ and $r$, in order to indicate the effect of quasi-wavelets parameter $W$ and $r$ to the numerical results. For the sake of the effects of different $W$, we set the parameters $W = 12; r = 3.4$ compared with $W = 20; r = 3.4$. We can see that the numerical results become more accurate as the $W$ increases from 12 to 20. At the same time, we also set the parameters $r = 3.4, W = 35$ compared with $r = 2, W = 35$, we could see that the parameters $r = 3.4$ has a better results than $r = 2$. The accuracy of the algorithm depend on the choice of $r$ and $W$, as the quasi-wavelet algorithm is a local method. A careful comparison indicates that the parameter $W$ and $r$ used in the quasi-wavelet algorithm affects the accuracy of the method.

Besides, Fig. 1 plot the figure with $N = 10; M = 100; W = 8; r = 3.4$ We can see from the fig. 1-(a) and fig. 1-(b) that the computational solution is highly consistent with the exact solution. The fig. 1-(c) and the fig. 1-(d) show the evolution of $L_{\infty}$ error and absolute error for $N = 20, M = 100, \tau = 0.00001, r = 3.4, W = 8$, respectively.
Table 1

Example 1. $\tau = 0.000001$, $r = 3.4$, $W = 35.$

<table>
<thead>
<tr>
<th>N</th>
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<th>$L^\infty$</th>
<th>$L_2$</th>
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<tr>
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<td>1.6274e-006</td>
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$\tau = 0.00001$

<table>
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Table 2

Example 1. $\tau = 0.000001$, $W = 35$, $r = 3.4$.

<table>
<thead>
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<th>W</th>
<th>r</th>
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Example 2. We consider the following variable-order fractional advection-diffusion equation with the Riesz fractional derivative

$$\frac{\partial u}{\partial t} = -v(x, t) \frac{\partial u}{\partial x} - k(x, t)(-\Delta)^{\alpha/2} u + f(u, x, t),$$

$$x, t \in \Omega. \quad (39)$$

With the initial condition

$$v(x, 0) = x^2(1 - x)^2; 0 \leq x \leq 1, \quad (40)$$

and the boundary condition

$$v(0, t) = v(1, t) = 0; 0 \leq t \leq 2. \quad (41)$$

Figure 1: (a) Numerical solution in interval [0,1] for Example 1, for N=20 M=100 and (b) the exact solution $u(x, t) = e^t x^2(1 - x^2)$. (c) $L_\infty$ error estimation at 100th step, and (d) the global $L_\infty$ error estimation for N=10 M=100.
We choose
\[ v(x,t) = x^3(1-x)^3 e^t, \]
\[ f(u,x,t) = [1 - g_1(x,t) - g_2(x,t)]u + 2(1-2x)u^2; \]
\[ k(x,t) = -2 \cos(0.5 \pi \alpha(x,t)) \Gamma(5-\alpha(x,t))(x-x^2)^{2+\alpha(x,t)}, \]
\[ g_1(x,t) = 2x^2(1-x)^{2+\alpha(x,t)} [12x^2 + (4-\alpha(x,t))(-6x + 3 - \alpha(x,t))], \]
\[ g_2(x,t) = 2(1-x)^2x^{2+\alpha(x,t)} [12(1-x)^2 + (4-\alpha(x,t))(6x - 3 - \alpha(x,t))]. \]

The above problem has the exact solution \( u(x,t) = e^t x^2(1-x)^2 \). Table 3 and 4 give a series of numerical results for Example 2. In table 3, we provide the different values of \( r \) and \( W \) for \( \tau = 0.000001 \) and \( \tau = 0.00001 \). From the table 3, we can see that the numerical results are more accurate for \( r = 2.4 \), the results in Table 3 further verify the validity and accuracy of the proposed quasi-wavelet algorithm. In table 4, we offer the numerical results at \( t=1 \), the numerical results explain the quasi-wavelet based numerical method is a local method.

In Fig. 2 we also plot the figure with \( N = 20; M = 100; W = 3; r = 3.4; \tau = 0.000001 \), and we can see that the results are accurate.

### Table 3

<table>
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<th>( N )</th>
<th>( M )</th>
<th>( r )</th>
<th>( W )</th>
<th>( L_2 )</th>
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</table>

### Table 4

| Example 2. \( \tau = h, r = 3.4, W = 3 \) |
|---|---|---|
| \( N \) | \( M \) | \( L_\infty \) |
| 10 | 10 | 6.1655e-002 |
| 15 | 15 | 1.1708e-001 |
| 20 | 20 | 1.6184e-001 |
| 25 | 25 | 2.0841e-001 |
| 30 | 30 | 2.5125e-001 |

#### 5.2 The numerical example of double quasi-wavelet based numerical method

The advantage of the quasi-wavelet based numerical method described in [9, 10, 32] and in this paper is that it has the good ability to analyze the local characteristic of function, in other words, the time increment \( \tau \) must be small enough so that stability condition of the explicit Euler difference scheme can be satisfied. We proposed double quasi-wavelet based numerical scheme, which can relax restriction on the time step. we choose the parameters \( W = 3, W_1 = 3, r = 3.4 \) in table 5, in order to prove the effectiveness of the double quasi-wavelet based numerical method, we compare the numerical results in table 5 with table 4, and the results in [9, 10, 31, 32], we could have a conclusion that

1. Comparing the numerical results of Table 4 with table 5, we can see that the numerical results in table 5 are more accurate, it illustrates the quasi-wavelet is a local function, at the same time, it shows the double quasi-wavelet can relax restriction on the time step.
2. From the numerical results in [9, 10, 31, 32], we could see that the time increment \( \Delta t \), for example \( \Delta t = 10^{-6} \) or \( \Delta t = 10^{-5} \), must be small enough so that stability condition of the explicit Euler difference scheme can be satisfied, however, the numerical results in table 5 shows that the double quasi-wavelet is effective for prolonging calculating time. At the same time, Fig. 3 plot the figure with \( N = 20; M = 20; W = 3; W_1 = 3; r = 3.4 \). We can also see that the computational solution is consistent with the analytical solution, which also shows the proposed method is effective.
Figure 2: (a) Numerical solution for Example 2 when $N=20, M=100$. (b) The exact solution $u(x, t) = e^t x^2 (1 - x^2)$. (c) $L^\infty$ error estimation at 100 time point when $N=10$. (d) The global $L^\infty$ error estimation when $N=10, M=100$.

Figure 3: The double quasi-wavelet based numerical method for Example 2 when $N=15, M=15, r=3.4, t=1$. (a) Numerical solution. (b) The exact solution $u(x, t) = e^t x^2 (1 - x^2)$.

Table 5

<table>
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6 Conclusion

In this paper, the forward Euler scheme in combination with quasi-wavelet and double quasi-wavelet numerical methods are developed for solving the variable-order fractional advection-diffusion equation with a nonlinear source term. From the above tables and figures, we can see that quasi-wavelet method has good ability to analyze the local characteristic of functions and has high accuracy and feasible and valid for the variable-order fractional advection-diffusion equation with a nonlinear source term, double quasi-wavelet scheme in this paper can relax restriction on the time step, which is important in our further study. It is the first time that such a class of problems have been tackled with the quasi-wavelet based numerical method.

References:


[19] G.W. Wei, Y.B. Zhao, Y.Xiang, Discrete singular convolution and its application to the


