Integrability in Elementary Functions of Certain Classes of Nonconservative Systems

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Abstract: We study the non-conservative systems for which the methods for studying, for example, Hamiltonian systems is not applicable in general. Therefore, for such systems, it is necessary, in some sense, to “directly” integrate the main equation of dynamics. Herewith, we offer more universal interpretation of both obtained cases and new ones of complete integrability in transcendental functions in dynamics in a non-conservative force field.

Key–Words: Dynamical Systems With Variable Dissipation, Integrability

The results of the proposed work are a development of the previous studies, including a certain applied problem from rigid body dynamics [1], where complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. Later on, this circumstance allows us to perform a complete analysis of all phase trajectories and show those their properties which have a roughness and are preserved for systems of a more general form. The complete integrability of such systems are related to symmetries of latent type.

As is known, the concept of integrability is sufficiently fuzzy in general. In its construction, it is necessary to take into account the meaning in which it is understood(we mean a certain criterion with respect to which one makes a conclusion that the structure of trajectories of the dynamical system considered is especially simple and “attractive and simple”, in which class of functions, we seek for first integrals, etc. (see also [2]).

In this work, we accept the approach that as the class of functions for first integrals, takes transcendental functions, and, moreover, elementary ones. Here, the transcendence is understood not in the sense of elementary function theory (for example, trigonometric functions), but in the sense of existence of essentially singular points for them (according to the classification accepted in the theory of function of one complex variable). In this case, we need to formally continue the function considered to the complex domain (see also [3]).

1 Preliminary Arguments and Results

Of course, in the general case, it is sufficiently difficult to construct a certain theory of integrability for nonconservative systems (even of low dimension). But in a number of cases where the system studied have additional symmetries, we succeed in finding first integrals through finite combinations of elementary functions [4].

Proposed work appeared from the plane problem of motion of a rigid body in a resisting medium whose contact surface with the medium is a part of its exterior surface. In this case, the force field is constructed from the reasons of the medium action on the body under streamline (or separation) flow around under the quasi-stationarity conditions. It turns out that the study of motion of such classes of bodies reduces to systems with energy scattering ((purely) dissipative systems or systems in a dissipative force field) or systems with energy pumping (the so-called systems with anti-dissipation, or systems with dispersing forces). Note that such problems were already appeared in applied aerodynamics (see also [5]).

The previously considered problems have stimulated (and stimulate) the development of quality tools for studying, which essentially complement the qualitative theory of nonconservative systems with dissipation of both signs (see also [6]).

Also, we have qualitatively studied nonlinear effects of motion in plane and spatial rigid body dynamics and have justified on the qualitative level the necessity of introducing the definitions of relative rough-

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ness and relative non-roughness of various degrees (see also [7]).

The following series of results allowed us to prepare this work in a very definite style.

i) We have elaborated the methods for qualitative studying dissipative systems and systems with anti-dissipation that allo us to obtain the condition for bifurcation of birth of stable and unstable auto-oscillations and also the conditions for absence of any singular trajectories. We succeeded in generalizing the method for studying plane topographical Poincarè systems to higher dimensions. We have obtained sufficient Poisson stability conditions(density near itself) of certain classes of non-closed trajectories of dynamical systems [8].

ii) In two- and three-dimensional rigid body dynamics, we have discovered complete lists of first integrals of dissipative systems and systems with anti-dissipation that are transcendental (in the sense of classification of their singularities) functions that are expressed through elementary functions in a number of cases. We have introduced new definitions of relative roughness and relative non-roughness of various degrees, which have the integrated systems [9].

iii) We have obtained multiparameter families of topologically non-equivalent phase portraits arising in purely dissipative systems (i.e., systems with variable dissipation and nonzero (positive) mean). Almost every portrait of such families is (absolutely) rough [10].

iv) We have discovered new qualitative analogs between the properties of motion of free bodies in a resisting medium that is fixed at infinity, and bodies in the run-of medium flow [11].

Many results of this work were regularly reported at numerous workshops, including the workshop “Actual Problems of Geometry and Mechanics” named after professor V. V. Trofimov led by D. V. Georgievskii and M. V. Shamolin.

2 Variable Dissipation Dynamical Systems as a Class of Systems Admitting a Complete Integration

2.1 Visual characteristic of variable dissipation dynamical systems

Since, in the initial modelling of the medium action on a rigid body, we have used the experimental information about the properties of streamline flow, it becomes necessary to study the class of dynamical systems that has the property of (relative) structural stability (relative roughness). Therefore, it is quite natural to introduce these definitions for such systems. In this case, many of the systems considered are (absolutely) rough in the Andronov-Pontryagin sense [12, 13].

After some simplifications (for example, in the two-dimensional dynamics), the dynamical part of the general system of equations of plane-parallel motion can be reduced to a pendulum-like second-order system in which there is a linear nonconservative (sign-alternating) force with coefficients that have different signs for different values of the periodic phase coordinate in the system.

