Frequency Identification of Nonlinear Systems

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Abstract: - The problem of identifying nonlinear subsystems structured by Wiener-Hammerstein models is addressed. The two linear are of structure totally unknown. Presently, the nonlinear element is allowed to be noninvertible. The system identification problem is dealt by developing a two-stage frequency identification method such that a set of points of the nonlinearity are estimated first. Then, the frequency gains of the two linear subsystems are determined at a number of frequencies. The method involves Fourier series decomposition and only requires periodic excitation signals. All involved estimators are shown to be consistent.

Key-Words: - Hammerstein models, Wiener models, Nonlinear systems, Frequency identification, Fourier expansions

1 Introduction

Wiener-Hammerstein systems consist of a series connection including a nonlinear static element sandwiched with two linear subsystems (Fig.1). Clearly, this model structure is a generalization of Hammerstein and Wiener models and so it is expected to feature a superior modelling capability. This has been confirmed by several practical applications e.g. paralyzed skeletal muscle dynamics (Bai et al., 2009). As a matter of fact, Wiener-Hammerstein (WH) systems are more difficult to identify than the simpler Hammerstein and Wiener models. The complexity of the former lies in the fact that these systems involve two internal signals not accessible to measurements, whereas the latter only involve one. Then, it is not surprising that only a few methods are available that deal with WH system identification. The available methods have been developed following three main approaches i.e. iterative nonlinear optimization procedures (e.g. Marconato et al., 2012), stochastic methods (Pillonetto et al., 2011); frequency methods (Brouri et al., 2014).

In this paper, the problem of identifying WH systems is addressed, for simplicity, in the continuous-time. Unlike many previous works, the model structure of the two linear subsystems is entirely unknown. Furthermore, the static nonlinearity is also of unknown structure and is not required to be invertible. This is only supposed to be well approximated, within any subinterval belonging to the working interval, with a polynomial of unknown order and parameters. The order p and the parameters of the polynomial can vary from one subinterval to another. It turns out that the complexity of the identification problem lies in: (i) the fact that the internal signals u_i and u_o are not accessible to measurement (Fig.1); (ii) the nonparametric and nonlinear nature of the system. Given the system nonparametric nature, the identification problem is presently dealt with by developing a two-stage frequency identification method, involving periodic inputs. First, a set of points of the nonlinearity is identified using simple experiments; the size of this set is arbitrarily chosen by the user. Then, the frequency responses of the two linear subsystems are estimated for a number of frequencies; in turn, this number can be made arbitrarily large. The frequency gain estimator design relies on input/output Fourier series expansions.

This paper is organized as follows: the identification problem is formally described in Section 2; then, the identification of the nonlinearity is coped with in Section 3; the identification of the linear subsystems is dealt with in Section 4. Simulation examples are provided in Section 5 to illustrate the performances of the whole identification method.

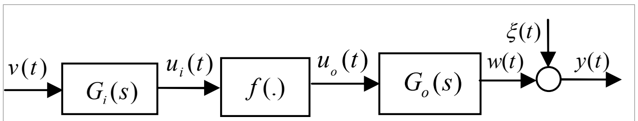

Fig.1. Wiener-Hammerstein System Model

2 Identification problem statement

We are interested in systems that can be described by the Wiener-Hammerstein model of Fig.1 where the different blocs are analytically described as follows:

$$y = w + \xi \quad (1)$$

$$w = G_o(s)u_o \quad (2)$$

$$u_o = f(u_i)\,;\;\; u_i = G_i(s)v \quad (3)$$

where $G_i(s)$ and $G_o(s)$ are the transfer functions of the linear subsystems, $f(.)$ denotes the static nonlinearity, and ξ is an external noise. As, the signals u_i, u_o, w and ξ are not accessible to measurements, the identification procedure of the nonlinear system must only relay on the external signals v and y. The signal ξ is supposed to be a zero-mean stationary sequence of independent random variables and ergodic. The static nonlinear element $f(.)$ has any shape and, in particular, may be noninvertible. It is only assumed that $f(.)$ is smooth so that it can be accurately represented, within any finite interval, with a polynomial of finite order. Of course the polynomial order depends on the interval length.

