Editors
Valeri Mladenov
Nikos Mastorakis

Some Integral Equations with Modified Argument

by Dr. Maria Dobritoiu

ISBN: 978-1-61804-372-6

Some Integral Equations with Modified Argument

Editors
Prof. Valeri Mladenov
Prof. Nikos Mastorakis

Author

Maria Dobritoiu

Published by WSEAS Press
www.wseas.org

ISBN: 978-1-61804-372-6

Some Integral Equations with Modified Argument

Published by WSEAS Press
www.wseas.org

All papers of the present volume were peer reviewed by two independent reviewers. Acceptance was granted when both reviewers' recommendations were positive.

ISBN: 978-1-61804-372-6

World Scientific and Engineering Academy and Society

Preface

The theory of integral equations is an important part in applied mathematics. The first books with theme of study, the integral equations appeared in the 19th century and early 20th century, and they have been authored by some of the famous mathematicians: N. Abel (1802-1829), A. Cauchy (1789-1857), E. Goursat (1858-1936), M. Bocher (1867-1918), David Hilbert (1862-1943), Vito Volterra (1860-1940), Ivar Fredholm (1866-1927), E. Picard (1856-1941), T. Lalescu (1882-1929). The first treatise in this field appeared in 1910 (T. Lalescu 1911, M. Bocher 1912, D. Hilbert 1912, V. Volterra 1913) (see I.A. Rus [100]). In the 20th century, the theory of integral equations had a spectacular development, both in terms of mathematical theories that may apply, and in terms of effective approximation of solutions.

The main methods that apply to the study of integral equations are: fixed point methods, variational methods, iterative methods and numerical methods. In this book was applied a fixed point method by applying the contraction principle. By this approach, the study of an integral equation represents the development of a fixed point theory, which contains the results on existence and uniqueness of the solution, the integral inequalities (lower-solutions and upper-solutions), the theorems of comparison, the theorems of data dependence of the solution (continuous data dependence and the differentiability of the solution with respect to a parameter) and an algorithm for approximating its solution.

The integral equations, in general, and the integral equations with modified argument, in particular, have been the basis of many mathematical models from various fields of science, with high applicability in practice, e.g., the integral equation from theory of epidemics and the Chandrasekhar's integral equation.

In this book, the Picard operators technique has been used for all the stages of this type of study.

This book is a monograph of integral equations with modified argument and contains the results obtained by the author in a period that began in the years of study in college and ended up with years of doctoral studies, both steps being carried out under the scientific coordination of Prof. Dr. Ioan A. Rus from Babes-Bolyai University of Cluj-Napoca. It is addressed to all who are concerned with the study of integral equations with modified argument and of knowledge of results and/or of obtaining new results in this area. The book is useful, also, to those concerned with the study of mathematical models governed by integral equations, generally, and by integral equations with modified argument, in particular.

Finally, we mention several authors of the used basic treatises having the theme of integral equations: T. Lalescu, I. G. Petrovskii, K. Yosida, Gh. Marinescu, A. Haimovici, C. Corduneanu, Gh. Coman, I. Rus, G. Pavel, I. A. Rus, W. Walter, D. Guo, V. Lakshmikantham, X. Liu, W. Hackbusch, D. V. Ionescu, Şt. Mirică, V. Mureşan, A. D. Polyanin, A. V. Manzhirov, R. Precup, I. A. Rus, M. A. Şerban, Sz. András.

I dedicate this book to my parents *Ana* and *Alexandru*.

Dr. Maria Dobrițoiu
University of Petroșani, Romania

Acknoledgements

This book was published with the support of WSEAS. I would like to expresss my sincere thanks to the WSEAS Team for their support in entering the results obtained in the study of integral equations with modified argument into the international scientific circuit. I wholeheartedly appreciate their help. Thank you WSEAS.

The Author

Table of contents

Overview of the book

The integral equations, in general, and those with modified argument, in particular, form an important part of applied mathematics, with links with many theoretical fields, specially with practical fields. The first papers that treated the integral equations had as authors renowned mathematicians, such as: N. H. Abel, J. Liouville, J. Hadamard, V. Volterra, I. Fredholm, E. Goursat, D. Hilbert, E. Picard, T. Lalescu, E. Levi, A. Myller, F. Riez, H. Lebesgue, G. Bratu, H. Poincaré, P. Levy, E. Picone. T. Lalescu was the author of the first book about integral equations (Bucharest 1911, Paris 1912).

This book is a study of some of the integral equations with modified argument and it focuses mostly on the study of the following integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) \;, \quad t \in [a,b] \;, \tag{1}$$

where $K : [a,b]\times[a,b]\times\boldsymbol{B}^4 \to \boldsymbol{B}$, $f : [a,b] \to \boldsymbol{B}$, $g : [a,b] \to [a,b]$, and $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ is a Banach space.

Starting with the Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t) \;, \quad t \in [a,b] \;, \tag{2}$$

which is a mathematical model from the turbo-reactors industry, we have also considered a modification of the argument through a continuous function $g : [a,b] \to [a,b]$, thus obtaining the integral equation with modified argument (1). It is an example of a nonlinear Fredholm integral equation with modified argument.

The integral equations (1) and (2) have been studied by the author, laying down the conditions of existence and uniqueness of the solution, the conditions of the continuous data dependence of the solution, and also, of differentiability of the solution with respect to a parameter and the conditions of approximating the solution, and the obtained results were published in papers [2], [22], [23], [24], [26], [29], [31], [33], [34], [35], [37], [38].

The book contains results of existence and uniqueness, of comparison, of data dependence, of differentiability with respect to a parameter and of approximation for the solution of the integral equation with modified argument (1) and a few results related to the solution of a well known equation from the epidemics theory.

Chapter 1, entitled "*Preliminaries*", that has eight paragraphs, is an introductory chapter which presents the notations and a few classes of operators that are used in this book, the basic notions and the abstract results of the fixed point theory and also, the notions from the Picard operators theory on L-spaces and the fiber contractions principle.

There are also presented the quadrature formulas (the trapezoids formula, the rectangles formula and Simpson's quadrature formula) that were used for the calculus of the integrals that appear in the terms of the successive approximations sequence from the obtained method of approximating the solution of the integral equation (1).

The seventh paragraph contains a very brief overview of Fredholm and Volterra nonlinear integral equations and the basic results regarding the existence and uniqueness of the solutions of these equations (see [10]).

In the eighth paragraph there are presented two mathematical models governed by functional-integral equations: an integral equation from physics and a mathematical model of the spreading of an infectious disease.

The first model refers to equation (2), and the results of existence and uniqueness, data dependence and approximation of the solution (theorems 1.8.1, 1.8.2 and 1.8.3), presented in this paragraph, were obtained by the author and published in the papers [2], [22], [23], [24], [26] and [29].

The presentation of the mathematical model of the spreading of an infectious disease, which refers to the following equation from the epidemics theory

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds \ , \qquad (3)$$

contains results obtained by K.L. Cooke and J.L. Kaplan [18], D. Guo, V. Lakshmikantham [42], I. A. Rus [88], [93], Precup [73], [75], R. Precup and E. Kirr [78], C. Iancu [47], [48], I. A. Rus, M. A. Şerban and D. Trif [114].

The fiber generalized contractions theorem 1.5.2, theorem which is a result obtained by I.A. Rus in paper [100], was used to lay down theorem 1.5.3 in this chapter, theorem that was published in paper [27].

Chapter 2, entitled "*Existence and uniqueness of the solution*" has five paragraphs. Three of them contain the conditions of existence and uniqueness of the integral equation with modified argument (1), in the space $C([a,b],\boldsymbol{B})$ and in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$, in a general case and in two particular cases for $\boldsymbol{B}$: $\boldsymbol{B} = \boldsymbol{R}^{\boldsymbol{m}}$ and $\boldsymbol{B} = \boldsymbol{l}^{2}(\boldsymbol{R})$. In order to prove these results, the following theorems have been used: *the Contraction Principle* 1.3.1 and *Perov's theorem* 1.3.4.

The fourth paragraph of this chapter contains three examples: two integral equations with modified argument and a system of integral equations with modified argument and for each of these examples the conditions of existence and uniqueness, which were obtained by using some of the results presented in the previous paragraphs, are given.

In the fifth paragraph was studied the existence and uniqueness of the solution of the integral equation with modified argument

$$x(t) = \int_{\Omega} K(t, s, x(s), x(g(s)), x|_{\partial\Omega})ds + f(t) \ , \quad t \in \overline{\Omega} \ , \qquad (4)$$

where $\Omega \subset \boldsymbol{R}^{\boldsymbol{m}}$ is a bounded domain, $K : \overline{\Omega} \times \overline{\Omega} \times R^m \times R^m \times C(\partial\Omega, R^m) \to R^m$, $f : \overline{\Omega} \to R^m$ and $g : \overline{\Omega} \to \overline{\Omega}$. This equation is a generalization of the integral equation (1).

Some of the author's results that are presented in this chapter, were published in papers [31] and [37].

Chapter 3, entitled "*Gronwall lemmas and comparison theorems*" has three paragraphs. Several Gronwall lemmas, comparison theorems and a few examples for the integral equation with modified argument (1) are presented. These results represent the properties of the solution of this integral equation. In order to prove the results presented in this chapter, the following theorems were used: *the abstract Gronwall lemma* 1.4.1 and *the abstract comparison lemmas* 1.4.4 and 1.4.5. The third paragraph of this chapter contains examples which are applications of the results given in the first two paragraphs. These results were obtained by the author and published in the papers [35] and [38].

In chapter 4, entitled "*Data dependence*", which has four paragraphs, the author present the theorems of data dependence, the differentiability theorems with respect to a and b (limits of integration), and theorems of differentiability with respect to a parameter, of the solution of the integral equation with modified argument (1) and also, a few examples.

In order to prove the results presented in this chapter, the following theorems were used: *the abstract data dependence theorem* 1.3.5 and *the fiber generalized contractions theorem* 1.5.2. These results were published in the papers [31], [33], [34] and [37].

In chapter 5, entitled "*Numerical analysis of the Fredholm integral equation with modified argument* (2.1)", following the conditions of one of the existence and uniqueness theorems given in the second chapter, a method of approximating the solution of the integral equation (1) is given, using the successive approximations method. For the calculus of the integrals that appear in the successive approximations sequence, the following quadrature formulas were used*: the trapezoids formula*, *Simpson's formula* and *the rectangles formula*.

This chapter has five paragraphs. The first paragraph presents the statement of the problem and the conditions under which it is studied. In paragraphs 2, 3 and 4 there are presented the results obtained related to the method of approximating the solution of the integral equation (1). The results obtained in paragraphs 2, 3 and 4 are used in the fifth paragraph to approximate the solution of an integral equation with modified argument, given as example.

The MatLab software was used to calculate the approximate value of the integral which appears in the general term of the successive approximations sequence, with trapezoids formula, rectangles formula and Simpson's formula; for each of these cases was obtained the approximation of the solution of the integral equation given as example. In appendices 1, 2 and 3 one can find the results obtained by these programs written in MatLab.

Some of the results obtained by the author for equation (1), that were presented in this chapter, were published in paper [31]. The results obtained for the numerical analysis of equation (2) were published in the papers [22], [23], [24] and [26].

Chapter 6, entitled "*An equation from the theory of epidemics*", has four paragraphs and contains the results obtained through a study of the solution of the integral equation (3), using the Picard operators. This study was carried out by the author in collaboration with I.A. Rus and M.A. Şerban, and the results obtained, refering to the existence and uniqueness of the solution in a subset of the space $C(\boldsymbol{R},I)$, lower and upper solutions, data dependence and differentiability of the solution of the integral equation (3), with respect to a parameter, are published in paper [36].

The bibliography used to write this book contains several important basic treatises from the theory of integral equations, scientific papers on this topic, of some known authors and scientific articles which contains the author's own results.

Each of the six chapters has its own bibliography and all these references are listed in a bibliography at the end of the book.

The basic treatises used for the study in this book are the following: T. Lalescu [56], I. G. Petrovskii [69], K. Yosida [129], Gh. Marinescu [59] and [60], A. Haimovici [45], C. Corduneanu [20], Gh. Coman, I. Rus, G. Pavel and I. A. Rus [15], D. Guo, V. Lakshmikantham and X. Liu [43], W. Hackbusch [44], C. Iancu [48], D. V. Ionescu [49] and [50], V. Lakshmikantham and S. Leela [55], Şt. Mirică [61], D. S. Mitrinović, J. E. Pečarić and A. M. Fink [62], V. Mureşan [65], B. G. Pachpatte [66], A. D. Polyanin and A. V. Manzhirov [72], R. Precup [74] and [81], I. A. Rus [88], [89], [95], [106], I. A. Rus, A. Petruşel and G. Petruşel [109], D. D. Stancu, Gh. Coman, O. Agratini and R. Trîmbiţaş [119], D. D. Stancu, Gh. Coman and P. Blaga [120], M. A. Şerban [124], Sz. András [6].

This book is a monograph of some of the integral equations with modified argument and it contains the results on which the author had been working, starting with the university years and ending with the years of Ph.D. studies, under the the scientific coordination of professor Ioan A. Rus from the "Babeş-Bolyai" University of Cluj-Napoca, Romania.

The purpose of this book is to help those who wish to study the integral equations with modified argument, to learn about these results and to obtain new results in this field.

This book is also useful for those who would like to study the mathematical models governed by integral equations, in general, and integral equations with modified argument, in particular.

1 Preliminaries

In this chapter we present the principal notions and results which were used in this book. It is an introductive chapter composed of seven paragraphs, which contains the notations that were used, several classes of the used operators, the basic notions and the abstract results from the fixed point theory, some notions from the theory of Picard operators on L-spaces and *the fiber contractions principle*, that represent the basis of the obtained results which were presented in this book.

There are also presented the quadrature formulas: the trapezoids formula, the rectangle formula and Simpson's quadrature formula which were used for the calculus of the integrals that appear in the terms of the successive approximations sequence from the method of approximation of the solution of integral equation (1).

In the last paragraph of this chapter there are presented two mathematical models governed by functional-integral equations: an integral equation from physics and a mathematical model of the spreading of an infectious disease.

The theorems 1.5.3, 1.8.1, 1.8.2, 1.8.3 and the algorithm for approximating the solution of the integral equation (1.31) presented in this chapter, are the results obtained by the author. These results were published in the papers [1], [19] and [20].