Therefore, in this case, we will speak of systems with the so-called variable dissipation, where the term “variable” refers not to the value of the dissipation coefficient, but to the possible alternation of its sign (therefore, it is more reasonable to use the term “sign-alternating”).

On the average, for the period of the existing periodic coordinate, the dissipation can be either positive (“purely” dissipative systems), or negative (systems with dispersing forces), and also it can be equal to zero (but not identically in this case). In the latter case, we speak of the zero mean variable dissipation systems (such systems can be associated with “almost” conservative systems).

As was already mentioned previously, we have detected important mechanical analogs arising in the comparison of qualitative properties of the stationary motion of a free body and the equilibrium of a pendulum in a medium flow. Such analogs have a deep support meaning since they allow us to extend the properties of nonlinear dynamical systems for a pendulum to dynamical systems for a free body. Both these systems belong to the class of the so-called zero mean variable dissipation pendulum-like dynamical systems.

Under additional conditions, the equivalence described above is extended to the case of spatial motion, which allows us to speak of the common character of symmetries in a zero mean variable dissipation system under the plane-parallel and spatial motions (for the plane and spatial variants of a pendulum in the medium flow, see also [1–3]).

In this connection, previously, we have introduced the definitions of relative structural stability (relative roughness), and relative structural instability (relative nonroughness) of various degrees. The latter properties were proved for systems that arise, e.g., in [4, 5].

The purely dissipative systems (as well as (purely) anti-dissipative ones), which in our case can belong to nonzero zero mean variable dissipation systems, are, as a rule, structurally stable ((absolutely) rough), whereas zero mean variable dissipation systems (which, as a rule, have additional symmetries) are either structurally unstable (nonrough) or only relatively structurally stable (relatively nonrough). In the general case, it is difficult to prove the latter assertion.

For example, a dynamical system of the form

$$\begin{gathered}\dot{\alpha}=\Omega+\beta\sin\alpha,\\ \dot{\Omega}=-\beta\sin\alpha\cos\alpha\end{gathered}\tag{1}$$

is relatively structurally stable (relatively rough) and is topologically equivalent to the system describing a clamped pendulum in an over-run medium flow [6, 7].

Below we present its first integral, which is a transcendental (in the sense of the theory of functions of one complex variable having essentially singular points after its continuation to the complex domain) function of phase variables expressed through a finite combination of elementary functions. The phase cylinder $\mathbf{R}^2\{\alpha,\Omega\}$ of quasi-velocities of the system considered has an interesting topological structure of partition into trajectories (for more detail, see [9]).

Although the dynamical system considered is not conservative, in the rotation domain (and only in it) of the phase plane $\mathbf{R}^2\{\alpha,\Omega\}$, it admits the preservation of an invariant measure with variable density. This property characterizes the system considered as a zero mean variable dissipation system.

2.2 One of the definitions of a zero mean variable dissipation system

We will study systems of ordinary differential equations having a periodic phase coordinate. The systems under study have those symmetries under which, on the average, for the period in the periodic coordinates, their phase volume is preserved. So, for example, the following pendulum-like system with smooth and periodic in α of period T right-hand side $\mathbf{V}(\alpha,\omega)$ of the form

$$\begin{gathered}\dot{\alpha}=-\omega+f(\alpha),\\ \dot{\omega}=g(\alpha),\\ f(\alpha+T)=f(\alpha),\ g(\alpha+T)=g(\alpha),\end{gathered}\tag{2}$$

preserves its phase area on the phase cylinder over the period T:

$$\begin{gathered}\int_0^T \mathrm{div}\mathbf{V}(\alpha,\omega)d\alpha=\\ =\int_0^T\left(\frac{\partial}{\partial\alpha}(-\omega+f(\alpha))+\frac{\partial}{\partial\omega}g(\alpha)\right)d\alpha=\\ =\int_0^T f'(\alpha)d\alpha=0.\end{gathered}\tag{3}$$

The system considered is equivalent to the pendulum equation

$$\ddot{\alpha}-f'(\alpha)\dot{\alpha}+g(\alpha)=0,\tag{4}$$

in which the integral of the coefficient $f'(\alpha)$ standing by the dissipative term $\dot{\alpha}$ is equal to zero on the average for the period.

It is seen that the system considered has those symmetries under which it becomes the so-called *zero mean variable dissipation system* in the sense of the following definition.

Definition 1 *Consider a smooth autonomous system of the $(n+1)$th order of normal form given on the cylinder $\mathbf{R}^n\{x\}\times\mathbf{S}^1\{\alpha \bmod 2\pi\}$, where α is a periodic coordinate of period $T>0$. The divergence of the right-hand side $\mathbf{V}(x,\alpha)$ (which, in general, is a function of all phase variables and is not identically equal to zero) of this system is denoted by div$\mathbf{V}(x,\alpha)$. This system is called a zero (nonzero) mean variable dissipation system if the function*

$$\int_0^T \mathit{div}\mathbf{V}(x,\alpha)d\alpha\tag{5}$$

is equal (not equal) to zero identically. Moreover, in some cases (for example, when singularities arise at separate points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$), this integral is understood in the sense of principal value.