The transfer functions $G_i(s)$ and $G_o(s)$ are of unknown structures. There are only supposed to be asymptotically stable and with nonzero static gain (i.e. $G_i(0) \neq 0$ and $G_o(0) \neq 0$). System stability is coherent with open-loop system identification. Also, note that the nonzero static-gain requirement is satisfied by most real life systems. In fact, only derivative systems make an exception that can be coped with using ad-hoc adaptations of the method developed in this paper. The problem complexity also lies in the fact that the (unavailable) internal signals (u_i, u_o, w and ξ) are not uniquely defined from an input-output viewpoint. In effect, if $(G_i(s), f(x), G_o(s))$ is representative of the system then, any model of the form $(G_i(s)/k_1, k_2 f(k_1 x), G_o(s)/k_2)$ is also representative whatever the real numbers $k_1 \neq 0,\ k_2 \neq 0$. Therefore, the fact that $G_i(0) \neq 0$ and $G_o(0) \neq 0$ implies that, without reducing the problem generality, one can assume $G_i(0) = G_o(0) = 1$.

3 Identification of static nonlinearity

In this section, we want to treat the problem of identifying a set of points belonging to non-linearity. In Section 2 it was shown that, if k_1 and k_2 are any nonzero real numbers, then any model of the form $(G_i(s)/k_1, k_2 f(k_1 x), G_o(s)/k_2)$ is representative of the system. Accordingly, the system to be identified is described by the transfer functions:

$$\bar{G}_i(s) = G_i(s)/G_i(0)\,;\;\; \bar{G}_o(s) = G_o(s)/G_o(0) \quad (4a)$$

$$\bar{f}(x) = G_o(0) f\left(G_i(0)x\right) \quad (4b)$$

Then, $\bar{G}_i(0) = \bar{G}_o(0) = 1$. Under these conditions, if $v(t)$ is constant then the steady-state undisturbed output $w(t)$ depends only on the input value and the nonlinearity $\bar{f}(.)$. The number n of points is arbitrary. Let $V_1 = v_m < V_2 < \cdots < V_n = v_M$ be the selected abscissas. To determine the points $(V_j, \bar{f}(V_j))$, letting:

$$v(t) = V_j \quad (t \geq 0) \;\text{ for }\; j = 1 \ldots n \quad (5)$$

As the linear subsystem $G_i(s)$ is asymptotically stable, therefore the internal signal $u_i(t)$ is constant, and one has $u_i(t) \underset{t\to\infty}{\rightarrow} U_i^{\,j}$, then in the steady-state:

$$U_i^{\,j} = G_i(0)V_j \;\text{ for }\; j = 1 \ldots n \;\text{ and }\; t \geq NT_r \quad (6)$$

where T_r should be comparable to the system rise time i.e. the time that is necessary for a system step response to reach 90% of its final value. Then, as the system is asymptotically stable, its step response settles down (i.e. gets very close to final value) after a transient period of NT_r seconds with $N \geq 1$.

As the linear subsystem $G_o(s)$ is asymptotically stable, it follows that the steady-state of the internal signal $u_o(t)$ is constant i.e. $u_o(t) \underset{t\to\infty}{\rightarrow} U_o^{\,j}$, and is written for $t \geq NT_r$:

$$U_o^{\,j} = f(U_i^{\,j}) = f(G_i(0)V_j) \;\text{ for }\; j = 1 \ldots n \quad (7)$$

In which case, the undisturbed output $w(t)$ is also constant (in the steady-state) i.e. $w(t) \underset{t\to\infty}{\rightarrow} W_j$. It readily follows from (4b) and (7) that W_j can be expressed as follows:

$$W_j = G_o(0)U_o^{\,j} = \bar{f}(V_j) \;\text{ for }\; j = 1 \ldots n \quad (8)$$

Finally, notice that the steady-state undisturbed output W_j $(j = 1 \ldots n)$ can simply be estimated using the fact that $y(t) = w(t) + \xi(t)$ and $\xi(t)$ is zero-mean. Specifically, W_j can be recovered by averaging $y(t)$ on a sufficiently large interval. Hence, a number of points of the nonlinear function $\bar{f}(.)$ can thus be accurately estimated by repeating the above experiment successively for V_1 to V_n.