1.1 Notations and notions

Let X be a nonempty set and $A : X \to X$ an operator. Then, we denote:

$P(X) := \{ Y \subset X \,/\, Y \neq \varnothing \}$ – the set of all nonempty subsets of X

$A^0 := 1_X$, $A^1 := A, \ldots, A^{n+1} := A \circ A^n$, $n \in \mathbf{N}$ – the iterates operators of the operator A

$I(A) := \{ Y \in P(X) \,/\, A(Y) \subset Y \}$ – the family of the nonempty subsets of X, invariants for the operator A

$F_A := \{ x \in X \,/\, A(x) = x \}$ – the fixed points set of the operator A .

Let X be an ordered set and $A : X \to X$ an operator. Then, we denote:

$(UF)_A := \{ x \in X \,/\, A(x) \le x \}$ – the upper fixed points set of the operator A

$(LF)_A := \{ x \in X \,/\, A(x) \ge x \}$ – the lower fixed points set of the operator A .

Let (X,d) be a metric space, $x_0 \in X$, $r \in \mathbf{R}_+$ and $A : X \to X$ an operator. Then

$B(x_0;r) := \{ x \in X \,/\, d(x,x_0) < r \}$ – the opened sphere with center x_0 and radius r

$\overline{B}(x_0;r) := \{ x \in X \,/\, d(x,x_0) \le r \}$ – the closed sphere with center x_0 and radius r.

Also, we denote:

$P_b(X) := \{ Y \in P(X) \,/\, Y$ is bounded set$\}$ – the set of all nonempty and bounded subsets of X

$P_{cl}(X) := \{ Y \in P(X) \,/\, Y = \overline{Y} \}$ – the set of all nonempty and closed subsets of X

$P_{b,cl}(X)$: $=\{Y\in P(X)$ / Y is bounded and $Y=\overline{Y}\}$ – the set of all nonempty, bounded and closed subsets of X

$P_{cp}(X)$: $=\{Y\in P(X)$ / Y is compact set$\}$ – the set of all nonempty and compact subsets of X

$I_b(A)$: $=\{Y\in I(A)$ / Y is bounded set$\}$ – the family of all bounded subsets of X, invariants for the operator A

$I_{cl}(A)$: $=\{Y\in I(A)$ / $Y=\overline{Y}\}$ – the family of all closed subsets of X, invariants for the operator A

$I_{b,cl}(A)$: $=\{Y\in I_b(A)$ / $Y=\overline{Y}\}$ – the family of all bounded and closed subsets of X, invariants for the operator A

$CT(X,X)$: $=\{f : X\to X$ / f is contraction$\}$

$Lip[a,b]$: $=\{f : [a,b]\to \mathbf{R}$ / f satisfies the Lipschitz condition$\}$.

In what follows, we present a few basic notions which were used in this book.

Definition 1.1.1. ([53]) Let X be a nonempty set. A functional $d : X\times X\to \mathbf{R}$ that has the following properties:

(i) $d(x, y)\geq 0$, for all $x, y\in X$; $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y\in X$;

(iii) $d(x, y)\leq d(x, z) + d(z, y)$, for all $x, y, z\in X$,

is called *metric on X*.

The conditions (i), (ii) and (iii) are called *the axioms of the metric*.

Definition 1.1.2. ([53]) A pair (X, d) consisting of a set X and a metric d on X, is called *metric space*.

Definition 1.1.3. ([53]) A sequence $(x_n)_{n\in \mathbf{N}}$ of elements in a metric space (X, d) *converges* to an element $x_0\in X$, if for each $\varepsilon > 0$ there exists $n_0(\varepsilon)\in \mathbf{N}$ such that

$d(x_n , x_0) < \varepsilon$, for each $n > n_0(\varepsilon)$.

Definition 1.1.4. ([53]) A sequence $(x_n)_{n\in \mathbf{N}}$ of elements in a metric space (X, d) is called *fundamental sequence* or *Cauchy sequence*, if for each $\varepsilon > 0$ there exists $n_0(\varepsilon)\in \mathbf{N}$ such that

$n\in \mathbf{N}$, $m > n_0(\varepsilon)$ imply $d(x_n,x_m) < \varepsilon$.

The following theorem is true.

Theorem 1.1.1. ([41]) *Any convergent sequence is a Cauchy sequence*.

Proof. Let $\varepsilon > 0$ and $(x_n)_{n\in \mathbf{N}}$ be a sequence which converges to x_0. Therefore, for this $\varepsilon > 0$ there exists $n_0(\varepsilon)\in \mathbf{N}$ such that $d(x_n, x_0) < \frac{\varepsilon}{2}$, for each $n > n_0(\varepsilon)$ and $d(x_m, x_0) < \frac{\varepsilon}{2}$, for each $m > n_0(\varepsilon)$. Now, we have

$d(x_n, x_m)\leq d(x_n, x_0) + d(x_0, x_m) < \varepsilon$

and the proof is complete.

The reciprocal theorem of the theorem 1.1.1, generally is not true.

Definition 1.1.5. ([53]) The metric space in which every fundamental sequence is convergent is called a *complete metric space*.

Example 1.1.1. ([53]) Let $\boldsymbol{R^m}$ be the set X , i.e. $X = \boldsymbol{R^m}$. The functionals d, δ, $\rho : \boldsymbol{R^m}\times\boldsymbol{R^m} \to \boldsymbol{R}_+$ defined by the following relations:

$$d(x,y) = \left(\sum_{i=1}^{m} (x_i - y_i)^2 \right)^{\frac{1}{2}}, \quad x, y \in \boldsymbol{R^m} ,$$

$$\delta(x,y) = \sum_{i=1}^{m} | x_i - y_i |, \quad x, y \in \boldsymbol{R^m} ,$$

$$\rho(x,y) = \max_{1 \le i \le m} | x_i - y_i |, \quad x, y \in \boldsymbol{R^m}$$

are metrics on X, and the metric spaces (X, d), (X, δ) and (X, ρ) are complete metric spaces.

Let $(X,+,\boldsymbol{R},|\cdot|)$ be a normed linear space, i.e. a real linear space $(X,+,\boldsymbol{R})$ endowed with a norm $|\cdot|$.

Definition 1.1.6. ([53]) A functional $|\cdot| : X \to \boldsymbol{R}$ that satisfies the following conditions:

(*i*) $|x| \ge 0$, for all $x \in X$; $|x| = 0$ if and only if $x = 0$;

(*ii*) $|\lambda x| = |\lambda| \cdot |x|$, for all $\lambda \in \boldsymbol{R}$ and $x \in X$;

(*iii*) $|x+y| \le |x| + |y|$, for all $x, y \in X$

is called *norm on X*.

The functional $d : X\times X \to \boldsymbol{R}$, defined by $d(x, y) = |x- y|$, represent a metric on the set X. This metric is called *metric induced by the norm* $|\cdot|$.

Definition 1.1.7. ([53]) A normed linear space is called *Banach space* (or *complete normed linear space*) if this space is complete with respect to the metric induced by the norm.

Let $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ be a Banach space. In this book were considered the following cases: $\boldsymbol{B} = \boldsymbol{R^m}$ and $\boldsymbol{B} = \boldsymbol{l^2}(\boldsymbol{R})$.

In the particular case $\boldsymbol{B} = \boldsymbol{R^m}$, $|\cdot|$ is one of the following norms (see [53]):

- *Euclidean norm* $\|\cdot\|_E : \boldsymbol{R^m} \to \boldsymbol{R}$, defined by the relation:

$$\| x \|_E := \left(\sum_{i=1}^{m} | x_i |^2 \right)^{\frac{1}{2}}, \quad x \in \boldsymbol{R^m} , \tag{1.1}$$

- *Minkowski's norm* $\|\cdot\|_M : \boldsymbol{R^m} \to \boldsymbol{R}$, defined by the relation:

$$\| x \|_M := \sum_{i=1}^{m} | x_i | , \quad x \in \boldsymbol{R^m} , \tag{1.2}$$

- *norm of Chebyshev* $\|\cdot\|_C : \boldsymbol{R^m} \to \boldsymbol{R}$, defined by the relation:

$$\| x \|_C := \max_{1 \le i \le m} | x_i | \ , \quad x \in \boldsymbol{R^m} \ , \tag{1.3}$$

and the spaces ($\boldsymbol{R^m}$, + , $\boldsymbol{R}$, $\|\cdot\|_E$), ($\boldsymbol{R^m}$, + , $\boldsymbol{R}$, $\|\cdot\|_M$) and ($\boldsymbol{R^m}$, + , $\boldsymbol{R}$, $\|\cdot\|_C$) are Banach spaces.

In the particular case $\boldsymbol{B} = \boldsymbol{l^2}(\boldsymbol{R})$,

$$l^2(\boldsymbol{R}) := \left\{ (x_n)_{n \in N} \ / \ x_n \in R, \ \sum_{n \in N} x_n^2 < +\infty \right\},$$

the norm $|\cdot|$ is the functional $\|\cdot\|_{l^2(R)} : \boldsymbol{l^2}(\boldsymbol{R}) \to \boldsymbol{R}_+$, defined by the relation:

$$\| x \|_{l^2(R)} := \left(\sum_{i=1}^{\infty} x_i^2 \right)^{\frac{1}{2}} \ , \quad x \in \boldsymbol{l^2}(\boldsymbol{R}) \tag{1.4}$$

and the space ($\boldsymbol{l^2}(\boldsymbol{R})$, + , $\boldsymbol{R}$, $\|\cdot\|_{l^2(R)}$) is a Banach space (see [53]).

In this book we will consider *the metric of Chebyshev* $d : C[a,b] \times C[a,b] \to \boldsymbol{R}$, on the set $C[a,b] = \{ f : [a,b] \to \boldsymbol{R} \ / \ f$ is a continuous function $\}$, defined by the relation:

$$d(f, g) := \max_{x \in [a,b]} | f(x) - g(x) | \ , \quad \text{for all } f, g \in C[a,b] \ , \tag{1.5}$$

and the norm induced by this metric, i.e. *the norm of Chebyshev*

$$\| f \|_C := \max_{x \in [a,b]} | f(x) | \ , \quad \text{for all } f \in C[a,b] \ . \tag{1.6}$$

Also, we will consider *the norm of Chebyshev* on the space $C([a,b],\boldsymbol{R^m})$, defined by the relation:

$$\| x \|_C := \begin{pmatrix} \| x_1 \|_C \\ \ldots \\ \| x_m \|_C \end{pmatrix} , \quad \text{for all } x = \begin{pmatrix} x_1 \\ \ldots \\ x_m \end{pmatrix} \in C([a,b],\boldsymbol{R^m}) \ , \tag{1.7}$$

where $\| x_k \|_C = \max_{t \in [a,b]} | x_k(t) |$, $k = \overline{1,m}$.

1.2 Classes of operators

The successive approximations method is a basic tool in the theory of operatorial equations, generally, and in the fixed point theory, in particular, and the evolution of this method occurred in three periods.

In the first period, represented by L. A. Cauchy, J. Liouville, R. Lipschitz, G. Peano, E. I. Fredholm and E. Picard, for each fixed point equation are studied:

(*i*) the uniqueness of the solution,

(*ii*) the convergence of the successive approximations sequence,

(*iii*) the limit is a solution of a given equation.

In the second period that begins with the papers of S. Banach and R. Caccioppoli, are given, in an abstract case, conditions that include (*i*), (*ii*) and (*iii*). Thus, with S. R. Banach and R. Caccioppoli begins the metric theory of the fixed point, for which we mention also the papers of W. A. Kirk and B. Sims [35], I. A. Rus [54], [55], [64] and V. Berinde [8].

During the third period, the conclusion of a fixed point metric theorem is used as a means of definition. This way introduce new classes of operators: Picard operators and weakly Picard operators (I. A. Rus [55]).

In what follows we give the problem underlying *the successive approximations method* (see [53]).

Let (X, d) be a metric space, $f : X \to X$ an operator and $x_0 \in X$. Relative to the operator f the following problem is formulated:

Under what conditions on f and X, the sequence of the successive approximations $\left(f^n(x_0)\right)_{n\in N}$, $x_0 \in X$, *converges and its limit is a fixed point of operator* f ?

Next, we present several classes of operators from the metric theory of fixed point ([10], [54], [64]).

Let (X, d) and (Y, ρ) be two metric spaces.

Definition 1.2.1. An operator $f : X \to Y$ is *continuous in point* $x_0 \in X$, if for each sequence $(x_n)_{n\in \mathbf{N}}$, $x_n \in X$, which converges to x_0 , the sequence $(f(x_n))_{n\in \mathbf{N}}$ is convergent and its limit is $f(x_0)$, i.e.:

$$\forall (x_n)_{n\in \mathbf{N}} \ , x_n \in X , \ \lim_{n\to\infty} d(x_n, x_0) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \rho(f(x_n), f(x_0)) = 0 \ .$$

The operator f is *continuous on* X if f is continuous at any point $x_0 \in X$.

Definition 1.2.2. An operator $f : X \to Y$ is a *bounded operator* if

$$A \in P_b(X) \quad \Rightarrow \quad f(A) \in P_b(Y) \ .$$

Definition 1.2.3. An operator $f : X \to Y$ is a *compact operator* if

$$A \in P_b(X) \quad \Rightarrow \quad \overline{f(A)} \in P_{cp}(Y) \ .$$

Definition 1.2.4. An operator $f : X \to Y$ is a *complete continuous operator* if f is compact and continuous operator.

Example 1.2.1. ([41]) A linear operator $f : \mathbf{R}^m \to \mathbf{R}^m$ is a complete continuous operator.

Example 1.2.2. ([41]) Let $K : [a,b]\times [a,b] \to \mathbf{R}$ be a continuous operator. The integral operator A : $C[a,b] \to C[a,b]$, $A(x) \mapsto x$, where

$$A(x)(t) = \int_a^b K(t,s)x(s)ds \ ,$$

is a complete continuous operator.

Definition 1.2.5. An operator $f : X \to Y$ is *uniformly continuous operator on X* if for any $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that from

$$d(x', x'') < \delta \text{ , for all } x', x'' \in X \ ,$$

it results that $\rho(f(x'), f(x'')) < \varepsilon$.

Definition 1.2.6. An operator $f : X \to Y$ is a *closed operator* if the graph of f ,

$G(f)=\{(x,f(x))\in X\times Y\,/\,x\in X\}\subset X\times Y$

is a closed set.