It should be noted that giving a general definition of a zero (nonzero) mean variable dissipation system is not simple. The definition just presented uses the concept of divergence (as is known, the divergence of the right-hand side of a system in normal form characterizes the variation of the phase volume in the phase space of this system).

3 Systems with Symmetries and Zero Mean Variable Dissipation

Let us consider systems of the form (the dot denotes the derivative in time)

$$\begin{gathered}\dot{\alpha}=f_\alpha(\omega,\sin\alpha,\cos\alpha),\\ \dot{\omega_k}=f_k(\omega,\sin\alpha,\cos\alpha),\ k=1,\dots,n,\end{gathered}\tag{6}$$

given on the set

$$\mathbf{S}^1\{\alpha \bmod 2\pi\}\backslash K\times\mathbf{R}^n\{\omega\},\tag{7}$$

$\omega=(\omega_1,\dots,\omega_n)$, where the smooth functions $f_\lambda(u_1,u_2,u_3)$, $\lambda=\alpha,1,\dots,n$, of three variables u_1,u_2,u_3 are as follows:

$$\begin{gathered}f_\lambda(-u_1,-u_2,u_3)=-f_\lambda(u_1,u_2,u_3),\\ f_\alpha(u_1,u_2,-u_3)=f_\alpha(u_1,u_2,u_3),\\ f_k(u_1,u_2,-u_3)=-f_k(u_1,u_2,u_3),\end{gathered}\tag{8}$$

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herewith, the functions $f_k(u_1,u_2,u_3)$ *are defined* for $u_3=0$ for any $k=1,\dots,n$.

The set K is either empty or consists of finitely many points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$.

The latter two variables u_2 and u_3 in the functions $f_\lambda(u_1,u_2,u_3)$ depend on one parameter α, but they are allocated into different groups because of the following reasons. First, not in the whole domain, they are one-to-one expressed through each other, and, second, the first of them is an odd function and the second is an even function of α, which influences the symmetries of system (6) in different ways.

To the system (6), let us put in correspondence the following nonautonomous system

$$\frac{d\omega_k}{d\alpha}=\frac{f_k(\omega,\sin\alpha,\cos\alpha)}{f_\alpha(\omega,\sin\alpha,\cos\alpha)},\ k=1,\dots,n, \tag{9}$$

which via the substitution $\tau=\sin\alpha$, reduces to the form

$$\frac{d\omega_k}{d\tau}=\frac{f_k(\omega,\tau,\varphi_k(\tau))}{f_\alpha(\omega,\tau,\varphi_\alpha(\tau))},\ k=1,\dots,n, \tag{10}$$

$$\varphi_\lambda(-\tau)=\varphi_\lambda(\tau),\ \lambda=\alpha,1,\dots,n.$$

The latter system can have in particular an algebraic right-hand side (i.e., it can be the ratio of two polynomials); sometimes this helps us to find its first integrals in explicit form.

The following assertion embeds the class of systems (6) in the class of zero mean variable dissipation systems. The inverse embedding does not hold in general.

Theorem 2 *Systems of the form (6) are zero mean variable dissipation dynamical systems.*

This theorem is proved by using the certain symmetries (8) of system (6) only, listed above, and uses the periodicity of the right-hand side of the system on α.

Indeed, let calculate the specified divergence of the vector field of system (6). It equals to

$$\frac{\partial f_\alpha(\omega,\sin\alpha,\cos\alpha)}{\partial u_2}\cos\alpha-$$
$$-\frac{\partial f_\alpha(\omega,\sin\alpha,\cos\alpha)}{\partial u_3}\sin\alpha+$$
$$+\sum_{k=1}^{n}\frac{\partial f_k(\omega,\sin\alpha,\cos\alpha)}{\partial u_1}. \tag{11}$$

The following integral on two first summands (11) is equal to zero:

$$\int_0^{2\pi}\left\{\frac{\partial f_\alpha(\omega,\sin\alpha,\cos\alpha)}{\partial u_2}d\sin\alpha+\right.$$
$$\left.+\frac{\partial f_\alpha(\omega,\sin\alpha,\cos\alpha)}{\partial u_3}d\cos\alpha\right\}=$$
$$=\int_0^{2\pi}\frac{\partial f_\alpha(\omega,\sin\alpha,\cos\alpha)}{\partial\alpha}d\alpha=h_\alpha(\omega)\equiv 0, \tag{12}$$

since the function $f_\alpha(\omega,\sin\alpha,\cos\alpha)$ is periodic one on α.