These ideas are formalized in the two-stage identification procedure of Table 1.

TABLE I. NONLINEARITY IDENTIFICATION (NI)

1. Data acquisition
Apply the piecewise signal analytically defined as follows, for all $t \in \left[(j-1)NT_r \;\; jNT_r\right]$ with $j = 1 \ldots n$ $v(t) = V_j$ (9)   Record the resulting output $y(t)$, $t \in \left[0 \; nNT_r\right]$.
2. Nonlinearity points estimation
Compute the (undisturbed output) mean value on each interval $\left[(j-1)NT_r \;\; jNT_r\right]$ with $j = 1 \ldots n$ $\hat{W}_j(N) = \frac{1}{NT_r}\int_{(j-1)NT_r}^{jNT_r} y(t)dt$ (10)   Then, the set of couples $\left(V_j, \hat{W}_j(N)\right)$ with $j = 1 \ldots n$, are estimates of n points all belonging to the trajectory of nonlinearity $\bar{f}(.)$.

Proposition 1. The points of coordinates $\left(V_j, \hat{W}_j(N)\right)$, for $j = 1 \ldots n$, obtained from the data collected on the time interval $\left[0 \; nNT_r\right]$, converge (in probability) to the trajectory of nonlinearity $\bar{f}(.)$ as $N \to \infty$.

4 Linear subsystem identification

In this section, an identification method is proposed to obtain estimates of the complex gain corresponding to the two linear subsystems $\bar{G}_i(s)$ and $\bar{G}_o(s)$ at the frequencies $k\omega$ $(k = 0,1,\ldots)$ whatever $\omega > 0$. From the NI procedure (Table 1), one gets estimates of n different points on the path of nonlinearity $\bar{f}(.)$. Furthermore, the larger the parameter N is, the better the estimation accuracy. For simplicity, we presently suppose that the estimated points have been exactly determined.

Recall that, the static nonlinearity $f(.)$ can be accurately represented, within any finite interval, with a polynomial of finite order, where the polynomial order depends on the interval length.Then, one has for all $x \in \left[v_m \; v_M\right]$:

$$f(x) \approx \sum_{l=0}^{p} c_l x^l \quad \text{with } p < \infty \tag{11a}$$

Then, it is readily seen, using (4b) and (11a), that $\bar{f}(.)$ can be developed as follows:

$$\bar{f}(x) \approx \sum_{l=0}^{p} \bar{c}_l x^l \tag{11b}$$

with $\bar{c}_l = G_o(0) G_i(0)^l c_l$ $(l = 0 \ldots p)$. All along this Section, the identified system is submitted to a given sine input:

$$v(t) = x_0 + V \sin(\omega t) \quad (\omega > 0) \tag{12}$$

where the amplitude $V > 0$ and x_0 is any point in the working interval, this latter may be chosen equal to zero. Let T be the corresponding period $(T = 2\pi/\omega)$. As the linear subsystem $\bar{G}_i(s)$ is asymptotically stable with unit static gain, it follows from (3)-(4a) that the internal signal $u_i(t)$ turns out to be (in steady state):

$$u_i(t) = x_0 + V\left|\bar{G}_i(j\omega)\right| \sin\left(\omega t + \varphi_i(\omega)\right) \tag{13}$$

with $\varphi_i(\omega) = \arg\left(\bar{G}_i(j\omega)\right)$. Also, it is readily obtained using (3), (11a-b) and (13):

$$u_o(t) \approx \sum_{l=0}^{p} \bar{c}_l \left(x_0 + V\left|\bar{G}_i(j\omega)\right| \sin\left(\omega t + \varphi_i(\omega)\right)\right)^l \tag{14}$$