Definition 1.2.7. An operator $f:X\to X$ is called:

(*i*) *Lipschitz operator* (*α–Lipschitz operator*) if there exists $\alpha\in \boldsymbol{R}_+$ such that

$$d(f(x),f(y))\le \alpha\cdot d(x,y)\ ,\ \textit{for all}\ \ x,y\in X\ .$$

(*ii*) *contraction* (*α–contraction*) i.e. $f\in CT(X,X)$, if there exists $\alpha\in(0,1)$ such that f is *α–Lipschitz*.

(*iii*) *contractive operator* if $d(f(x),f(y))<d(x,y)$, for all $x,y\in X$, $x\ne y$.

(*iv*) *non-expansive operator* if f is a 1–*Lipschitz operator*, i.e.

$$d(f(x),f(y))\le d(x,y)\ ,\ \text{for all}\ \ x,y\in X\ .$$

(*v*) *non-contractive operator* if $d(f(x),f(y))\ge d(x,y)$, for all $x,y\in X$.

(*vi*) *expansive operator* if $d(f(x),f(y))>d(x,y)$, for all $x,y\in X$, $x\ne y$.

(*vii*) *expansion operator* (*α–expansion operator*) if there exists $\alpha>1$ such that

$$d(f(x),f(y))\ge \alpha\cdot d(x,y)\ ,\ \text{for all}\ \ x,y\in X\ .$$

(*viii*) *isometry* if $d(f(x),f(y))=d(x,y)$, for all $x,y\in X$.

Example 1.2.3. ([64])

a) The operator $f:\boldsymbol{R}\to\boldsymbol{R}$, $f(x)=\dfrac{1}{2}x$ is a contraction.

b) The operator $f:\boldsymbol{R}\to\boldsymbol{R}$, $f(x)=2x$ is an expansion operator.

c) The operator $f:\boldsymbol{R}\to\boldsymbol{R}$, $f(x)=x$ is an isometry.

d) The operator $f:[1,+\infty)\to[1,+\infty)$, $f(x)=x+\dfrac{1}{x}$ is a contractive operator.

According to the above definitions we have:

Theorem 1.2.1. (*Gh. Coman, I. Rus, G. Pavel and I. A.Rus* [10]) *The following implications are true*:

$(ii)\ \Rightarrow\ (i)$

$(ii)\Rightarrow(iii)\Rightarrow(iv)$.

In this book we use the continuous, bounded, Lipschitz and contractions operators.

1.3 Fixed point theorems

In order to establish some of the results presented in this book, were used several basic theorems of fixed point theory, which we present below.

Thus, in order to obtain the existence and uniqueness results of the chapter 2, was used following fixed point theorem.

Theorem 1.3.1. (*Contraction Principle*) *Let* (X, d) *be a complete metric space and* $A : X \to X$ *an* α*-contraction* ($\alpha < 1$). *Under these conditions we have*:

(*i*) *A has a unique fixed point* x^*, *i.e.* $F_A = \{x^*\}$;

(*ii*) *the successive approximations sequence considered for an* $x_0 \in X$

$$x_0,\ x_1=A(x_0),\ x_2=A(x_1)=A(A(x_0))=A^2(x_0),\ \ldots,\ x_n=A(x_{n-1})=A^n(x_0),\ \ldots$$

converges to x^*, *i.e.*

$$x^* = \lim_{n\to\infty} A^n(x_0), \quad \textit{for all } x_0 \in X ;$$

(*iii*) $d(x^*, A^n(x_0)) \le \dfrac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

The proof of this theorem, which became classic, can be found in [53] and for this reason is omitted. Also, we mention the following fixed point theorem in a set with two metrics:

Theorem 1.3.2. (*M. G. Maia*) *Let* X *be a nonempty set,* d *and* ρ *two metrics defined on* X *and* $A : X \to X$ *an operator. Suppose that*

(*i*) $d(x,y) \le \rho(x,y)$, *for all* $x, y \in X$;

(*ii*) (X, d) *is a complete metric space* ;

(*iii*) $A : (X, d) \to (X, d)$ *is a continuous operator* ;

(*iv*) *the operator* $A : (X, \rho) \to (X, \rho)$ *is an* α*–contraction*.

Then

(*a*) $F_A = \{x^*\}$;

(*b*) $A^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, *for all* $x \in X$.

In the paper [53] I. A. Rus makes the following remark:

Remark 1.3.1. (I. A. Rus [53]) The Maia's theorem remains true if the condition (*i*) is replaced by the following condition:

(*i'*) *there exists* $c > 0$ *such that* $d(f(x),f(y)) \le c\rho(x,y)$, *for all* $x, y \in X$.

Another variant of the Maia's theorem, presented by I. A. Rus in the paper [70], is as follows:

Theorem 1.3.2'. (I. A. Rus [70]) *Let X be a nonempty set,* d *and* ρ *two metrics defined on* X *and* $A : X \to X$ *an operator. Suppose that*

(*i*) (X, d) *is a complete metric space* ;

(*ii*) *there exists* $k \in \mathbf{N}$, *sucht that the operator* $A^k: (X, \rho) \to (X, d)$ *is uniformly continuous*;

(*iii*) $A : (X, d) \to (X, d)$ *is a closed operator* ;

(*iv*) *the operator* $A : (X, \rho) \to (X, \rho)$ *is an* α*–contraction*.

Then

(a) $F_A = \{x_A^*\}$;

(b) $A^n(x) \xrightarrow{d} x_A^*$ *as* $n \to \infty$, *for all* $x \in X$;

(c) $A^n(x) \xrightarrow{\rho} x_A^*$ *as* $n \to \infty$, *for all* $x \in X$, *and*

$$\rho(A^n(x),\ x_A^*) \le \alpha^n \cdot \rho(x,\ x_A^*)\ ,\quad \textit{for all } n \in \mathbf{N}^* \textit{ and } x \in X;$$

(d) $\rho(x,\ x_A^*) \le \dfrac{1}{1-\alpha}\rho(x, A(x))$, *for all* $x \in X$.

Also, in the paper [70], I. A. Rus makes a few remarks:

Remark 1.3.2. (I. A. Rus [70]) The implication (i) – (iv) $\Rightarrow$ (a) + (b) is the fixed point theorem of Maia, 1.3.2. (see M. G. Maia [38], I. A. Rus [52], R. Precup [51]).

Remark 1.3.3. Using the Picard operators (see [66] and [67]) we have the following conclusions of theorem 1.3.2':

(a) + (b) – The operator $A : (X, d) \to (X, d)$ is a Picard operator.

(c) + (d) – The operator $A : (X, \rho) \to (X, \rho)$ is an $\dfrac{1}{1-\alpha}$–Picard operator.

Remark 1.3.4. ([66]) The condition $d(A^k(x), A^k(y)) \le C\rho(x, y)$ implies the condition (ii) of the theorem 1.3.2'.

In $\mathbf{R}^m$ we consider the natural ordering, i.e. if $x, y \in \mathbf{R}^m$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, then $x \le y$ if and only if $x_i \le y_i$, $i = \overline{1,m}$.

Definition 1.3.1. ([53]) Let X be a nonempty set. An operator $d : X \times X \to \mathbf{R}^m$ that satisfies the conditions:

(i) $d(x, y) \ge 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$, ($0 = (0, 0, \ldots, 0)$);

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$,

is called *generalized metric on X*.

Definition 1.3.2. ([53]) A pair (X, d) which consists of a set X and a generalized metric d, defined on X is called *generalized metric space*.

Example 1.3.1. ([53]) Let $\mathbf{R}^m$ be the set X i.e. $X = \mathbf{R}^m$. The operator $d : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m$, defined by the relation:

$$d(x, y) = (|x_1 - y_1|, \ldots, |x_m - y_m|)$$

is a generalized metric defined on X.

Example 1.3.2. ([53]) Let $C([a,b], \boldsymbol{R}^m)$ be the set X i.e. $X = C([a,b],\boldsymbol{R}^m)$. The operator $d : X \times X \to \boldsymbol{R}^m$, defined by the relation:

$$d(f,g) = \left(\max_{x\in[a,b]} \left| f_1(x) - g_1(x) \right|, \ldots, \max_{x\in[a,b]} \left| f_m(x) - g_m(x) \right| \right), \quad f, g \in X,$$

is a generalized metric defined on X, and (X, d) is a generalized metric space.

Remark 1.3.5. ([53]) The notions of convergent sequence, fundamental sequence, complete generalized metric space, generalized metric induced by a generalized norm are defined similarly as for ordinary metric spaces.

Definition 1.3.3. ([53]) Let (X, d) be a generalized metric space. An operator A: $X \to X$ satisfies a Lipschitz condition, if there exists a matrix $Q\in M_{m\times m}(\boldsymbol{R}_+)$, such that

$$d(A(x), A(y)) \leq Q\cdot d(x, y)\ , \ \textit{for all}\ \ x, y \in X\ .$$

Definition 1.3.4. ([53]) A matrix $Q\in M_{n\times n}(\boldsymbol{R})$ is called *matrix convergent to zero* if the matrix Q^k converges to the null matrix as $k \to \infty$.

The next theorem gives three equivalent conditions of convergence to zero of a matrix $Q\in M_{n\times n}(\boldsymbol{R}_+)$ and was used in example 2.4.2 of chapter 2, in example 3.3.2 of chapter 3, and in examples 4.4.2 and 4.4.3 of chapter 4.

Theorem 1.3.3. (see [53]) *Let $Q \in M_{m\times m}(\boldsymbol{R}_+)$ be a matrix. The following statements are equivalent:*

(*i*) $Q^k \to 0$ *as* $k \to \infty$;

(*ii*) *The eigenvalues* λ_k, $k = \overline{1,m}$, *of the matrix Q, satisfies the condition* $|\lambda_k| < 1$, $k = \overline{1,m}$;

(*iii*) *The matrix* $I_m - Q$ *is non-singular and*

$$(I_m - Q)^{-1} = I + Q + Q^2 + \ldots \quad .$$

Theorem 1.3.4. (*A. I. Perov*) *Let* (X, d) *be a complete generalized metric space, with the metric* $d(x,y) \in \boldsymbol{R}^m$ *and* $A : X \to X$ *an operator. Suppose that there exists a matrix* $Q\in M_{m\times m}(\boldsymbol{R}_+)$, *such that*

(*i*) $d(A(x), A(y)) \leq Q\, d(x, y)$, *for all* $x, y \in X$;

(*ii*) $Q^n \to 0$ *as* $n \to \infty$.

Then

(*a*) *A has a unique fixed point* x^*, *i.e.* $F_A = \{x^*\}$;

(*b*) *the successive approximations sequence* $x_n = A^n(x_0)$, *converges to* x^* *for all* $x_0\in X$, *i.e.*

$$x^* = \lim_{n\to\infty} A^n(x_0)\ , \quad \textit{for all}\ \ x_0\in X\ .$$

In addition, the following estimation

$$d(A^n(x), x^*) \leq (I_m - Q)^{-1}Q^n\, d(x_0, A(x_0)),\ \ n\in \boldsymbol{N}^*$$

is accomplished.

Some consequences of this theorem can be found in [53].

In the chapter 4 we study the data dependence of the solution of integral equation (2.1) and the following theorem was used.

Theorem 1.3.5. (*Abstract data dependence theorem*) *Let* (X, d) *be a complete metric space and* A, $B : X \to X$ *two operators. Suppose that*

(*i*) *A is a contraction. Let* $\alpha < 1$ *a Lipschitz constant of* A *and* $F_A = \{x_A^*\}$;

(*ii*) $x_B^* \in F_B$;

(*iii*) *there exists* $\eta > 0$ *such that*

$$d(A(x), B(x)) \le \eta, \quad \text{for all } x \in X.$$

Under these conditions we have:

$$d(x_A^*, x_B^*) \le \frac{\eta}{1-\alpha}.$$

The proof of this theorem can be found in [53] and for this reason is omitted.

Also, we mention the following theorem of data dependence of fixed points in a set with two metrics (I. A. Rus [70]).

Theorem 1.3.6. (I. A. Rus [70]) *Let X be a nonempty set, d and* ρ *two metrics on X,* $A : X \to X$ *an operator and suppose that the conditions of theorem 1.3.2' are satisfied.*

Let $B : X \to X$ *be an operator and* $\eta > 0$ *such that*

$$\rho(A(x), B(x)) \le \eta, \quad \text{for all } x \in X.$$

Then

$$x_B^* \in F_B \quad \Rightarrow \quad \rho(x_A^*, x_B^*) \le \frac{\eta}{1-\alpha}.$$

1.4 Picard operators on *L*-spaces

In order to establish some of the results presented in the chapter 2 and in the chapter 3, were used a few results from the Picard operators theory on *L*-spaces, *the abstract Gronwall's lemma* 1.4.1 and *the abstract comparison lemmas* 1.4.4 and 1.4.5.

In the third period of development of the successive approximations method, as mentioned in paragraph 1.2, were introduced the Picard and weakly Picard operators (I. A. Rus [55]). The weakly Picard operators theory is useful in studying some properties of the solutions of those equations for which can be used the method of successive approximations.

In what follows, we present the general area where is acting the method of successive approximations (problem formulated in 1975 by K. Iseki) and some results of Picard and weakly Picard operators theory.

Let X be a nonempty set and the following set:

$$s(X) := \{(x)_{n\in \mathbf{N}} \mid x_n \in X,\ n \in \mathbf{N}\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$, an operator.

Definition 1.4.1. ([55]) The triplet $(X, c(X), Lim)$ is called *L-space* if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbf{N}$, then $(x_n)_{n\in \mathbf{N}} \in c(X)$ and $Lim(x_n)_{n\in \mathbf{N}} = x$;

(ii) If $(x_n)_{n\in \mathbf{N}} \in c(X)$ and $Lim(x_n)_{n\in \mathbf{N}} = x$, then for all subsequences $(x_{n_i})_{i\in N}$ of the sequence $(x_n)_{n\in N}$ we have $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

Definition 1.4.2. ([55]) An element $(x_n)_{n\in \mathbf{N}} \in c(X)$ is *convergent sequence*, *his limit* is $x := Lim(x_n)_{n\in \mathbf{N}}$ and we will write

$x_n \to x$, $n \to \infty$.

In what follows we denote by $(X, \to)$ an L–space.

In general, all set endowed with a structure involving a notion of convergence for sequences, from an L–space. Such structures are metric spaces, generalized metric spaces ($d(x,y) \in R_+^m$; $d(x,y) \in \mathbf{R}_+ \cup \{+\infty\}$), endowed with two metrics.