So, by virtue of the third equation (8), the following property holds for any $k=1,\dots,n$:

$$\frac{\partial f_k(\omega,\sin\alpha,\cos\alpha)}{\partial u_1}=\cos\alpha\cdot\frac{\partial g_k(\omega,\sin\alpha)}{\partial u_1}, \tag{13}$$

herewith, the function $g_k(u_1,u_2)$ is rather smooth for any number $k=1,\dots,n$.

Then the integral on period 2π from the right-hand side of the equation (13) gives

$$\int_0^{2\pi}\frac{\partial g_k(\omega,\sin\alpha)}{\partial u_1}d\sin\alpha=h_k(\omega)\equiv 0 \tag{14}$$

for any $k=1,\dots,n$. And we get the assertion of the theorem 2 from the equalities (12), (14).

The converse assertion is not true in general since we can present a set of dynamical systems on the two-dimensional cylinder, being zero mean variable dissipation systems that have none of the above-listed symmetries.

In this work, we are mainly concerned with the case where the functions $f_\lambda(\omega,\tau,\varphi_k(\tau))$ ($\lambda=\alpha,1,\dots,n$) are polynomials in ω,τ.

Example 1. We consider, in particular, pendulum-like systems on the two-dimensional cylinder $\mathbf{S}^1\{\alpha \bmod 2\pi\}\times\mathbf{R}^1\{\omega\}$ with parameter $b>0$ from rigid body dynamics:

$$\begin{aligned}\dot\alpha&=-\omega+b\sin\alpha,\\ \dot\omega&=\sin\alpha\cos\alpha,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\dot\alpha&=-\omega+b\sin\alpha\cos^2\alpha+b\omega^2\sin\alpha,\\ \dot\omega&=\sin\alpha\cos\alpha-b\omega\sin^2\alpha\cos\alpha+\\ &+b\omega^3\cos\alpha,\end{aligned} \tag{16}$$

in the variables (ω,τ), to these systems we can put in correspondence the following equations with algebraic right-hand sides:

$$\frac{d\omega}{d\tau}=\frac{\tau}{-\omega+b\tau}, \tag{17}$$

and

$$\frac{d\omega}{d\tau}=\frac{\tau+b\omega[\omega^2-\tau^2]}{-\omega+b\tau+b\tau[\omega^2-\tau^2]} \tag{18}$$

which have the form (4.2). Moreover, these systems are zero mean variable dissipation dynamical systems, which is easily directly verified.

Indeed, the divergences of its right-hand sides are equal to

$$b\cos\alpha$$

and

$$b\cos\alpha[4\omega^2+\cos^2\alpha-3\sin^2\alpha],$$

respectively. It is easy to verify that they belong to the class of systems (6).

Moreover, each of these systems has a first integral that is a transcendental (in the sense of the theory of functions of one complex variable) function expressed through a finite combination of elementary functions.

Let us present one more important example of a higher-order system having the listed properties.

Example 2. To the following system

$$\begin{aligned}\dot\alpha &= -z_2+b\sin\alpha,\\ \dot z_2 &= \sin\alpha\cos\alpha - z_1^2\tfrac{\cos\alpha}{\sin\alpha},\\ \dot z_1 &= z_1z_2\tfrac{\cos\alpha}{\sin\alpha},\end{aligned} \tag{19}$$

with parameter b, considered in the three-dimensional domain

$$\mathbf{S}^1\{\alpha \bmod 2\pi\}\backslash\{\alpha=0,\alpha=\pi\}\times\mathbf{R}^2\{z_1,z_2\} \tag{20}$$

(such a system can also be reduced to an equivalent system on the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$) and describing the spatial motion of a rigid body in a resisting medium, we put in correspondence the following nonautonomous system with algebraic right-hand side: ($\tau=\sin\alpha$):

$$\frac{dz_2}{d\tau}=\frac{\tau-z_1^2/\tau}{-z_2+b\tau},\ \frac{dz_1}{d\tau}=\frac{z_1z_2/\tau}{-z_2+b\tau}. \tag{21}$$

It is seen that system (19) is a zero mean variable dissipation system; in order to achieve complete correspondence with the definition, it suffices to introduce a new phase variable

$$z_1^*=\ln|z_1|. \tag{22}$$

If we calculate the divergence of the right-hand side of the system (19) in Cartesian coordinates α, z_1^*, z_2, then we shall get that it is equal to $b\cos\alpha$. Herewith, if we consider (20), we shall have in the sense of principal value:

$$\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi-\varepsilon} b\cos\alpha+\lim_{\varepsilon\to0}\int_{\pi+\varepsilon}^{2\pi-\varepsilon} b\cos\alpha=0. \tag{23}$$

Moreover, the system (19) has two first integrals (i.e., a complete list) that are transcendental functions and we expressed through a finite combination of elementary functions, which, as was mentioned above, becomes possible after putting in correspondence to it a (nonautonomous in general) system of equations with algebraic (polynomial) right-hand (21).

The above-presented systems (15), (16), and (19), along with the property that they belong to the class of systems (6) and are zero mean variable dissipation systems, also have a complete list of transcendental first integrals expressed through a finite combination of elementary functions.