The factor multiplying $\bar{c}_l$ in (14) is written as:

$$\begin{aligned}&\left(x_0 + V\left|\bar{G}_i(j\omega)\right| \sin\left(\omega t + \varphi_i(\omega)\right)\right)^l \\ &\quad = \sum_{r=0}^{l} C_r^l x_0^{\,r} \left(V\left|\bar{G}_i(j\omega)\right| \sin\left(\omega t + \varphi_i(\omega)\right)\right)^{l-r}\end{aligned} \tag{15a}$$

where the value of the binomial coefficient C_r^l is given explicitly by:

$$C_r^l = \frac{l!}{(l-r)!\,r!} \tag{15b}$$

Indeed, it is readily obtained by combining (14)-(15b):

$$u_o(t) = \sum_{l=0}^{p} \bar{c}_l \sum_{r=0}^{l} C_r^l x_0^{\,r} \left(V\left|\bar{G}_i(j\omega)\right|\right)^{l-r} \left(\sin(\omega t + \varphi_i(\omega))\right)^{l-r} \tag{16}$$

Furthermore, the power formulas $(\sin\theta)^{2l}$ and $(\sin\theta)^{2l+1}$ can also be given respectively as

$$(\sin\theta)^{2l} = \frac{1}{2^{2l}} C_l^{2l} + \frac{(-1)^l}{2^{2l-1}} \sum_{r=0}^{l-1} (-1)^r C_r^{2l} \cos\left(2(l-r)\theta\right) \tag{17a}$$

$$(\sin\theta)^{2l+1} = \frac{(-1)^l}{4^l} \sum_{r=0}^{l} (-1)^r C_r^{2l+1} \sin\left((2l+1-2r)\theta\right) \tag{17b}$$

Finally, the internal signal $u_o(t)$ can be expressed as:

$$u_o(t)=\sum_{l=0}^{p}\bar{c}_l\sum_{k=0}^{l}A_k\left(\left|\bar{G}_i(j\omega)\right|\right)\sin\left(k\omega t+\alpha_k(k\varphi_i(\omega))\right) \quad (18)$$

where the amplitude A_k particularly depends on $\left|\bar{G}_i(j\omega)\right|$. The phase α_k depends on the term $\varphi_i(\omega)$. We note that, (18) may have the following form:

$$u_o(t)=\sum_{k=0}^{p}B_k\left(\left|\bar{G}_i(j\omega)\right|\right)\sin\left(k\omega t+\beta_k\left(\varphi_i(\omega),\left|\bar{G}_i(j\omega)\right|\right)\right) \quad (19)$$

The amplitude B_k particularly depends on $\left|\bar{G}_i(j\omega)\right|$ and coefficients $\bar{c}_l$ $(l=0,...,p)$. The phase β_k depends on all these parameters in addition to the phase $\varphi_i(\omega)$. As the linear subsystem $\bar{G}_o(s)$ is asymptotically stable with $\bar{G}_o(0)=1$, let $\varphi_o(\omega)=\arg\left(\bar{G}_o(j\omega)\right)$. The final output of system is written:

$$y(t)=\sum_{k=0}^{p}B_k\left|\bar{G}_o(jk\omega)\right|\sin\left(k\omega t+\beta_k+\varphi_o(k\omega)\right)+\xi(t) \quad (20)$$

On the other hand, one can notice that the steady-state undisturbed output $w(t)$ is periodic of same period as the input, it can be developed in Fourier series:

$$w(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(k\omega t)+b_k\sin(k\omega t) \quad (21)$$

with:

$$a_k=\frac{2}{T}\int_0^T w(t)\cos(k\omega t)dt\,,\ b_k=\frac{2}{T}\int_0^T w(t)\sin(k\omega t)dt \quad (22)$$

where $k=0,1,2,...$. One immediately gets from (21):

$$y(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(k\omega t)+b_k\sin(k\omega t)+\xi(t) \quad (23)$$