Let X and Y be two metric spaces and $M(X,Y)$ the set of operators defined from X, to Y. We denote by $\xrightarrow{p}$ the punctual convergence in $M(X,Y)$, by $\xrightarrow{unif}$ the uniform convergence in $M(X,Y)$ and by $\xrightarrow{cont}$ the convergence with continuity in $M(X,Y)$. The spaces $(M(X,Y), \xrightarrow{p})$, $(M(X,Y), \xrightarrow{unif})$ and $(M(X,Y), \xrightarrow{cont})$ are L–spaces (see I. A. Rus [67]).

1.4.1 Picard operators

Let $(X,\to)$ be an L-space.

Definition 1.4.3. ([55]) An operator $A : X \to X$ is *Picard operator* if

(i) $F_A = \{x_A^*\}$;

(ii) $A^n(x) \to x_A^*$, as $n \to \infty$, for all $x \in X$.

Remark 1.4.1. (see [53]) If A is a *Picard operator*, then A is a *Bessaga operator*, i.e.

$F_{A^n} = F_A = \{x^*\}$, for all $n \in \mathbf{N}^*$.

The metric fixed point theory gives us examples of Picard operators.

In the paper [53] are given examples of Picard operators defined on different L-spaces, of those present the following two examples:

a) (*Banach-Cacciopoli*) Let (X, d) be a complete metric space and $A : X \to X$ an α–contraction. Then A is a Picard operator.

b) (*Perov*) Let (X, d) be a complete generalized metric space with the metric $d(x,y) \in R_+^m$ and let $Q \in M_{m\times m}(\mathbf{R}_+)$ be a matrix such that $Q^n \to 0$ as $n \to \infty$. If an operator $A : X \to X$ is Q–contraction, i.e.

$d(A(x), A(y)) \leq Q\, d(x, y)$, for all $x, y \in X$, then A is a Picard operator.

1.4.2 Weakly Picard operators

Let $(X,\to)$ be an L–space.

Definition 1.4.4. ([55]) An operator $A : X \to X$ is *weakly Picard operator* if the sequence $\left(A^n(x)\right)_{n\in N}$ converges for all $x \in X$ and his limit (which may depend on x) is a fixed point of A.

A weakly Picard operator A for which $F_A = \{x^*\}$ is Picard operator.

If A is a weakly Picard operator, then $F_{A^n} = F_A \neq \varnothing$, for all $n \in \boldsymbol{N}^*$.

Examples of weakly Picard operators and their properties are presented and studied by many mathematicians, among which: I. A. Rus [55], [57], [58], [62], [65], I. A. Rus, S. Mureşan and V. Mureşan [69], A. Petruşel and I. A. Rus [46], M. A. Şerban [76], [77], M. A. Şerban, I. A. Rus and A. Petruşel [78] and many others.

If A is a weakly Picard operator, then defines the operator $A^\infty : X \to X$ by the relation:

$$A^\infty(x) := \lim_{n\to\infty} A^n(x)$$

and observe that $A^\infty(X) = F_A$.

We present below a generic example of weakly Picard operator.

Let (X_i, d_i), $i \in I$, be a family of metric spaces, $A_i : X_i \to X_i$ a family of Picard operators and x_i^* the unique fixed point of the operator A_i.

Let $X := \bigcup_{i\in I} X_i$ be the disjoint union of the sets of family $(X_i)_{i\in I}$ and $d : X \times X \to \boldsymbol{R}_+$, a metric on X, defined by the relation:

$$d(x,y) := \begin{cases} d_i(x,y) & if \quad x,y \in X_i, \quad i \in I \\ d_i(x,x_i^*) + d_j(y,x_j^*) + 1 & if \quad i \neq j, \quad x \in X_i, \quad y \in X_j \end{cases}.$$

Then the operator A is a weakly Picard operator (see I. A. Rus [57]).

A basic result in the weakly Picard operators theory is the following theorem.

Theorem 1.4.1. (*Theorem of characterization of weakly Picard operators*) *Let* $(X,\to)$ *be an L–space and* $A : X \to X$ *an operator. The operator A is weakly Picard operator if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda ,$$

such that

(a) $X_\lambda \in I(A)$, $\lambda \in \Lambda$;

(b) *the restriction of* A *to* X_λ, $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all* $\lambda \in \Lambda$.

This theorem is useful to prove that certain operators are weakly Picard operators.

To study the data dependence of the fixed points of an operator, is considered another class of weakly Picard operators defined on a metric space (X, d).

Definition 1.4.5. ([55]) By definition, a weakly Picard operator A is *c–weakly Picard operator*, $c > 0$, if and only if

$$d(x, A^{\infty}(x)) \le c\, d(x, A(x)), \quad \text{for all } x \in X.$$

For example, the operator $A : X \to X$ of α–contraction type, defined on the metric space (X, d) is c–weakly Picard operator with $c = (1-\alpha)^{-1}$.

Also, if (X, d) is a complete metric space, $A : X \to X$ is an operator and if we suppose that:

(i) there exists $\alpha \in (0,1)$ such that

$$d(A^2(x), A(x)) \le \alpha\, d(x, A(x)), \quad \text{for all } x \in X;$$

(ii) the operator A is a closed operator,

then A is *c–weakly Picard operator with* $c = (1-\alpha)^{-1}$.

The next result is a theorem of data dependence of the fixed points set of an operator and it is useful in the study of data dependence of the solutions of integral equations (I. A. Rus [67]).

Theorem 1.4.2. (I. A. Rus [67]) *Let (X, d) be a metric space and $A, B : X \to X$ two operators. Suppose that*

(i) *A is c–Picard operator* ;

(ii) x_B^* *is a fixed point of the operator B, i.e.* $x_B^* \in F_B$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(A(x), B(x)) \le \eta, \quad \text{for all } x \in X.$$

If we denote by x_A^*, *the unique fixed point of the operator A, then*

$$d(x_A^*, x_B^*) \le \eta c.$$

1.4.3 Picard operators on ordered *L*–spaces

Let $(X,\to)$ be an L–space and $\le$ an order relation on X. If the following implication is true

$$x_n \le y_n, \quad x_n \to x^*, \quad y_n \to y^*, \quad \text{as } n \to \infty \quad \Rightarrow \quad x^* \le y^*,$$

then, by definition, $(X, \to, \le)$ is an ordered L–space. The following lemma is true.

Lemma 1.4.1. (*Abstract Gronwall's lemma*, [64]) *Let $(X, \to, \le)$ be an ordered L-space and $A : X \to X$ an operator. Suppose that*

(i) *A is Picard operator* ;

(ii) *the operator A is increasing.*

If we denote by x_A^* *the unique fixed point of the operator A, then*

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

Gronwall's Lemma 1.4.1 for an ordered metric space $(X, d, \leq)$ is useful to determine the results of the chapter 3.

From Lemma 1.4.1 we have (see [67]):

Lemma 1.4.2. (I. A. Rus [67]) *Let* $(X, \to, \leq)$ *be an ordered L-space and* $A : X \to X$ *an increasing operator. Then*

(a) $(UF)_A \in I(A)$, $(LF)_A \in I(A)$;

(b) *if the restriction of* A *to* $(UF)_A \cup (LF)_A$, $A|_{(UF)_A \cup (LF)_A}$ *is Picard operator, then*

$$x \leq x_A^* \leq y, \quad \text{for all } x \in (LF)_A \text{ and } y \in (UF)_A .$$

In lemmas 1.4.1 and 1.4.2 may be replaced ”$(X, \to, \leq)$ *an ordered L-space*” with ”$(X, d, \leq)$ *an ordered metric space*” and the condition ”A is Picard operator” it may be replaced by a requirement to ensure that A is Picard operator in the ordered metric space $(X, d, \leq)$. One thus obtains the following results (see I. A. Rus [65], [67]):

Theorem 1.4.3. (I. A. Rus [65], [67]) *Let* $(X, d, \leq)$ *be an ordered and complete metric space and* A : $X \to X$ *an operator such that* A^k *is contraction for* $k \in \mathbf{N}^*$. *If we denote by* x_A^* *the unique fixed point of the operator* A, *then*

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

Theorem 1.4.4. (I. A. Rus [65], [67]) *Let* $(X, d, \leq)$ *be an ordered and complete metric space. Let* A : $X \to X$ *be an operator such that for all* $0 < a < b < +\infty$, *there exists* $L(a,b) \in (0,1)$ *such that*

$$x, y \in X, \quad a \leq d(x, y) \leq b \;\Rightarrow\; d(A(x), A(y)) \leq L(a,b) \cdot d(x, y) .$$

Then

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$,

where x_A^* *is the unique fixed point of the operator A.*

Theorem 1.4.5. (I. A. Rus [65], [67]) *Let* X *be a nonempty set, d and* ρ *two metrics defined on* X, $\leq$ *an order relation on* X *and* $A : X \to X$ *an operator. We suppose that*

(i) $(X, d, \leq)$ *is an ordered and complete metric space* ;

(ii) *there exists* $c > 0$ *such that* $d(A(x), A(y)) \leq c \cdot \rho(x, y)$, *for all* $x, y \in X$;

(iii) $A : \left(X, \xrightarrow{d}\right) \to \left(X, \xrightarrow{d}\right)$ *is continuous operator* ;

(iv) *the operator* $A : (X, \rho) \to (X, \rho)$ *is an l–contraction.*

Then A is a Picard operator on $\left(X, \xrightarrow{d}\right)$ *and if denote by* x_A^* *the unique fixed point of the operator A,we have*:

$(a)\ x \leq A(x) \ \Rightarrow\ x \leq x_A^*$;

$(b)\ x \geq A(x) \ \Rightarrow\ x \geq x_A^*$.

Theorem 1.4.6. (I. A. Rus [65], [67]) *Let* $(X, d, \leq)$ *be an ordered metric space with two metrics defined on X and* $A : X \to X$ *an operator. Suppose that*

(i) *A is* α*–contraction* ;

(ii) *the operator A is increasing.*

If denote by x_A^* *the unique fixed point of operator A, then*

$(a)\ x \leq A(x) \ \Rightarrow\ x \leq x_A^*$;

$(b)\ x \geq A(x) \ \Rightarrow\ x \geq x_A^*$.

The above abstract results have applications in the theory of differential and integral inequalities.

For Gronwall inequalities proved by classical methods, we mention: D. Bainov and P. Simeonov [5], P. R. Beesak [6], A. Constantin [12], S. S. Dragomir [29], V. Laksmikantham and S. Leela [36], D. S. Mitrinović, J. E. Pečarić and J. E. Fink [40], B. G. Pachpatte [43], W. Walter [79], M. Zima [80], and for concrete Gronwall inequalities, proved using above abstract results we mention: Sz. András [2], A. Buică [9], C. Crăciun [14], M. Dobriţoiu [27], N. Lungu and I. A. Rus [37], V. Mureşan [42], R. Precup and E. Kirr [50], I. A. Rus [58], [59], [60], [64], [65], [67], [68], M. A. Şerban [77], M. A. Şerban, I. A. Rus and A. Petruşel [78].

1.4.4 Weakly Picard operators on ordered *L*–spaces

We begin our considerations of these operators with the following lemma given by I. A. Rus in his paper [67].

Lemma 1.4.3. (I. A. Rus [67]) *Let* $(X, \to, \leq)$ *be an ordered L-spaces and* $A : X \to X$ *an operator, such that*

(i) *A is weakly Picard operator*.

(ii) *A is increasing operator*.

Then the operator A^∞ *is increasing*.

The following lemma is a comparison abstract result for an ordered *L*-spaces.

Lemma 1.4.4. (I. A. Rus [67]) *Let* $(X, \to, \leq)$ *be an ordered L-space and the operators* $A, B, C : X \to X$, *such that*

$(i)\ A \leq B \leq C$;

(ii) A, B, C *are weakly Picard operators* ;

(iii) *the operator* B *is increasing*.

Then

$$x \le y \le z \quad \Rightarrow \quad A^{\infty}(x) \le B^{\infty}(y) \le C^{\infty}(z) .$$

The next result is given by I. A. Rus [65], in the case of an ordered metric space.

Lemma 1.4.5. (*Abstract comparison lemma*, [65]) *Let* $(X, d, \le)$ *be an ordered metric space and the operators* $A, B, C : X \to X$, *such that*

(i) $A \le B \le C$;

(ii) A, B, C *are weakly Picard operators* ;

(iii) *the operator* B *is increasing* .

Then

$$x \le y \le z \quad \Rightarrow \quad A^{\infty}(x) \le B^{\infty}(y) \le C^{\infty}(z) .$$

In the same paper we find makes the following useful remark.

Remark 1.4.2. Let A, B, C be the operators defined in lemma 1.4.5. Moreover, suppose that $F_B = \{x_B^*\}$, i.e. B is Picard operator. Then

$$A^{\infty}(x) \le x_B^* \le C^{\infty}(x), \quad \text{for all } x \in X.$$

But $A^{\infty}(X) = F_A$, $C^{\infty}(X) = F_C$. Thus, we have

$$F_A \le x_B^* \le F_C .$$

Now, the following theorem is also an interesting and useful result.

Theorem 1.4.7. (Sz. András [4]) *If* $(X, \|\cdot\|, \le)$ *is an ordered normed space, and* A:$X \to X$ *is an increasing and weakly Picard operator*, *then the following implications are true*:

(a) *If* $x \in X$ *and* $x \le \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}(x)$, *then* $x \le A^{\infty}(x)$;

(b) *If* $x \in X$ *and* $x \ge \sum_{i=0}^{p-1} \alpha_i \cdot A^{i+1}(x)$, *then* $x \ge A^{\infty}(x)$,

where the numbers $\alpha_i \in (0,1)$, $i = \overline{0, p-1}$ *satisfies the relation*:

$$\sum_{i=0}^{p-1} \alpha_i = 1 .$$

The above lemmas are useful to study the properties of the solutions of differential and integral equations.

1.5 Fiber contractions principle

The theorem of fiber Picard operators is a fixed point theorem for operators defined on cartesian product and is useful to prove the differentiability of the solution of a functional-integral equation with respect to a parameter. This theorem, given by I. A. Rus in [63], is a generalization of the result given by M.W. Hirsch and C.C. Pugh in [31].

In the paper [63] I. A. Rus studied *the problem of the fiber Picard operators*:

Let $(X, \to)$ *be an L-space and* (Y, ρ) *a metric space,* $B : X \to X$, $C : X \times Y \to Y$, *two operators and* $A : X \times Y \to X \times Y$ *a triangular operator, such that*

$$A(x, y) = (B(x), C(x, y)), \quad x \in X, y \in Y . \tag{1.8}$$

Suppose that

(*i*) *B is Picard operator* (*weakly Picard operator*) ;

(*ii*) $C(x, \cdot)$ *is Picard operator, for all* $x \in X$.