Therefore, to search for the first integrals of the system considered, it is better to reduce systems of the form (6) to systems (10) with polynomial right-hand sides, on whose form the possibility of integration in elementary functions of the initial system depends. Therefore, we proceed as follows: we seek sufficient conditions for integrability in elementary functions of systems of equations with polynomial right-hand sides studying systems of the most general form in this process.

4 Systems on Tangent Bundle of Two-Dimensional Sphere

Let consider the following dynamic system

$$\begin{aligned}\ddot\theta+b\dot\theta\cos\theta+\sin\theta\cos\theta-\dot\psi^2\tfrac{\sin\theta}{\cos\theta}&=0,\\ \ddot\psi+b\dot\psi\cos\theta+\dot\theta\dot\psi\left[\tfrac{1+\cos^2\theta}{\sin\theta\cos\theta}\right]&=0\end{aligned} \tag{24}$$

on tangent bundle $T_*\mathbf{S}^2$ of two-dimensional sphere $\mathbf{S}^2\{\theta,\psi\}$. This system describes the spherical pendulum, placed in the accumulating medium flow. Herewith, both conservative moment is present in the system

$$\sin\theta\cos\theta, \tag{25}$$

and the force moment depending on the velocity as linear one with the variable coefficient:

$$b\begin{pmatrix}\dot\theta\\ \dot\psi\end{pmatrix}\cos\theta. \tag{26}$$

The coefficients remaining in the equations are the coefficients of connectedness, i.e.,

$$\Gamma^{\theta}_{\psi\psi}=-\frac{\sin\theta}{\cos\theta},\ \Gamma^{\psi}_{\theta\psi}=\frac{1+\cos^2\theta}{\sin\theta\cos\theta}. \tag{27}$$

The system (24) has an order 3 practically, since the variable ψ is a cyclic, herewith, the derivative $\dot\psi$ is present in the system only.

Proposition 3 *The equation*

$$\dot\psi=0 \tag{28}$$

define the family of integral planes for the system (24).

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Furthermore, the equation (28) reduces the system (24) to an equation describing the cylindric pendulum which placed in the accumulating medium flow.

Proposition 4 *The system (24) is equivalent to the following system:*

$$\begin{array}{c} \dot\theta = -z_2 + b\sin\theta, \\ \dot z_2 = \sin\theta\cos\theta - z_1^2\frac{\cos\theta}{\sin\theta}, \\ \dot z_1 = z_1 z_2 \frac{\cos\theta}{\sin\theta}, \\ \dot\psi = z_1\frac{\cos\theta}{\sin\theta} \end{array} \tag{29}$$

on the tangent bundle $T_*\mathbf{S}^2\{z_1, z_2, \theta, \psi\}$ *of two-dimensional sphere* $\mathbf{S}^2\{\theta, \psi\}$.

Moreover, the first three equations of the system (29) form the closed three order system and coincide with the equations of the system (19) (if we denote $\alpha = \theta$). The separation of fourth equation of the system (29) has also occurred by the reason of cyclicity of the variable ψ.

On the construction of phase pattern of the system (24), expressed in Pic. **??**, see [9].

Example 3. Let us study a system of the form (19), which reduces to (21), and also the system

$$\begin{array}{c} \dot\alpha = -z_2 + b(z_1^2 + z_2^2)\sin\alpha + \\ +b\sin\alpha\cos^2\alpha, \\ \dot z_2 = \sin\alpha\cos\alpha + bz_2(z_1^2+z_2^2)\cos\alpha - \\ -bz_2\sin^2\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot z_1 = bz_1(z_1^2+z_2^2)\cos\alpha - \\ -bz_1\sin^2\alpha\cos\alpha + z_1z_2\frac{\cos\alpha}{\sin\alpha}, \end{array} \tag{30}$$

which also arises in the three-dimensional dynamics of a rigid body interacting with a medium and which corresponds to the following system with algebraic right-hand side:

$$\begin{array}{l} \frac{dz_2}{d\tau} = \frac{\tau + bz_2(z_1^2+z_2^2) - bz_2\tau^2 - z_1^2/\tau}{-z_2 + b\tau(z_1^2+z_2^2) + b\tau(1-\tau^2)}, \\ \frac{dz_1}{d\tau} = \frac{bz_1(z_1^2+z_2^2) - bz_1\tau^2 + z_1z_2/\tau}{-z_2 + b\tau(z_1^2+z_2^2) + b\tau(1-\tau^2)}. \end{array} \tag{31}$$

Therefore, as before, we consider a pair of systems: the initial system (30) and the algebraic system (31) corresponding to it.