The right side of (23) simplifies to:

$$y(t)=s_0+\sum_{k=1}^{\infty}s_k\sin(k\omega t+\phi_k)+\xi(t) \quad (24)$$

with:

$$s_k=\sqrt{{a_k}^2+{b_k}^2} \quad \text{for } k=1,\,2,\,... \quad (25a)$$

$$\phi_k=\tan^{-1}\left(\frac{a_k}{b_k}\right) \quad \text{for } k=1,\,2,\,... \quad (25b)$$

$$s_0=\frac{a_0}{2}=\frac{1}{T}\int_0^T w(t)dt \quad (26c)$$

Knowing that:

$$\lim_{k\to\infty}s_k=0 \quad (26d)$$

Remark 1. a) Considering the above assumptions, practically, it is reasonable to limit the development in Fourier series of $y(t)$ to those frequencies for which the Fourier series coefficients are significant. Furthermore, it readily follows from (24)-(26d) that $\sum_{k=p+1}^{\infty}\frac{s_k^2}{2}=0$ as $p\to\infty$.

b) The choice of the polynomial order p is as follows:
For any point x_0 in the working interval and a given error ε $(=1\%,\ 2\%,\ ...)$, let P_t and P_p denote, respectively, the total power and the power of the first p components of the output signal. Using the Parseval's identity (e.g. Ljung, 1987):

$$P_t=\frac{a_0^2}{4}+\sum_{k=1}^{\infty}\frac{a_k^2+b_k^2}{2}=s_0^2+\sum_{k=1}^{\infty}\frac{s_k^2}{2}=\frac{1}{T}\int_0^T y^2(t)dt<\infty \quad (27a)$$

$$P_p=\frac{a_0^2}{4}+\sum_{k=1}^{p}\frac{a_k^2+b_k^2}{2}=s_0^2+\sum_{k=1}^{p}\frac{s_k^2}{2}<\infty \quad (27b)$$

Finally, for $p=1,2,...$, we seek the minimum order p satisfying the following condition:

$$P_p\geq\left(\frac{100\ -\varepsilon}{100}\right)P_t \quad (28)$$

for some $0<\varepsilon<<100\,\%$ that is chosen by the user. The amplitude V of the input signal is reduced if necessary. ☐

If the condition (28) holds, then the right side of (24) simplifies to

$$y(t)=s_0+\sum_{k=1}^{p}s_k\sin(k\omega t+\phi_k)+\xi(t) \quad (29)$$

By using (20) and (29), we deduce the following relations:

$$B_k\left(\left|\bar{G}_i(j\omega)\right|\right)\left|\bar{G}_o(jk\omega)\right|=s_k \quad \text{for } k=1...p \quad (30a)$$

$$\beta_k\left(\varphi_i(\omega),\left|\bar{G}_i(j\omega)\right|\right)+\varphi_o(k\omega)=\phi_k \quad \text{for } k=1...p \quad (30b)$$

Accordingly, equations (20) and (29) show how to obtain the complex amplitudes $\bar{G}_i(j\omega)$ and $\bar{G}_o(jk\omega)$ using the two couples (B_k,β_k) and (s_k,ϕ_k) $(k=0,1,...,p)$. This is performed noticing that the right side of (20) is nothing other than the Fourier series expansion of the output signal $y(t)$, up to noise $\xi(t)$. Consequently, the procedure to estimate the output $y(t)$ is as follows: First, we assume that the condition (28) holds, so the equation (29) is maintained. Next, given that all deterministic terms on the right side of (29) are periodic, with common period T, and $\xi(t)$ is a zero-mean ergodic white noise, the effect of the latter can be filtered considering the following trans-period averaging of the output:

$$y_f(t,M)=\frac{1}{M}\sum_{i=1}^{M} y(t+(i-1)T)\ ;\quad 0\le t<T \tag{31}$$

Indeed, it is readily obtained using (29) and (31):

$$\lim_{M\to\infty} y_f(t,M)=s_0+\sum_{k=1}^{p} s_k \sin(k\omega t+\phi_k) \tag{32}$$