Under what conditions A is Picard operator (*weakly Picard operator*) ?

and establishes the following theorem:

Theorem 1.5.1. (*Fiber Picard operators theorem*, I. A. Rus [63]) *Let* $(X, \to)$ *be an L-space,* (Y, ρ) *a metric space,* $B : X \to X$ *and* $C : X \times Y \to Y$ *two operators and* $A : X \times Y \to X \times Y$, $A = (B, C)$, *a triangular operator. Suppose that*

(*i*) (Y, ρ) *is a complete metric space* ;

(*ii*) $B : X \to X$ *is weakly Picard operator* ;

(*iii*) *there exists* $\alpha \in [0,1)$ *such that* $C(x, \cdot)$ *is an* α*–contraction, for all* $x \in X$;

(*iv*) *if* $(x^*, y^*) \in F_A$, *then* $C(x, y^*)$ *is continuous in* x^* .

Then A is a weakly Picard operator. Moreover, if B is Picard operator, then A is Picard operator too.

Another generalization of the result given by M.W. Hirsch and C.C. Pugh in [31], is *the fiber generalized contractions theorem* , given by I. A. Rus in [61].

Theorem 1.5.2. (*Fiber generalized contractions theorem* , I. A. Rus [61]) *Let* (X,d) *be a metric space* (*generalized or not*) *and* (Y,ρ) *a complete generalized metric space* ($\rho(x,y) \in \boldsymbol{R}^{\boldsymbol{m}}$).

Let $B : X \to X$ *and* $C : X \times Y \to Y$ *be two operators and* $A : X \times Y \to X \times Y$ *a continuous operator. Suppose that*

(*i*) $A(x, y) = (B(x), C(x,y))$, *for all* $x \in X$, $y \in Y$;

(*ii*) $B : X \to X$ *is a weakly Picard operator* ;

(*iii*) *there exists a matrix* $Q \in M_{m\times m}(\boldsymbol{R}_+)$, $Q^n \to 0$ *as* $n \to \infty$, *such that*

$$\rho(C(x, y_1), C(x, y_2)) \le Q\, \rho(y_1, y_2), \quad \textit{for all } x \in X,\ y_1, y_2 \in Y .$$

Under these conditions A is a weakly Picard operator. In addition, if B is a Picard operator, then A is a Picard operator too.

Theorems 1.5.1 and 1.5.2 are useful to study the differentiability with respect to a parameter, of the solutions of integral equations and of systems of integral equations, respectively. Some results in this regard were given by J. Sotomayor [72], V. Berinde [8], I. A. Rus [61], [62], [63], [64], [65], [67], M. A. Şerban, I. A. Rus and A. Petruşel [78], M. A. Şerban [75], [76], [77], Sz. András [3], [4], M. Dobriţoiu, I. A. Rus and M. A. Şerban [25], M. Dobriţoiu [19], [21], [22], [23], [26] and others.

Next, we present an application of *the fiber contractions principle* at the following integral equation

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)))ds+f(t),\quad t\in[\alpha,\beta], \tag{1.9}$$

where $\alpha,\beta\in\mathbf{R}$, $\alpha\le\beta$, $a, b\in[\alpha,\beta]$, $K\in C([\alpha,\beta]\times[\alpha,\beta]\times\mathbf{R}^m\times\mathbf{R}^m,\mathbf{R}^m)$,

$f\in C([\alpha,\beta],\mathbf{R}^m)$, $g\in C([\alpha,\beta],[\alpha,\beta])$, $x\in C([\alpha,\beta],\mathbf{R}^m)$.

The obtained result is the following theorem published in [19].

Theorem 1.5.3. (M. Dobriţoiu [19]) *Suppose that there exists a matrix* $Q\in M_{m\times m}(\mathbf{R}_+)$ *such that*

(*i*) $[2(\beta-\alpha)Q]^n\to 0$ *as* $n\to\infty$;

$$(ii)\begin{pmatrix}|K_1(t,s,u_1,u_2)-K_1(t,s,v_1,v_2)|\\ \cdot\ \cdot\ \cdot\\ |K_m(t,s,u_1,u_2)-K_m(t,s,v_1,v_2)|\end{pmatrix}\le Q\begin{pmatrix}|u_{11}-v_{11}|+|u_{21}-v_{21}|\\ \cdot\ \cdot\ \cdot\\ |u_{1m}-v_{1m}|+|u_{2m}-v_{2m}|\end{pmatrix}$$

for all $t, s\in[\alpha,\beta]$, $u_i, v_i\in\mathbf{R}^m$, $i = 1, 2$.

Then

(*a*) *the integral equation* (1.9) *has a unique solution,* $x^*(\cdot,a,b)\in C([\alpha,\beta],\mathbf{R}^m)$;

(*b*) *for all* $x^0\in C([\alpha,\beta],\mathbf{R}^m)$, *the sequence* $(x^n)_{n\in\mathbf{N}}$, *defined by the relation*:

$$x^{n+1}(t;a,b):=\int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b))ds+f(t),$$

converges uniformly to x^*, *for all t, a,* $b\in[\alpha,\beta]$ *and*

$$\begin{pmatrix}|x_1^n(t;a,b)-x_1^*(t;a,b)|\\ \cdot\ \cdot\ \cdot\\ |x_m^n(t;a,b)-x_m^*(t;a,b)|\end{pmatrix}\le[I_m-2(\beta-\alpha)Q]^{-1}[2(\beta-\alpha)Q]^n\begin{pmatrix}|x_1^0(t;a,b)-x_1^1(t;a,b)|\\ \cdot\ \cdot\ \cdot\\ |x_m^0(t;a,b)-x_m^1(t;a,b)|\end{pmatrix};$$

(*c*) *the function* $x^*:[\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta]\to\mathbf{R}^m$, $(t,a,b)\mapsto x^*(t;a,b)$ *is continuous*;

(*d*) *if* $K(t,s,\cdot,\cdot)\in C^1(\mathbf{R}^m\times\mathbf{R}^m,\mathbf{R}^m)$ *for all t,* $s\in[\alpha,\beta]$, *then* $x^*(t;\cdot,\cdot)\in C^1([\alpha,\beta]\times[\alpha,\beta],\mathbf{R}^m)$ *for all* $t\in[\alpha,\beta]$.

Proof. We denote $X := C([\alpha,\beta]^3,\mathbf{R}^m)$ and we consider the generalized norm on *X*, defined by the relation (1.7) in the paragraph 1.

We consider the operator $B:X\to X$ defined by the relation:

$$B(x)(t;a,b) := \int_a^b K(t,s,x(s;a,b),x(g(s);a,b))ds\,, \quad \text{for all } t,\, a,\, b \in [\alpha,\beta].$$

From the conditions (*i*), (*ii*) and applying *the Perov's theorem* 1.3.4, it results that the conlusions (*a*), (*b*) and (*c*) are fulfilled.

(*d*) We prove that there exists $\frac{\partial x^*}{\partial a}, \frac{\partial x^*}{\partial b} \in X$.

If suppose that there exists $\frac{\partial x^*}{\partial a}$, then from (1.9) it results that

$$\frac{\partial x^*(t;a,b)}{\partial a} = -K(t,a,x^*(a;a,b),x^*(g(a);a,b)) +$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b))}{\partial x_i^*(s;a,b)} \right) \cdot \left(\frac{\partial x^*(s;a,b)}{\partial a} \right) + \right.$$

$$\left. + \left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b))}{\partial x_i^*(g(s);a,b)} \right) \cdot \left(\frac{\partial x^*(g(s);a,b)}{\partial a} \right) \right] ds\;.$$

This relation leads us to consider the operator $C : X \times X \to X$ defined by the relation:

$$C(x,y)(t;a,b) := -K(t,a,x(a;a,b),x(g(a);a,b)) + \tag{1.10}$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b))}{\partial x_i(s;a,b)} \right) y(s;a,b) + \right.$$

$$\left. + \left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b))}{\partial x_i(g(s);a,b)} \right) y(g(s);a,b) \right] ds\;.$$

From the condition (*ii*), we obtain that

$$\left(\left| \frac{\partial K_j(t,s,u,w)}{\partial u_i} \right| \right)_{i,j=1}^m \le Q \qquad \text{and} \qquad \left(\left| \frac{\partial K_j(t,s,u,w)}{\partial w_i} \right| \right)_{i,j=1}^m \le Q\;, \tag{1.11}$$

for all $t, s \in [\alpha,\beta]$, $u, w \in \boldsymbol{R}^m$.

From (1.10) and (1.11) it results that

$$\left\| C(x,y_1) - C(x,y_2) \right\| \le 2(\beta-\alpha)Q\left\| y_1 - y_2 \right\|, \quad \text{for all } x, y_1, y_2 \in X\,.$$

Now, if we consider the operator $A : X \times X \to X \times X$, $A=(B, C)$, then the conditions of *the fiber generalized contractions theorem* 1.5.2 are satisfied. From this theorem it results that A is a Picard operator, and the sequences

$$x^{n+1}(t;a,b):=\int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b))ds+f(t)\ ,$$

$$y^{n+1}(t;a,b):=-K(t,a,x^n(s;a,b),x^n(g(s);a,b))+$$

$$+\int_a^b\left[\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b))}{\partial x_i^n(s;a,b)}\right)y^n(s;a,b)+\right.$$

$$\left.+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b))}{\partial x_i^n(g(s);a,b)}\right)y^n(g(s);a,b)\right]ds$$

converge uniformly (with respect to t, a, $b\in[\alpha,\beta]$) to $(x^*,y^*)\in F_A$, for all x^0, $y^0\in X$.

If we take $x^0 = y^0 = 0$, then $y^1=\dfrac{\partial x^1}{\partial a}$ and it proves by induction that $y^n=\dfrac{\partial x^n}{\partial a}$. Thus we obtain

$$x^n \xrightarrow{uniformly} x^* \quad \text{as} \quad n\to\infty\ , \ \text{and}$$

$$\frac{\partial x^n}{\partial a} \xrightarrow{uniformly} y^* \quad \text{as} \quad n\to\infty\ ,$$

and hence it results that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a}=y^*$.

By a similar reasoning it proves that there exists $\dfrac{\partial x^*}{\partial b}$.

1.6 Quadrature formulas

The chapter 5 gives a method for approximating the solution of the integral equation with modified argument (2.1)

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t)\ , \quad t\in[a,b]\ ,$$

by using the successive approximations method and a quadrature formula. To obtain the terms of the successive approximations sequence, must calculate the integrals that appear in the terms of this sequence. The calculus of these integrals is most often a very difficult problem and this is the reason of establishing an approximate calculation methods of these integrals. This problem is studied in the literature, being the subject of chapter "Numerical quadratures" in the Numerical Analysis.

For the calculus of the integrals that appear in the terms of the successive approximations sequence from the approximating algorithm for the solution of integral equation with modified argument (2.1) the following quadrature formulas were used: trapezoids formula, rectangles formula and Simpson's formula (see [10] [11], [34], [39], [53], [74]).

Next, we present these methods for calculating the approximate value of the integral of a function *f*.

1.6.1 Trapezoids formula

The trapezoids method for approximate integration of a function f consists of approximating the function f with a polygonal function, i.e. to approximate the given function f with a polygonal line with vertices on the graph.

Let $f \in C^2[a,b]$ be a function. A formula of approximate calculus of the integral $\int_a^b f(t)dt$ is:

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(a)+f(b)\right]+R(f) \ , \tag{1.12}$$

where $R(f)$ represent the rest of the formula. Due to the geometrical interpretation of the formula (1.12), this formula is called *the trapezoids formula*. Following the delimitation of $R(f)$, the trapezoids formula becomes:

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(a)+f(b)\right]-\int_a^b \frac{(s-a)(b-s)}{2} f''(s)ds \ . \tag{1.13}$$

To get a better result, it is considered a division Δ of the interval $[a,b]$ into n equal parts by the points $a = t_0 < t_1 < \ldots < t_n = b$ and we apply the trapezoids formula (1.13), to each subinterval $[t_{i-1}, t_i]$.

Under these conditions we obtain the following *trapezoids formula* (see [10], [11], [34], [39], [53], [74]):

$$\int_a^b f(t)dt = \frac{b-a}{2n}\left[f(a)+2\sum_{i=1}^{n-1} f(t_i)+f(b)\right]+R(f) \ , \tag{1.14}$$

where $R(f) = \sum_{i=1}^{n} R_i(f)$, and

$$R_i(f) = \int_{x_{i-1}}^{x_i} \varphi_i(s) f''(s)ds \quad , \qquad \varphi_i(s) = -\frac{(s-x_{i-1})(x_i-s)}{2} \quad , \quad i=\overline{1,n} \ .$$

For the rest of the formula (1.14) we have the estimate given by the relation:

$$\left|R(f)\right| \le M^T \frac{(b-a)^3}{12n^2} \quad , \tag{1.15}$$

where by M^T we denote

$$M^T = \max_{t\in[a,b]}\left| f''(t)\right| \ . \tag{1.16}$$

Because the calculation error of the integral $\int_a^b f(t)dt$ is less than ε > 0, it is sufficient that the number n of subintervals of the equidistant division Δ of interval $[a,b]$ to verify the relation:

$$n \ \ge \ \sqrt{M^T \frac{(b-a)^3}{12\varepsilon}} \ . \tag{1.17}$$

Also, in the case of an equidistant division Δ of interval $[a,b]$, for the rest $R(f)$ of the formula (1.14), we have the following estimates (see [7]):

$$|R(f)| \le \begin{cases} \dfrac{(b-a)^2}{4n} L & \text{if} \quad f \in Lip[a,b] \\ \dfrac{(b-a)^2}{4n} \| f' \| & \text{if} \quad f \in C^1[a,b]\,, \end{cases} \tag{1.18}$$

and

$$|R(f)| \le \frac{(b-a)^3}{12n^2} L'\,, \tag{1.19}$$

if $f \in C^1[a,b]$ and f' satisfy a Lipschitz condition on [*a*,*b*], with the constant $L' > 0$.

1.6.2 Rectangles formula

The rectangles method for approximate integration of a function *f* consists of approximating the function *f* with a constant function on intervals, i.e. to approximate the graph of function *f* with a polygonal line with sides parallel to the coordinate axes.