In a similar way, we pass to homogeneous coordinates u_k, $k = 1, 2$, by the formulas

$$z_k = u_k\tau. \tag{32}$$

Using the latter change, we reduce system (21) to the form

$$\begin{array}{l} \tau\frac{du_2}{d\tau} + u_2 = \frac{\tau - u_1^2\tau}{-u_2\tau + b\tau}, \\ \tau\frac{du_1}{d\tau} + u_1 = \frac{u_1u_2\tau}{-u_2\tau + b\tau}, \end{array} \tag{33}$$

corresponding to the equation

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}. \tag{34}$$

This equation is integrated in elementary functions since the identity

$$d\left(\frac{1 - \beta u_2 + u_2^2}{u_1}\right) + du_1 = 0, \tag{35}$$

is integrated, and in the coordinates (τ, z_1, z_2), it has a first integral of the form (compare with [9])

$$\frac{z_1^2 + z_2^2 - \beta z_2\tau + \tau^2}{z_1\tau} = \text{const.}$$

System (30) after its reduction corresponds to the system

$$\begin{array}{l} \tau\frac{du_2}{d\tau} + u_2 = \frac{\tau + bu_2\tau^3(u_1^2+u_2^2) - bu_2\tau^3 - u_1^2\tau}{-u_2\tau + b\tau^3(u_1^2+u_2^2) + b\tau(1-\tau^2)}, \\ \tau\frac{du_1}{d\tau} + u_1 = \frac{bu_1\tau^3(u_1^2+u_2^2) - bu_1\tau^3 + u_1u_2\tau}{-u_2\tau + b\tau^3(u_1^2+u_2^2) + b\tau(1-\tau^2)}, \end{array} \tag{36}$$

which also reduces to the form (34).

5 Certain Generalizations

Let us pose the following question: What are the possibilities of integrating in elementary functions the system

$$\begin{array}{l} \frac{dz}{dx} = \frac{ax + by + cz + c_1z^2/x + c_2zy/x + c_3y^2/x}{d_1x + ey + fz}, \\ \frac{dy}{dx} = \frac{gx + hy + iz + i_1z^2/x + i_2zy/x + i_3y^2/x}{d_1x + ey + fz}, \end{array} \tag{37}$$

of a more general form, which includes the systems (21) and (31) considered above in three-dimensional phase domains and which has a singularity of the form $1/x$?

Previously, a number of results concerning this question were already obtained (see also [6]). Let us present these results and complement them by original arguments.

As previously, introducing the substitutions

$$y = ux, \; z = vx, \tag{38}$$

we obtain that system (37) reduces to the following system

$$x\frac{dv}{dx} + v = \frac{ax + bux + cvx + c_1v^2x + c_2vux + c_3u^2x}{d_1x + eux + fvx}, \tag{39}$$

$$x\frac{du}{dx} + u = \frac{gx + hux + ivx + i_1v^2x + i_2vux + i_3u^2x}{d_1x + eux + fvx}, \tag{40}$$

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which is equivalent to

$$x\frac{dv}{dx} = = \frac{ax+bux+(c-d_1)vx+(c_1-f)v^2x+(c_2-e)vux+c_3u^2x}{d_1x+eux+fvx}, \quad (41)$$

$$x\frac{du}{dx} = = \frac{gx+(h-d_1)ux+ivx+i_1v^2x+(i_2-f)vux+(i_3-e)u^2x}{d_1x+eux+fvx}, \quad (42)$$

we put in correspondence the following equation with algebraic right-hand side:

$$\frac{dv}{du} = \frac{a+bu+cv+c_1v^2+c_2vu+c_3u^2-v[d_1+eu+fv]}{g+hu+iv+i_1v^2+i_2vu+i_3u^2-u[d_1+eu+fv]}. \quad (43)$$

The integration of the latter equation reduces to that of the following equation in total differentials:

$$[g+hu+iv+i_1v^2+i_2vu+i_3u^2-d_1u-eu^2-fuv]dv = = [a+bu+cv+c_1v^2+ +c_2vu+c_3u^2-d_1v-euv-fv^2]du. \quad (44)$$

We have (in general) a 15-parameter family of equations of the form (44). To integrate the latter identity in elementary functions as a homogeneous equation, it suffices to impose seven relations

$$g=0,\ i=0,\ i_1=0, \quad e=c_2,\ h=c,\ i_2=2c_1-f. \quad (45)$$

Introduce nine parameters $\beta_1,\ldots,\beta_9$ and consider them as independent parameters:

$$\beta_1=a,\ \beta_2=b,\ \beta_3=c,\ \beta_4=c_1,\ \beta_5=c_2, \quad \beta_6=c_3,\ \beta_7=d_1,\ \beta_8=f,\ \beta_9=i_3. \quad (46)$$

Therefore, under the group of conditions (45), and (46), Eq. (44) reduces to the form

$$\frac{dv}{du} = \frac{\beta_1+\beta_2u+(\beta_3-\beta_7)v+(\beta_4-\beta_8)v^2+\beta_6u^2}{(\beta_3-\beta_7)u+2(\beta_4-\beta_8)vu+(\beta_9-\beta_5)u^2}, \quad (47)$$

and the system (41), (42), respectively, to the form

$$x\frac{dv}{dx} = \frac{\beta_1+\beta_2u+(\beta_3-\beta_7)v+(\beta_4-\beta_8)v^2+\beta_6u^2}{\beta_7+\beta_5u+\beta_8v}, \quad (48)$$

$$x\frac{du}{dx} = \frac{(\beta_3-\beta_7)u+2(\beta_4-\beta_8)vu+(\beta_9-\beta_5)u^2}{\beta_7+\beta_5u+\beta_8v}, \quad (49)$$

after that, the equation (47) is integrated in finite combination of elementary functions.