That is, the s_k's and ϕ_k's turn out to be (w.p.1) the limits of Fourier series parameters of $y_f(t,M)$ as $M\to\infty$. These parameters are given by the usual expressions:

$$s_k(M)=\sqrt{a_k(M)^2+b_k(M)^2}\quad (k=1,2,\dots,p) \tag{33a}$$

$$\phi_k(M)=\tan^{-1}\left(\frac{a_k(M)}{b_k(M)}\right)\quad (k=1,2,\dots,p) \tag{33b}$$

$$s_0(M)=\frac{1}{T}\int_0^T y_f(t,M)dt=\frac{a_0(M)}{2} \tag{33c}$$

where:

$$a_k(M)=\frac{2}{T}\int_0^T y_f(t,M)\cos(k\omega t)dt$$
$$\qquad (k=0\dots p) \tag{34}$$
$$b_k(M)=\frac{2}{T}\int_0^T y_f(t,M)\sin(k\omega t)dt$$

Then, it follows from (32)-(34) that:

$$\lim_{M\to\infty} s_k(M)=s_k\ \text{(w.p.1)}\ \text{for}\ (k=0\dots p) \tag{35a}$$

$$\lim_{M\to\infty} \phi_k(M)=\phi_k\ \text{(w.p.1)}\ \text{for}\ (k=1\dots p) \tag{35b}$$

Proposition 2. The estimates $\hat{G}_i(j\omega,M)$ and $\hat{G}_o(jk\omega,M)$ obtained by the FGI procedure are consistent i.e. $\hat{G}_i(j\omega,M)\to\bar{G}_i(j\omega)$ and $\hat{G}_o(jk\omega,M)\to\bar{G}_o(jk\omega)$ w.p.1 as $M\to\infty$. □

5 Simulation

The system to be identified is analytically described by equations (1)-(3) with:

$$G_i(s)=\frac{0.01}{(s+0.1)(s+0.5)}\ ;\ G_o(s)=\frac{0.1}{(s+0.2)(s+0.01)} \tag{36a}$$

$$f(x)=\exp(x) \tag{36b}$$

Then, the parameterized system will be identified:

$$\bar{G}_i(s)=\frac{0.05}{(s+0.1)(s+0.5)}\ ;\ \bar{G}_o(s)=\frac{0.002}{(s+0.2)(s+0.01)} \tag{37a}$$

$$\bar{f}(x)=G_o(0)f\left(G_i(0)\,x\right)=50\exp(0.2\,x) \tag{37b}$$

The above system is submitted to piecewise constant input (Fig.2). Fig.2 also shows the output signal. Fig.3 shows the nonlinearity $\bar{f}(.)$ considered in simulation and the couples estimates $\left(V_j,\hat{W}_j(N)\right)$.

TABLE II. FREQUENCY GAINS IDENTIFICATION (FGI)