Let $f \in C^1[a,b]$ be a function, Δ a division of interval [*a*,*b*] by points $a = t_0 < t_1 < \ldots < t_n = b$ and $\sigma_\Delta(f)$ an integral sum corresponding to this division:

$$\sigma_\Delta(f) = \sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i)\,, \qquad t_i \le \xi_i \le t_{i+1}\ .$$

If Δ is a sufficiently fine division, i.e. the norm of division is sufficiently small, then you can approximate the integral with the integral sum, i.e.

$$\int_a^b f(t)dt \approx \sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i)\ . \tag{1.20}$$

To simplify the calculations, it is considered that the division Δ of interval [*a*,*b*] is equidistant, i.e.

$$t_{i+1} - t_i = \frac{b-a}{n}\ .$$

Under these conditions the following two formulas of approximation are obtained (see [11], [34], [39], [74]):

(*a*) If we consider the intermediary points of the division of interval [*a*,*b*] on the left end of partial intervals $[t_i, t_{i+1}]$, $\xi_i = t_i$, then we obtain the formula:

$$\int_a^b f(t)dt = \frac{b-a}{n}\left[f(a) + \sum_{i=1}^{n-1} f(t_i) \right] + R(f)\,; \tag{1.21}$$

(*b*) If we consider the intermediary points of the division of interval [*a*,*b*] on the right end of partial intervals $[t_i, t_{i+1}]$, $\xi_i = t_{i+1}$, then we obtain the formula:

$$\int_a^b f(t)dt = \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b) \right] + R(f)\,, \tag{1.21'}$$

and each of these two formulas is called *the rectangles fomula*.

For the rest of the formula (1.21) or (1.21'), $R(f)=\sum_{i=1}^{n}R_i(f)$, we have the estimate given by the relation:

$$\left|R(f)\right| \le M^D \frac{(b-a)^2}{n} \quad , \tag{1.22}$$

where, by M^D we denote

$$M^D = \max_{t\in[a,b]}\left| f'(t)\right| \ . \tag{1.23}$$

Because the calculation error of the integral $\int_a^b f(t)dt$ is less than ε > 0, it is sufficient that the number n of subintervals of the equidistant division Δ of interval [a,b] to verify the relation:

$$n \ \ge \ M^D \frac{(b-a)^2}{\varepsilon} \ . \tag{1.24}$$

It is noted that to obtain a better approximations, the rectangles method requires a large number of points of division of the interval [a,b] (see [11], [34], [39], [74]).

1.6.3 Simpson's formula

The approximation method that results using the Simpson's formula, consists in approximation of the given function on certain intervals with a second degree polynomial, i.e. to approximate the graph of the function on certain intervals with a parable.

Let $f \in C^4[a,b]$ be a function. A formula of approximate calculation of the integral $\int_a^b f(t)dt$ is:

$$\int_a^b f(t)dt = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b \varphi(s) f^{(IV)}(s)ds \ , \tag{1.25}$$

where

$$\varphi(s) = \begin{cases} \dfrac{(s-a)^4}{4!} - \dfrac{b-a}{6}\cdot\dfrac{(s-a)^3}{3!}, & daca \quad s \in \left[a, \dfrac{a+b}{2}\right] \\[2ex] \dfrac{(b-s)^4}{4!} - \dfrac{b-a}{6}\cdot\dfrac{(b-s)^3}{3!}, & daca \quad s \in \left[\dfrac{a+b}{2}, b\right] \end{cases} \ . \tag{1.26}$$

To get a better result, it is considered a division Δ of interval [a,b] into n equal parts by the points $a = t_0 < t_1 < \ldots < t_n = b$ and we apply the Simpson's formula (1.25) to each subinterval $[t_{i-1}, t_i]$.

Under these conditions we obtain the following *Simpson's quadrature formula* (see [11], [34], [39], [73], [74]):

$$\int_a^b f(t)dt = \frac{b-a}{6n}\left[f(a) + 2\sum_{i=1}^{n-1} f(t_i) + 4\sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) + f(b)\right] + R(f), \tag{1.27}$$

with the estimate of the rest $R(f)=\sum_{i=1}^{n} R_i(f)$ given by the relation:

$$|R(f)| \le M^S \frac{(b-a)^5}{2880n^4} , \tag{1.28}$$

where by M^S we denote

$$M^S = \max_{t\in[a,b]} \left| f^{(4)}(t) \right| . \tag{1.29}$$

Because the calculation error of the integral $\int_a^b f(t)dt$ is less than ε > 0, it is sufficient that the number n of subintervals of the equidistant division Δ of interval $[a,b]$ to verify the relation:

$$n \ \ \ge \ \ \sqrt[4]{M^S \frac{(b-a)^5}{2880\varepsilon}} . \tag{1.30}$$

It is noted that to obtain the desired approximation, the Simpson's formula requires generally fewer calculations than the previous formulas (see [11], [34], [39], [73], [74]).

1.7 Integral equations, basic results

An equation in which the unknown function appears under the integral sign is an *integral equation*.

In what follows we present several basic results regarding the existence and uniqueness for the solution of a Fredholm integral equation and of a Volterra integral equation, respectively.

1.7.1 Fredholm integral equation

The Fredholm integral equation is one of the most well known integral equations. This integral equation was studied by Ivar Fredholm.

Let $\Omega\subset \boldsymbol{R}^n$ be a bounded domain. Also, let $K : \overline{\Omega}\times\overline{\Omega}\times J \to \boldsymbol{R}$, $J\subset \boldsymbol{R}$ a closed interval and $f : \overline{\Omega} \to \boldsymbol{R}$ be two continuous functions.

An integral equation of the form

$$x(t) = \int_{\Omega} K(t,s,x(s))ds + f(t) \ , \quad t\in\Omega, \tag{1.31}$$

where the unknown function is $x\in C(\overline{\Omega})$ and the functions K and f are given, is called *Fredholm integral equation*.

If $\Omega = (a,b)$, then the Fredholm integral equation (1.31) is written as:

$$x(t) = \int_a^b K(t,s,x(s))ds + f(t) \ , \quad t\in(a,b), \tag{1.31'}$$

where the unknown function is $x\in C[a,b]$ and the functions K and f are given.

The function K is called *the kernel function* and the function f is called *the free term* of the integral equation.

The above two equations are *nonlinear Fredholm integral equations*.

In what follows, we present two theorems of existence and uniqueness of the solution of integral equation (1.31'), in the space $C[a,b]$ and in the sphere $\overline{B}(f;r)\subset C[a,b]$, respectively. These theorems can be found in the book [10].

Theorem 1.7.1. ([10]) *Suppose that*

(*i*) $K\in C([a,b]\times[a,b]\times\mathbf{R})$, $f\in C[a,b]$;

(*ii*) *there exists* $L>0$, *such that*

$$|K(t,s,u)-K(t,s,v)| \le L\,|u-v|\,,\quad \text{for all } t, s\in[a,b],\ u, v\in\mathbf{R}\,;$$

(*iii*) $L(b-a) < 1$. (*contraction condition*)

Under these conditions the Fredholm integral equation (1.31') *has a unique solution* $x^{*}\in C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0\in C[a,b]$. *Moreover, if* x_n *is the n-th successive approximation, then the following estimation is proved:*

$$\|x^{*}-x_n\|_{C[a,b]} \le \frac{L^n(b-a)^n}{1-L(b-a)}\cdot\|x_0-x_1\|_{C[a,b]}\,. \tag{1.32}$$

Theorem 1.7.2. ([10]) *Suppose that*

(*i*) $K\in C([a,b]\times[a,b]\times J)$, $J\subset\mathbf{R}$ *is a compact interval*, $f\in C[a,b]$;

(*ii*) *there exists* $L>0$, *such that*

$$|K(t,s,u)-K(t,s,v)| \le L\,|u-v|\,,\quad \text{for all } t, s\in[a,b],\ u, v\in J\,;$$

(*iii*) $L(b-a) < 1$. (*contraction condition*)

If $r>0$ *is a positive real number such that*

$$x\in\overline{B}(f;r) \quad\Rightarrow\quad x(t)\in J\subset\mathbf{R}\,,$$

and the following condition is fulfilled:

(*iv*) $M(b-a) \le r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$)

where denote by M *a positive constant, such that*

$$|K(t,s,u)| \le M,\quad \text{for all } t, s\in[a,b],\ u\in J\subset\mathbf{R},\ J \text{ is a compact interval}, \tag{1.33}$$

then the Fredholm integral equation (1.31') *has a unique solution* $x^{*}\in\overline{B}(f;r)\subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in\overline{B}(f;r)$. *Moreover, if* x_n *is the n-th successive approximation, then the estimation* (1.32) *is satisfied.*

The proofs of the two theorems above were obtained by applying *the Contraction Principle*, they can be found in [10] and for this reason are omitted.

Remark 1.7.1. In the study of the solution of Fredholm integral equation were used *the metric of Chebyshev* defined by the relation (1.5) and also *the norm of Chebyshev* defined by the relation (1.6).

1.7.2 Volterra integral equation

The Volterra integral equations were introduced by Vito Volterra and then studied by Traian Lalescu in 1908, in his doctoral thesis in 1908, *Sur les équations de Volterra*, written under the direction of Émile Picard. In 1911, Lalescu wrote the first book ever written about integral equations, entitled *Introduction to the theory of integral equations* (in Romanian).

The Volterra integral equations have application in demography, in epidemics, in the study of viscoelastic materials and in insurance mathematics.

The integral equation of the form

$$x(t) = \int_a^t K(t,s,x(s))ds + f(t) \ , \quad t \in (a,b) \ , \tag{1.34}$$

where the unknown function is $x \in C[a,b]$ and the functions K and f are given, is called *Volterra integral equation*.

The functions K and f are called *the kernel function* and *the free term of integral equation*, respectively.

The equation (1.34) is a *nonlinear Volterra integral equation*.

Now, we present two theorems of existence and uniqueness of the solution of integral equation (1.34), in the space $C[a,b]$ and in the sphere $\overline{B}(f;r) \subset C[a,b]$, respectively. These theorems can be found in the book [10].

Theorem 1.7.3. ([10]) *Suppose that*

(i) $K \in C([a,b]\times[a,b]\times\boldsymbol{R})$, $f \in C[a,b]$;

(ii) *there exists* $L > 0$, *such that*

$$|K(t,s,u) - K(t,s,v)| \le L\,|u-v| \ , \quad \text{for all } t, s \in [a,b], \ u, v \in \boldsymbol{R} .$$

Under these conditions the Volterra integral equation (1.34) *has a unique solution* $x^* \in C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0 \in C[a,b]$. *Moreover, if* x_n *is the n-th successive approximation, then the following estimation is proved:*

$$\left\| x^* - x_n \right\|_{C[a,b]} \le \frac{L^n}{\tau^{n-1}(\tau - L)} \cdot \left\| x_0 - x_1 \right\|_{C[a,b]} \ , \tag{1.35}$$

where τ is an arbitrary positive number chosen such that $\tau > L$.

Theorem 1.7.4. ([10]) *Suppose that*

(i) $K \in C([a,b]\times[a,b]\times J)$, $J \subset \boldsymbol{R}$ *is a compact interval*, $f \in C[a,b]$;

(ii) *there exists* $L > 0$, *such that*

$$|K(t,s,u) - K(t,s,v)| \le L\,|u-v| \ , \quad \text{for all } t, s \in [a,b], \ u, v \in J .$$

If $r > 0$ *is a positive real number such that*

$$x \in \overline{B}(f;r) \ \Rightarrow \ x(t) \in J \subset \boldsymbol{R} ,$$

and the following condition is fulfilled:

(iii) $M(b-a) \le r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$)

where denote by M *a positive constant, such that*

$$|K(t,s,u)| \leq M, \quad \text{for all } t, s\in [a,b],\ u\in J \subset \mathbf{R},\ J \text{ is a compact interval}, \tag{1.36}$$

then the Volterra integral equation (1.34) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r)$. *Moreover, if* x_n *is the n-th successive approximation*, *then the estimation* (1.35) *is satisfied.*

The proofs of the two theorems above were obtained also, by applying *the Contraction Principle*, they can be found in [10] and for this reason are omitted.

Remark 1.7.2. The existence and uniqueness of the solution of Volterra integral equation was studied in the space $B[a,b]$ using *the norm of Bielecki* defined by:

$$\| f \|_{B[a,b]} = \max_{a\le t\le b} | f(t) |\, e^{-\tau(t-a)}, \quad \text{for every } \tau > 0,\ f \in C[a,b], \tag{1.37}$$

where by $B[a,b]$ was denoted the space $C[a,b]$ endowed with *the metric of Bielecki* defined by:

$$d(f,g) := \max_{t\in[a,b]} |f(t) - g(t)| e^{-\tau(t-a)}, \quad \text{for every } \tau > 0,\ f, g\in C[a,b]. \tag{1.38}$$

Remark 1.7.3. *In a Fredholm integral equation the limits of integration are constant unlike the Volterra integral equation*.

1.8 Mathematical models governed by functional-integral equations

In this paragraph we present two mathematical models governed by functional-integral equations: an integral equation in physics and a mathematical model for studying the spread of an infectious disease.

1.8.1 An integral equation in physics

In the study of some problems from turbo-reactors industry, in the '70, a Fredholm integral equation with modified argument appears, having the following form:

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \quad t\in [a,b], \tag{1.39}$$

where $K : [a,b]\times[a,b]\times\mathbf{R}^3 \to \mathbf{R}$, $f : [a,b] \to \mathbf{R}$.

This integral equation is a mathematical model with reference to the turbo-reactors working.

We have obtained the conditions of existence and uniqueness and of data dependence of the solution, and, also, the conditions for approximating the solution of integral equation (1.39) and these results were published in papers [1], [15], [16], [17], [18] and [20].

A problem which leads to the equation (1.39) is as follows.

We consider the functional-integral equation

$$x(t) = \int_a^b K(t,s,x(s), \min_{a\le\varsigma\le s} x(\varsigma), \max_{s\le\varsigma\le b} x(\varsigma))ds + f(t), \quad t\in [a,b] \tag{1.40}$$

and note that if we are seeking for increasing solutions, then we obtain the equation (1.39).

Starting from the integral equation (1.39) we consider, in addition, a change in argument through a continuous function $g : [a,b] \to [a,b]$, thus obtaining the following integral equation with modified argument:

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) \ , \quad t \in [a,b] \ , \tag{1.41}$$

where $K : [a,b]\times[a,b]\times\boldsymbol{R}^4 \to \boldsymbol{R}$, $f : [a,b] \to \boldsymbol{R}$, $g : [a,b] \to [a,b]$.