Indeed, integrating the identity (44), we obtain the relation

$$d\left[\frac{(\beta_3-\beta_7)v}{u}\right] + +d\left[\frac{(\beta_4-\beta_8)v^2}{u}\right] + d[(\beta_9-\beta_5)v] + d\left[\frac{\beta_1}{u}\right] - -d[\beta_2\ln|u|] - d[\beta_6u] = 0, \quad (50)$$

which for the beginning allows us to obtain the following invariant relation:

$$\frac{(\beta_3-\beta_7)v}{u} + \frac{(\beta_4-\beta_8)v^2}{u} + (\beta_9-\beta_5)v + \frac{\beta_1}{u} - -\beta_2\ln|u| - \beta_6u = C_1 = \text{const}, \quad (51)$$

and in the coordinates (x,y,z), it allows us to obtain the first integral in the form

$$\frac{A}{yx} - \beta_2\ln\left|\frac{y}{x}\right| = \text{const}, \quad (52)$$

$$A = (\beta_4-\beta_8)z^2 - -\beta_6y^2 + (\beta_3-\beta_7)zx + (\beta_9-\beta_5)zy + \beta_1x^2.$$

Therefore, we can make a conclusion about the integrability in elementary functions of the following, in general, nonconservative third-order system depending on nine parameters:

$$\frac{dz}{dx} = \frac{\beta_1x+\beta_2y+\beta_3z+\beta_4z^2/x+\beta_5zy/x+\beta_6y^2/x}{\beta_7x+\beta_5y+\beta_8z}, \quad \frac{dy}{dx} = \frac{\beta_3y+(2\beta_4-\beta_8)zy/x+\beta_9y^2/x}{\beta_7x+\beta_5y+\beta_8z}. \quad (53)$$

Corollary 5 *The third-order system*

$$\begin{aligned} \dot\alpha &= \beta_7\sin\alpha+\beta_5z_1+\beta_8z_2,\\ \dot z_2 &= \beta_1\sin\alpha\cos\alpha+\beta_2z_1\cos\alpha+ \\ &+\beta_3z_2\cos\alpha+ \\ &+\beta_4z_2^2\frac{\cos\alpha}{\sin\alpha}+\beta_5z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_6z_1^2\frac{\cos\alpha}{\sin\alpha},\\ \dot z_1 &= \beta_3z_1\cos\alpha+(2\beta_4-\beta_8)z_1z_2\frac{\cos\alpha}{\sin\alpha}+ \\ &+\beta_9z_1^2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \quad (54)$$

on the set

$$\mathbf{S}^1\{\alpha \mathit{\ mod\ } 2\pi\}\backslash\{\alpha=0,\alpha=\pi\}\times\mathbf{R}^2\{z_1,z_2\}, \quad (55)$$

which depends on nine parameters, has a first integral, which is in general transcendental and expressed through elementary functions:

$$\frac{B}{z_1\sin\alpha} - \beta_2\ln\left|\frac{z_1}{\sin\alpha}\right| = \mathit{const}, \quad (56)$$

$$B = (\beta_4-\beta_8)z_2^2 - \beta_6z_1^2 + +(\beta_3-\beta_7)z_2\sin\alpha + (\beta_9-\beta_5)z_2z_1 + \beta_1\sin^2\alpha^2.$$

In particular, for $\beta_1=1,\ \beta_2=\beta_3=\beta_4=\beta_5=\beta_9=0$, $\beta_6=\beta_8=-1$, $\beta_7=b$ system (54) reduces to system (19).

To find the additional first integral of the nonautonomous system (37), we use the found first integral

(52), which is expressed through a finite combination of elementary functions.