1. Data acquisition
▪ Choose a frequency $\omega\in\{\omega_1,\omega_2,\dots,\omega_N\}$, let $T=2\pi/\omega$. ▪ Apply the sine signal, with period *T*, defined as follows: $v(t)=x_0+V\sin(\omega t)\quad t\ge 0$ (38) ▪ Take a sufficiently long record of the output signal.
2. Estimate of the filtered output
▪ Fix a given error ε (1%, 2%, ...), select an order p satisfying the following condition: $P_p\ge\left(\frac{100-\varepsilon}{100}\right)P_t$ (39) where P_t is the total power output signal and P_p is the power of the first p harmonics of $y(t)$. ▪ If necessary, vary the point x_0 and reduce the amplitude *V*. ▪ Generate the filtered output $y_f(t,M)$ using (31) and compute its Fourier series coefficients $a_k(M)$ and $b_k(M)$.
3. Data processing
▪ From the input sequence *v*(*t*), using the equations (11a)-(19), *w*(*t*) may have the following expression: $w(t)=\sum_{k=0}^{p} B_k\left\|\bar{G}_o(jk\omega)\right\|\sin\left(k\omega t+\beta_k+\varphi_o(k\omega)\right)$ (40) where the parameter B_k (resp. β_k), for $k=1\dots p$, depends on $\left\|\bar{G}_i(j\omega)\right\|$ (resp. $\varphi_i(\omega)$ and $\left\|\bar{G}_i(j\omega)\right\|$). ▪ Compute the estimates $\hat{G}_i(j\omega,M)$ and $\hat{G}_o(jk\omega,M)$: $B_k\left(\left\|\bar{G}_i(j\omega)\right\|\right)\left\|\bar{G}_o(jk\omega)\right\|=s_k(M)$ (41a) $\beta_k\left(\varphi_i(\omega),\left\|\bar{G}_i(j\omega)\right\|\right)+\varphi_o(k\omega)=\phi_k(M)$ (41b)
4. Estimation for all frequencies
▪ Repeat steps 1 through 3 for all frequencies $\{\omega_1,\dots,\omega_N\}$.

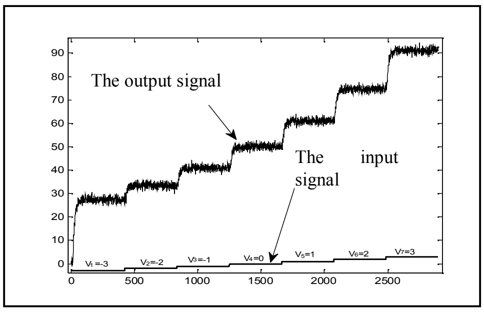

Fig.2. Shape of the resulting disturbed output signal.

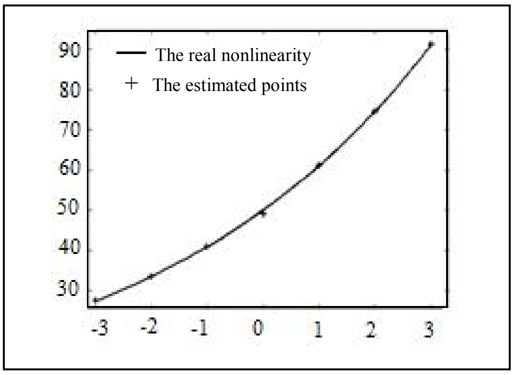

Fig.3. Nonlinear element $\bar{f}(.)$ and $\left(V_j, \hat{W}_j(N)\right)$.

For an error $\varepsilon = 0.6\%$ and $V \le 1.5$, the condition (39) is satisfied if $p \ge 3$. Let fix $p = 3$. Then, it follows from the power formulas (17a-b), the standard trigonometric formulas and using the procedure as explained in Section 4, one immediately gets:

$$\begin{aligned}
s_0 &= \bar{c}_0 + \frac{1}{2}\bar{c}_2 \left(V\left|\bar{G}_i(j\omega)\right|\right)^2 \\
s_1 &= V\left|\bar{G}_i(j\omega)\right|\left|\bar{G}_o(j\omega)\right|\left(\bar{c}_1 + \frac{3}{4}\bar{c}_3\left(V\left|\bar{G}_i(j\omega)\right|\right)^2\right) \\
s_2 &= -\frac{1}{2}\bar{c}_2\left(V\left|\bar{G}_i(j\omega)\right|\right)^2\left|\bar{G}_o(j2\omega)\right| \\
s_3 &= -\frac{1}{4}\bar{c}_3\left(V\left|\bar{G}_i(j\omega)\right|\right)^3\left|\bar{G}_o(j3\omega)\right|
\end{aligned} \tag{42a}$$

$$\begin{aligned}
&\phi_1(\omega) = \varphi_i(\omega) + \varphi_o(\omega); \\
&\phi_2(\omega) = 2\varphi_i(\omega) + \varphi_o(2\omega) + \frac{\pi}{2}; \phi_3(\omega) = 3\varphi_i(\omega) + \varphi_o(3\omega)
\end{aligned} \tag{42b}$$