For the beginning let transform the relation (51) as follows:

$$(\beta_4 - \beta_8)v^2 + \\ + [(\beta_9 - \beta_5)u + (\beta_3 - \beta_7)]\, v + f_1(u) = 0, \quad (57)$$

where

$$f_1(u) = \beta_1 - \beta_6 u^2 - \beta_2 u \ln |u| - C_1 u.$$

Herewith, the value v formally can be found from the equality

$$v_{1,2}(u) = \frac{1}{2(\beta_4 - \beta_8)} \times \\ \times \left\{ (\beta_5 - \beta_9)u + (\beta_7 - \beta_3) \pm \sqrt{f_2(u)} \right\}, \quad (58)$$

where

$$f_2(u) = A_1 + A_2 u + A_3 u^2 + A_4 u \ln |u|,$$

$$A_1 = (\beta_3 - \beta_7)^2 - 4\beta_1(\beta_4 - \beta_8),$$

$$A_2 = 2(\beta_9 - \beta_5)(\beta_3 - \beta_7) + 4C_1(\beta_4 - \beta_8),$$

$$A_3 = (\beta_9 - \beta_5)^2 + 4\beta_6(\beta_4 - \beta_8),$$

$$A_4 = 4\beta_2(\beta_4 - \beta_8).$$

Then the quadrature studied for the search of additional (in general) transcendental first integral (for example, of the system (48), (49) or (41), (42), herewith, the equation (49) is used) has the following form

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u + \beta_8 v_{1,2}(u)]du}{C} = \\ = \int \frac{[B_1 + B_2 u + B_3\sqrt{f_2(u)}]du}{B_4 u \sqrt{f_2(u)}}, \quad (59)$$

$$B_k = \text{const},\ k = 1, \ldots, 4,$$

$$C = (\beta_3 - \beta_7)u + (\beta_9 - \beta_5)u^2 + 2(\beta_4 - \beta_8)uv_{1,2}(u).$$

And the quadrature studied for the search of additional (in general) transcendental first integral (for example, of the system (48), (49) or (41), (42), herewith, the equation (48) is used) has the following form

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u(v) + \beta_8 v]dv}{D}, \quad (60)$$

$$D = \beta_1 + \beta_2 u(v) + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2(v).$$

herewith, the function $u(v)$ should be obtain as a result of resolving of implicit equation (51) respectively to u (that, in general case, is not always evident).

The following lemma gives the necessary conditions of the expression of integrals in (60) through the finite combination of elementary functions.

Lemma 6 *The indefinite integral in (60) is expressed through the finite combination of elementary functions for $A_4 = 0$, i.e., either*

$$\beta_2 = 0 \quad (61)$$

or

$$\beta_4 = \beta_8. \quad (62)$$

Theorem 7 *The system (54) under assumption of necessary conditions of lemma 6 (the property (61) holds in given case), has the compete set of first integrals, which expressing through the finite combination of elementary functions.*

Therefore, the dynamical systems considered in this work refer to zero mean variable dissipation systems in the existing periodic coordinate. Moreover, such systems often have a complete list of first integrals expressed through elementary functions.

So, it was shown above the certain cases of complete integrability in spatial dynamics of a rigid body motion in a nonconservative field. Herewith, we dealt with with three properties, which are independent for the first glance:

i) the distinguished class of systems (6) with the symmetries above;

ii) the fact that this class of systems consists of systems with zero mean variable dissipation (in the variable α), which allows us to consider them as "almost" conservative systems;

iii) in certain (although lower-dimensional) cases, these systems have the complete tuple of first integrals, which are transcendental in general (from the viewpoint of complex analysis).

The method for reducing the initial systems of equations with right-hand sides containing polynomials in trigonometric functions to systems with polynomial right-hand sides allows us to search for the first integrals (or to prove their absence) for systems of a more general form, but not only for systems that have the above symmetries (see also [9]).

6 Conclusion

The results of the presented work were appeared owing to the study the applied problem of the rigid body motion in a resisting medium, where we have obtained complete lists of transcendental first integrals expressed through a finite combination of elementary functions. This circumstance allows the author to carry out the analysis of all phase trajectories and show those their properties which have the roughness and are preserved for systems of a more general form. The complete integrability of such system is related

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to their symmetries of latent type. Therefore, it is of interest to study a sufficiently wide class of dynamical systems having analogous latent symmetries.

So, for example, the instability of the simplest body motion, the rectilinear translational drag, is used for methodological purposes, precisely, for finding the unknown parameters of the medium action on a rigid body under the quasi-stationarity conditions.

The experiment on the motion of homogeneous circular cylinders in the water carried out in Institute of Mechanics of M. V. Lomonosov State University justified that in modelling the medium action on the rigid body, it is also necessary to take into account an additional parameter that brings a dissipation to the system.

In studying the class of body drags with finite angle of attack, the principal problem is finding those conditions under which there exist auto-oscillations in a finite neighborhood of the rectilinear translational drag. Therefore, there arises the necessity of a complete nonlinear study.

Generally speaking, the dynamics of a rigid body interacting with a medium is just the field where there arise either nonzero mean variable dissipation systems or systems in which the energy loss in the mean during a period can vanish. In the work, we have obtained such a methodology owing to which it becomes possible to finally and analytically study a number of plane and spatial model problems.

In qualitative describing the body interaction with a medium, because of using the experimental information about the properties of the streamline flow around, there arises a definite dispersion in modelling the force-model characteristics. This makes it natural to introduce the definitions of relative roughness (relative structural stability) and to prove such a roughness for the system studied. Moreover, many systems considered are merely (absolutely) Andronov–Pontryagin rough.

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