The coefficients $\bar{c}_l \ (l = 0,\ldots,3)$ can be estimated using the NI procedure for 4 values $V_j \le 1.5 \ (j = 1,\ldots,4)$:

$$\hat{\bar{c}}_0 = 49.7;\ \hat{\bar{c}}_1 = 10.1;\ \hat{\bar{c}}_2 = 0.97;\ \hat{\bar{c}}_3 = 0.075 \tag{43}$$

Finally, for $\omega = 0.01 rd/s$ and $M = 40$, the following results are obtained:

$$\begin{aligned}
&\left|\hat{G}_i(j\omega, M)\right| = 1.03; \left|\hat{G}_o(j\omega, M)\right| = 0.74; \left|\hat{G}_o(j2\omega, M)\right| = 0.41; \\
&\left|\hat{G}_o(j3\omega, M)\right| = 0.29; \hat{\varphi}_i(j\omega, M) = -0.11;\ \hat{\varphi}_o(j\omega, M) = -0.85; \\
&\hat{\varphi}_i(j2\omega, M) = -0.27; \hat{\varphi}_o(j2\omega, M) = -1.18; \hat{\varphi}_i(j3\omega, M) = -0.33; \\
&\hat{\varphi}_o(j3\omega, M) = -1.42; \hat{\varphi}_o(j4\omega, M) = -1.55; \hat{\varphi}_o(j6\omega, M) = -1.64; \\
&\hat{\varphi}_o(j9\omega, M) = -1.85
\end{aligned} \tag{44}$$

6 Conclusion

We have developed a new two-stage frequency identification method to deal with WH systems identification. The originality of the present study lies in the fact that the phases of the two linear subsystems can be separated, also both linear subsystems are nonparametric and of unknown structure. Accordingly, the linear subsystems are not necessarily finite order. The nonlinear element $f(.)$ has any form and, in particular, may be noninvertible. This is only supposed to be accurately represented, within any finite interval, with a polynomial of finite order. Another feature of the method is the fact that the excitation signals are easily generated and the estimation algorithms can be simply implemented, compared with several published approaches. Finally, we note that the choice of the frequency band of interest is not required.

References:

[1] E.W. Bai, Z. Cai, S. Dudley-Javorosk, R.K. and Shields, Identification of a modified Wiener-Hammerstein system and its application in electrically stimulated paralyzed skeletal muscle modelling, *Automat. Contr.*, vol. 45, 2009, pp. 736-743.

[2] A. Marconato, J. Sjoberg, and J. Schoukens, Initialization of nonlinear state-space models applied to the Wiener-Hammerstein benchmark, *Control Engineering Practice*, vol. 20, 2012, pp. 1126-1132.

[3] G. Pillonetto, A. Chiuso, and G. De Nicolao, Prediction error identification of linear systems: a nonparametric Gaussian regression approach, *Automatica*, vol. 47, no. 2, 2011, pp. 291-305.

[4] A. Brouri, F. Giri, F. Ikhouane, F.Z. Chaoui, and O. Amdouri, Identification of Hammerstein- Wiener systems with Backlash input nonlinearity bordered by straight lines, in 19th IFAC, Cape Town, 2014, pp. 475-480.

[5] J. Paduart, L. Lauwers, R. Pintelon, and J. Schoukens, Identification of a Wiener–Hammerstein system using the polynomial nonlinear state space approach, *Control Engineering Practice*, vol. 20, 2012, pp. 1133-1139.

[6] L. Ljung, *System Identification – Theory for the User*, Prentice-Hall, Englewood Cliffs, N J, 1987.