The integral equation (1.41) was studied by author, establishing conditions of existence and uniqueness, of data dependence and of approximating the solution, and these results were published in papers [21], [22], [23], [24], [26] and [27] and will be presented in the following chapters.

Next, we present the results obtained by author in the study of integral equation (1.39).

I. Existence and uniqueness of the solution

Theorem 1.8.1. (M. Ambro [1]) *If*

(i) $K \in C([a,b]\times[a,b]\times\boldsymbol{R}^3)$, $f \in C[a,b]$;

(ii) *there exists* $L > 0$, *such that*

$$|K(t,s,u_1,v_1,w_1) - K(t,s,u_2,v_2,w_2)| \le L\,(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

for all $t, s \in [a,b]$, $u_1, u_2, v_1, v_2, w_1, w_2 \in \boldsymbol{R}$;

(iii) $3L(b-a) < 1$, ($contraction\ condition$)

then the integral equation (1.39) *has a unique solution* $x^* \in C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0 \in C[a,b]$.

Moreover, if x_n *is the n-th successive approximation, then the following estimation is proved:*

$$\|x^* - x_n\|_{C[a,b]} \le \frac{[3L(b-a)]^n}{1-3L(b-a)} \cdot \|x_0 - x_1\|_{C[a,b]} \ . \tag{1.42}$$

Proof. We consider the operator $A : C[a,b] \to C[a,b]$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t) \ , \quad t \in [a,b] \ . \tag{1.43}$$

The set of the solutions of integral equation (1.39) coincides with the fixed points set of the operator A.

To apply *the Contraction Principle* 1.3.1 and to obtain a theorem of existence and uniqueness of the solution of integral equation (1.39), the operator A must be contraction.

We have

$$|A(x_1)(t) - A(x_2)(t)| = \left| \int_a^b [K(t,s,x_1(s),x_1(a),x_1(b)) - K(t,s,x_2(s),x_2(a),x_2(b))]ds \right| \le$$

$$\le \left| \int_a^b |K(t,s,x_1(s),x_1(a),x_1(b)) - K(t,s,x_2(s),x_2(a),x_2(b))|\,ds \right| \ .$$

From the condition (ii) it results that

$$|A(x_1)(t) - A(x_2)(t)| \le L \left| \int_a^b \left[|x_1(s) - x_2(s)| + |x_1(a) - x_2(a)| + |x_1(b) - x_2(b)| \right] ds \right| ,$$

and using *the Chebyshev norm* in the right hand, we obtain

$$|A(x_1)(t) - A(x_2)(t)| \le$$

$$\le L(b-a)\cdot \max_{s\in[a,b]} \left[|x_1(s)-x_2(s)| + |x_1(a)-x_2(a)| + |x_1(b)-x_2(b)|\right] =$$

$$= 3L(b-a)\cdot \| x_1 - x_2 \|_{C[a,b]} .$$

Now using *the Chebyshev norm* in the left hand too, it results that

$$\| A(x_1) - A(x_2) \|_{C[a,b]} \le 3L(b-a)\cdot \| x_1 - x_2 \|_{C[a,b]} ,$$

and from the condition (*iii*) it results that the operator A is an α–contraction with the coefficient $\alpha = 3L(b-a)$.

Now, the conclusion of this theorem is obtained by applying *the Contraction Principle* 1.3.1 and the proof is complete.

The following theorem contains the conditions of existence and uniqueness of the solution of the integral equation (1.39) in the sphere $\overline{B}(f;r)\subset C[a,b]$.

Theorem 1.8.2. (M. Ambro [1]) *Suppose that*

(*i*) $K\in C([a,b]\times[a,b]\times J^3)$, $J\subset \boldsymbol{R}$ *is a compact interval*, $f\in C[a,b]$;

(*ii*) *there exists* $L > 0$, *such that*

$$|K(t,s,u_1,v_1,w_1) - K(t,s,u_2,v_2,w_2)| \le L\,(\,|u_1-u_2| + |v_1-v_2| + |w_1-w_2|\,)$$

for all $t, s\in[a,b]$, $u_1, u_2, v_1, v_2, w_1, w_2\in J$;

(*iii*) $3L(b-a) < 1$. (*contraction condition*)

If $r > 0$ *is a positive real number such that*

$$x\in \overline{B}(f;r) \ \Rightarrow\ x(t)\in J\subset \boldsymbol{R},$$

and

(*iv*) $M(b-a) \le r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$)

where denote by M *a positive constant, such that*

$$|K(t,s,u,v,w)| \le M, \quad \textit{for all } t, s\in[a,b],\ u, v, w\in J\subset \boldsymbol{R},\ J \textit{ compact interval}, \tag{1.44}$$

then the integral equation (1.39) *has a unique solution* $x^*\in \overline{B}(f;r)\subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in \overline{B}(f;r)$. *Moreover, if* x_n *is the nth successive approximation*, *then the estimation* (1.42) *is satisfied.*

Proof. We consider the operator $A:\overline{B}(f;r)\to C[a,b]$, defined by the relation (1.43).

The set of the solutions of integral equation (1.39) coincides with the fixed points set of the operator A.

In order to apply *the Contraction Principle* 1.3.1, we establish under what conditions the sphere $\overline{B}(f;r)$ is an invariant subset for the operator A. We have:

$$\left| A(x)(t) - f(t) \right| = \left| \int_a^b K(t,s,x(s),x(a),x(b))ds \right| \le \left| \int_a^b \left| K(t,s,x(s),x(a),x(b)) \right| ds \right|$$

and using (1.44) we obtain

$$\left| A(x)(t) - f(t) \right| \le M \cdot \int_a^b ds = M(b-a) \ ,$$

where according to condition (*iv*) it results that the sphere $\overline{B}(f;r) \subset C[a,b]$ is an invariant subset for the operator *A*, i.e. $\overline{B}(f;r) \in I(A)$. We can consider now the operator, noted also by *A*, $A\colon \overline{B}(f;r) \to \overline{B}(f;r)$, defined by the same relation (1.43), and $\overline{B}(f;r)$ is a closed subset of the complete metric space $C[a,b]$.

By an analogous reasoning to that of the proof of theorem 1.8.1 and using the condition (*ii*) it follows that the operator *A* satisfies the Lipschitz condition

$$\| A(x_1) - A(x_2) \|_{C[a,b]} \le 3L(b-a) \cdot \| x_1 - x_2 \|_{C[a,b]} \ ,$$

and according to condition (*iii*), it results that the operator *A* is an α–contraction with the coefficient $\alpha = 3L(b-a)$.

The conditions of *the Contraction Principle* 1.3.1 are fulfilled and therefore it results that the integral equation (1.39) has a unique solution $x^* \in \overline{B}(f;r) \subset C[a,b]$, and if x_n is the *nth* successive approximation, then the estimation (1.42) is satisfied.

The proof is complete.

II. Data dependence

Using *the abstract data dependence theorem* 1.3.5 we obtain a result of data dependence of the solution of integral equation (1.39) with respect to the functions *K* and *f*.

Now we consider a perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t) \ , \quad t \in [a,b] \ , \tag{1.45}$$

where $H : [a,b]\times[a,b]\times\mathbf{R}^3 \to \mathbf{R}$, $h : [a,b] \to \mathbf{R}$.

We have the following data dependence theorem.

Theorem 1.8.3. (M. Dobriţoiu [20]) *Suppose that*

(*i*) *the conditions of theorem* 1.8.1 *of existence and uniqueness of the solution of integral equation* (1.39) *in the space* $C[a,b]$ *are satisfied and denote by* x_A^*, *the unique solution of this equation*;

(*ii*) $H \in C([a,b]\times[a,b]\times \mathbf{R}^3)$, $h \in C[a,b]$;

(*iii*) *there exists* $\eta_1, \eta_2 > 0$ *such that*

$$|K(t,s,u,v,w) - H(t,s,u,v,w)| \le \eta_1 \ , \quad \text{for all } t, s \in [a,b],\ u, v, w \in \mathbf{R}$$

and

$$| f(t) - h(t)| \le \eta_2 \ , \quad \text{for all } t \in [a,b] \ .$$

Under these conditions, if x_B^* *is a solution of the perturbed integral equation* (1.45), *then*

$$\| x_A^* - x_B^* \|_{C[a,b]} \leq \frac{\eta_1(b-a)+\eta_2}{1-3L(b-a)} \ . \tag{1.46}$$

Proof. Consider the operator $A: C[a,b] \to C[a,b]$ from the proof of the theorem 1.8.1, defined by the relation (1.43).

To the perturbed equation (1.45) we attach the operator $B: C[a,b] \to C[a,b]$, defined by the relation:

$$B(y)(t) := \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t) \ , \quad t \in [a,b] \ . \tag{1.47}$$

We have

$$|A(x)(t) - B(x)(t)| =$$

$$= \left| \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t) - \int_a^b H(t,s,x(s),x(a),x(b))ds - h(t) \right| \leq$$

$$\leq \left| \int_a^b [K(t,s,x(s),x(a),x(b)) - H(t,s,x(s),x(a),x(b))]ds \right| + |f(t) - h(t)| \leq$$

$$\leq \left| \int_a^b |K(t,s,x(s),x(a),x(b)) - H(t,s,x(s),x(a),x(b))|ds \right| + |f(t) - h(t)| \ ,$$

where according to condition (*iii*) it results that

$$|A(x)(t) - B(x)(t)| \leq \eta_1(b-a) + \eta_2 \ , \quad \text{for all } t \in [a,b] \ ,$$

and using *the Chebyshev norm* in the left side, we obtain

$$\|A(x) - B(x)\|_{C[a,b]} \leq \eta_1(b-a) + \eta_2 \ .$$

Now, by applying *the abstract data dependence theorem* 1.3.5, the proof is complete.

III. Aproximation of the solution

In what follows we present an algorithm of approximating the solution of integral equation (1.39), that was published in paper [1].

Assume that the conditions of theorem 1.8.2 are satisfied and therefore the integral equation (1.39) has a unique solution in the sphere $\overline{B}(f;r) \subset C[a,b]$, which we denote by x^*. In order to obtain the solution x^* we apply the successive approximations method and we obtain the following successive approximations sequence:

$$x_0(t) = f(t)$$

$$x_1(t) = \int_a^b K(t,s,x_0(s),x_0(a),x_0(b))ds + f(t)$$

. . . . .

$$x_m(t) = \int_a^b K(t,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))ds + f(t)$$

. . . . .

Assume that $K\in C^2([a,b]\times[a,b]\times J^3)$, where $J\subset \mathbf{R}$ is a closed interval and $f\in C^2[a,b]$. We will approximate the terms of the successive approximations sequence using the trapezoids formula (1.14) with the estimation of the rest given by the relation (1.15).

We consider an equidistant division of interval $[a,b]$ by the points $a = t_0 < t_1 < \ldots < t_n = b$. In the general case for $x_m(t_k)$ we have:

$$x_m(t_k)=\frac{b-a}{2n}\Big[K(t_k,a,x_{m-1}(a),x_{m-1}(a),x_{m-1}(b))+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(a),x_{m-1}(b))+K(t_k,b,x_{m-1}(b),x_{m-1}(a),x_{m-1}(b))\Big]+$$

$$+f(t_k)+R_{m,k}\,,\quad k=\overline{0,n}\,,\quad m\in N\,, \tag{1.48}$$

and

$$\left|R_{m,k}\right|\le\frac{(b-a)^3}{12n^2}\cdot\max_{s\in[a,b]}\left|\big[K(t_k,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))\big]''_s\right|\,.$$

Since $K\in C^2([a,b]\times[a,b]\times J^3)$ it results that there exists the derivative of the function K from the expression of $R_{m,k}$, and therefore it has to be calculated. So, we have:

$$\big[K(t_k,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))\big]''_s=$$

$$=\frac{\partial^2K}{\partial s^2}+2\frac{\partial^2K}{\partial s\partial x_{m-1}}\cdot x'_{m-1}(s)+\frac{\partial^2K}{\partial x_{m-1}^2}\cdot\left(x'_{m-1}(s)\right)^2+\frac{\partial K}{\partial x_{m-1}}\cdot x''_{m-1}(s)\,,$$

where

$$x_{m-1}(t)=\int_a^b K(t,s,x_{m-2}(s),x_{m-2}(a),x_{m-2}(b))ds+f(t)\,,$$

$$x'_{m-1}(t)=\int_a^b\frac{\partial K(t,s,x_{m-2}(s),x_{m-2}(a),x_{m-2}(b))}{\partial t}ds+f'(t)\,,$$

$$x''_{m-1}(t)=\int_a^b\frac{\partial^2 K(t,s,x_{m-2}(s),x_{m-2}(a),x_{m-2}(b))}{\partial t^2}ds+f''(t)\,.$$

Denote

$$M_1=\max_{\substack{|\alpha|\le2\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u,v,w)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u^{\alpha_3}\partial v^{\alpha_4}\partial w^{\alpha_5}}\right|,\quad M_2=\max_{\substack{\alpha\le2\\ t\in[a,b]}}\left|f^{(\alpha)}(t)\right|.$$

Now, using the expressions of the derivatives of $x_{m-1}(t)$, it results that

$$\left|x_{m-1}(t)\right|\le(b-a)M_1+M_2\,,$$

$$|x'_{m-1}(t)| \le (b-a)M_1 + M_2 \ ,$$

$$|x''_{m-1}(t)| \le (b-a)M_1 + M_2 \ ,$$

and we obtain

$$\left|[K(t_k, s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))]''_s\right| \le M_1 + 2M_1[(b-a)M_1 + M_2] +$$

$$+ M_1[(b-a)M_1 + M_2]^2 + M_1[(b-a)M_1 + M_2] =$$

$$= M_1 + 3M_1[(b-a)M_1 + M_2] + M_1[(b-a)M_1 + M_2]^2 = M_0 \ .$$

It is obvious that M_0 doesn't depend on *m* and *k*, so we have the estimation of the rest:

$$|R_{m,k}| \le M_0 \cdot \frac{(b-a)^3}{12n^2}, \quad M_0 = M_0(K, D^\alpha K, f, D^\alpha f), \quad |\alpha| \le 2 \tag{1.49}$$

and we obtain a formula for the approximate calculation of integrals of the successive approximations sequence. Using the method of successive approximations and the formula (1.48) with the estimation of the rest resulted from (1.49), we suggest further on an algorithm in order to solve the integral equation (1.39) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we obtain the following result:

$x_0(t) = f(t)$

$$x_1(t_k) = \int_a^b K(t_k, s, x_0(s), x_0(a), x_0(b))ds + f(t_k) =$$

$$= \int_a^b K(t_k, s, f(s), f(a), f(b))ds + f(t_k) = \frac{b-a}{2n}[K(t_k, a, f(a), f(a), f(b)) +$$

$$+ 2\sum_{i=1}^{n-1} K(t_k, t_i, f(t_i), f(a), f(b)) + K(t_k, b, f(b), f(a), f(b))] + f(t_k) + R_{1,k} =$$

$$= \widetilde{x}_1(t_k) + R_{1,k} \ , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_a^b K(t_k, s, x_1(s), x_1(a), x_1(b))ds + f(t_k) =$$

$$= \frac{b-a}{2n}[K(t_k, a, x_1(a), x_1(a), x_1(b)) + 2\sum_{i=1}^{n-1} K(t_k, t_i, x_1(t_i), x_1(a), x_1(b)) +$$

$$+ K(t_k, b, x_1(b), x_1(a), x_1(b))] + f(t_k) + R_{2,k} =$$

$$= \frac{b-a}{2n}[K(t_k, a, \widetilde{x}_1(a) + R_{1,0}, \widetilde{x}_1(a) + R_{1,0}, \widetilde{x}_1(b) + R_{1,0}) +$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i,\widetilde{x}_1(t_i)+R_{1,i},\widetilde{x}_1(a)+R_{1,i},\widetilde{x}_1(b)+R_{1,i})+$$

$$+K(t_k,b,\widetilde{x}_1(b)+R_{1,n},\widetilde{x}_1(a)+R_{1,n},\widetilde{x}_1(b)+R_{1,n})\big]+f(t_k)+R_{2,k}=$$

$$=\frac{b-a}{2n}\big[K(t_k,a,\widetilde{x}_1(a),\widetilde{x}_1(a),\widetilde{x}_1(b))+2\sum_{i=1}^{n-1} K(t_k,t_i,\widetilde{x}_1(t_i),\widetilde{x}_1(a),\widetilde{x}_1(b))+$$

$$+K(t_k,b,\widetilde{x}_1(b),\widetilde{x}_1(a),\widetilde{x}_1(b))\big]+f(t_k)+\widetilde{R}_{2,k}=\widetilde{x}_2(t_k)+\widetilde{R}_{2,k},\quad k=\overline{0,n}\ ,$$

and the estimation of the rest

$$\left|\widetilde{R}_{2,k}\right| \le \frac{b-a}{2n}\cdot L\left(3\left|R_{1,0}\right|+6\sum_{i=1}^{n-1}\left|R_{1,i}\right|+3\left|R_{1,n}\right|\right)+\left|R_{2,k}\right| \le$$

$$\le 3(b-a)LM_0\cdot\frac{(b-a)^3}{12n^2}+M_0\cdot\frac{(b-a)^3}{12n^2} = \frac{(b-a)^3}{12n^2}\cdot M_0\left[3L(b-a)+1\right]\ .$$

The reasoning continues for $m = 3, \ldots$ and through induction we obtain

$$x_m(t_k)=\frac{b-a}{2n}\big[K(t_k,a,\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i,\widetilde{x}_{m-1}(t_i),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+K(t_k,b,\widetilde{x}_{m-1}(b),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))\big]+$$

$$+f(t_k)+\widetilde{R}_{m,k}=\widetilde{x}_m(t_k)+\widetilde{R}_{m,k},\quad k=\overline{0,n}$$

and

$$\left|\widetilde{R}_{m,k}\right| \le \frac{(b-a)^3}{12n^2}\cdot M_0\cdot\left[3^{m-1}(b-a)^{m-1}L^{m-1}+.\ \ .\ \ .+1\right],\quad k=\overline{0,n}\ .$$

Since $3L(b–a) < 1$, it results that the conditions of the existence and uniqueness theorem 1.8.2 are satisfied and we have the estimation:

$$\left|\widetilde{R}_{m,k}\right| \le \frac{(b-a)^3}{12n^2\left[1-3L(b-a)\right]}\cdot M_0\ . \tag{1.50}$$

Thus we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $\left(x_m\right)_{m\in N}$ using an equidistant division of interval $[a,b]$ by the points $a = t_0 < t_1 < \ldots < t_n = b$, with the following error in calculation:

$$\left|x_m(t_k)-\widetilde{x}_m(t_k)\right| \le \frac{(b-a)^3}{12n^2\left[1-3L(b-a)\right]}\cdot M_0\ . \tag{1.51}$$

Remark 1.8.1. If $K(t,s,x(s),x(a),x(b))\equiv K(t,s,x(s))$, then we obtain the existence and uniqueness theorems given in [10] and the method of approximation is the one given in [44] and [45].

Another integral equation of a similar type as the equation (1.39) is the following integral equation:

$$x(t) = \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))ds + f(t) \ , \quad t\in [a,b] \ , \qquad (1.52)$$

where $K : [a,b]\times[a,b] \to \boldsymbol{R}$, $h : [a,b]\times\boldsymbol{R}^3 \to \boldsymbol{R}$, $f : [a,b] \to \boldsymbol{R}$.

In the study of this equation were established several conditions for existence and uniqueness and for data dependence of the solution of integral equation (1.52) and these results were published in paper [28] and we present below them.

First we present two theorems of existence and uniqueness of the solution of the nonlinear Fredholm integral equation (1.52), in the space $C[a,b]$ and in the sphere $\overline{B}(f;r) \subset C[a,b]$, respectively.

Theorem 1.8.4. (M. Dobriţoiu [28]) *Suppose that*

(*i*) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times\boldsymbol{R}^3)$, $f\in C[a,b]$;

(*ii*) *there exists* $\alpha, \beta, \gamma > 0$, *such that*

$$|h(s,u_1,u_2,u_3) - h(s,v_1,v_2,v_3)| \le \alpha (| u_1-v_1 | + \beta | u_2-v_2 | + \gamma | u_3-v_3 |),$$

for all $s\in [a,b]$, $u_1, u_2, u_3, v_1, v_2, v_3\in \boldsymbol{R}$;

(*iii*) $M_K\cdot(\alpha + \beta + \gamma)\cdot(b - a) < 1$, (*contraction condition*)

where we denote by M_K *is a positive constant such that*

$$|K(t,s)| \le M_K , \quad \text{for all } t, s\in [a,b].$$

Under these conditions the integral equation (1.52) *has a unique solution* $x^*\in C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0\in C[a,b]$.

Moreover, if x_n *is the n-th successive approximation, then the following estimation is fulfilled:*

$$\|x^* - x_n\|_{C[a,b]} \le \frac{[M_K(\alpha+\beta+\gamma)\cdot(b-a)]^n}{1 - M_K(\alpha+\beta+\gamma)\cdot(b-a)} \cdot \|x_0 - x_1\|_{C[a,b]} \ . \qquad (1.53)$$

Theorem 1.8.5. (M. Dobriţoiu [28]) *Suppose that*

(*i*) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times J^3)$, $J \subset \boldsymbol{R}$ *is a closed interval*, $f\in C[a,b]$;

(*ii*) *there exists* $\alpha, \beta, \gamma > 0$, *such that*

$$|h(s,u_1,u_2,u_3) - h(s, v_1,v_2,v_3)| \le \alpha (| u_1-v_1 | + \beta | u_2-v_2 | + \gamma | u_3-v_3 |),$$

for all $s\in [a,b]$, $u_1, u_2, u_3, v_1, v_2, v_3\in J$, $J \subset \boldsymbol{R}$ *is a closed interval*;

(*iii*) $M_K\cdot(\alpha + \beta + \gamma)\cdot(b - a) < 1$. (*contraction condition*)

If there exists $r > 0$ *such that*

$$x\in \overline{B}(f;r) \quad \Rightarrow \quad x(t)\in J \subset \boldsymbol{R} ,$$

and the following condition id fulfilled:

(*iv*) $M_K M_h(b - a) \le r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$)

where we denote with M_h *a positive constant such that, for the restriction* $h|_{[a,b]\times J^3}$, $J\subset \boldsymbol{R}$ *compact, we have:*

$$|h(s,u,v,w)| \le M_h, \quad \text{for all } s\in [a,b], u, v, w\in J\subset \boldsymbol{R}, \qquad (1.54)$$

then the integral equation (1.52) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r)$. *Moreover, if* x_n *is the nth successive approximation*, *then the estimation* (1.53) *is satisfied.*

In order to study the data dependence of the solution of the integral equation (1.52) we consider the following perturbed integral equation:

$$y(t) = \int_a^b K(t,s) \cdot k(s, y(s), y(a), y(b))ds + g(t) \ , \quad t \in [a,b] \ , \tag{1.55}$$

where $K : [a,b] \times [a,b] \to \boldsymbol{R}$, $k : [a,b] \times \boldsymbol{R}^3 \to \boldsymbol{R}$, $g : [a,b] \to \boldsymbol{R}$.

The result presented below is a theorem of data dependence of the solution of integral equation (1.52).

Theorem 1.8.6. (M. Dobriţoiu [28]) *Suppose that*

(*i*) *the conditions of theorem* 1.8.4 *of existence and uniqueness of the solution of integral equation* (1.52) *in the space* $C[a,b]$ *are satisfied and denote by* x^* *the unique solution of this equation*;

(*ii*) $k \in C([a,b] \times \boldsymbol{R}^3)$, $g \in C[a,b]$;

(*iii*) *there exists* $\eta_1, \eta_2 > 0$ *such that*

$$|h(s,u,v,w) - k(s,u,v,w)| \leq \eta_1 \ , \quad \text{for all } s \in [a,b],\ u,\ v,\ w \in \boldsymbol{R}$$

and

$$|f(t) - g(t)| \leq \eta_2 \ , \quad \text{for all } t \in [a,b] \ .$$

Under these conditions, if y^* *is a solution of the perturbed integral equation* (1.55), *then*

$$\left\| x^* - y^* \right\|_C \leq \frac{M_K \eta_1 (b-a) + \eta_2}{1 - M_K(\alpha + \beta + \gamma) \cdot (b-a)} \ . \tag{1.56}$$

The proof of these three theorems above can be found in [28] and for this reason are omitted.

1.8.2 A mathematical model for studying the spread of an infectious disease

In the various problems that arise in connection with the development of populations can occurr some phenomena that occur periodically.

We consider an isolated population and suppose that:

- the population has a constant number of individuals, i.e. it is in the vicinity of a population of stable balance type;
- the population consists of two disjoint classes: individuals susceptible to infection and infected individuals;
- infection does not lead to death and does not provide immunity;
- infection period (duration of infection) is constant and denote by τ, $\tau > 0$.

Knowing the number of individuals who become infected at the time t_0, is required to determine the number of individuals infected at time t.

Denote by $x(t)$ the number of individuals of population which are infected at time t and let $f(t,x(t))$ be the number of newly infected individuals per unit of time ($f(t, 0) = 0$).

Under these conditions the following non-linear integral equation is an important mathematical model for studying the spread of an infectious disease with a periodical contact rate that varies seasonally:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds \ , \quad t \in \boldsymbol{R}. \tag{1.57}$$

Also, the integral equation (1.57) can be interpreted in the terms of population growth. Consider a single species population and assume that:

- $x(t)$ represents the number of individuals present in this single species population at time t (they assumed that the population is uniformly distributed in a given geographical area) ;
- $f(t,x(t))$ is the number of the new births per unit of time ;
- τ is the lifetime ; it is assumed that each individual lives to the age $\tau\,(\tau > 0)$ exactly, and then dies.

Under these conditions the equation (1.57) is the mathematical model of the growth of a single species population when the birth rate varies seasonally.

The mathematical model of the spread of an infectious disease, was studied by giving the conditions of existence and uniqueness of positive, non-trivial and periodic solutions of period $\omega > 0$, and highlighting some interesting properties of the solutions. Note the results obtained by K.L. Cooke and J.L. Kaplan [13], D. Guo and V. Lakshmikantham [30], I. A. Rus [53], [56], R. Precup [47], [48], [49], R. Precup and E. Kirr [50], C. Iancu [32], [33], I. A. Rus, M. A. Şerban and D. Trif [71].

We present below some results on positive, non-trivial and ω–periodic solution of integral equation (1.57), given by K.L. Cooke and J.L. Kaplan [13], I. A. Rus [56] and R. Precup [47].

I. Existence and uniqueness of the solution

Theorem 1.8.7. (K.L. Cooke, J.L. Kaplan [13]) *Suppose that*:

(*i*) $f \in C(\boldsymbol{R}\times\boldsymbol{R}_+)$ *and* $f(t, 0) = 0$, *for all* $t \in \boldsymbol{R}$;

(*ii*) *there exists* $\omega > 0$, *such that*

$$f(t+\omega, u) = f(t, u), \quad \text{for all } t \in \boldsymbol{R} \text{ and } u \in \boldsymbol{R}_+ ;$$

(*iii*) *there exists* $M > 0$, *such that*

$$0 \le f(t,u) \le M, \quad \text{for all } t \in \boldsymbol{R} \text{ and } u \in \boldsymbol{R}_+ ;$$

(*iv*) *there exists* $x_1 > 0$, *such that*

$$\frac{\partial f}{\partial u} \text{ there exists and is continuous, for } t \in \boldsymbol{R},\ 0 \le u \le x_1$$

and satisfies the condition

$$\inf_{t\in[0,+\infty)} \frac{\partial f}{\partial u}(t,0) = \alpha > 0 \ .$$

If $\alpha\tau > 1$, *then the integral equation* (1.57) *has a non-trivial and* ω*-periodic solution*.

This result was improved by I. A. Rus in [56], obtaining the following result of existence and uniqueness of an ω-periodic solution of integral equation (1.57).

Let $0 < m < M$ and $0 < \alpha < \beta$.

Theorem 1.8.8. (I. A. Rus [56]) *Suppose that*:

(i) $f \in C(\boldsymbol{R} \times [\alpha,\beta])$;

(ii) *there exists* $\omega > 0$, *such that*

$$f(t+\omega,u) = f(t,u), \quad \textit{for all } t \in \boldsymbol{R} \textit{ and } u \in [\alpha,\beta];$$

(iii) $m \le f(t,u) \le M$, *for all* $t \in \boldsymbol{R}$ *and* $u \in [\alpha,\beta]$;

(iv) $\alpha \le m\tau$, $\beta \ge M\tau$;

(v) *there exists L such that*

$$| f(t,u) - f(t,v) | \le L(t) \cdot | u - v |, \quad \textit{for all } t \in \boldsymbol{R} \textit{ and } u, v \in [\alpha,\beta];$$

(vi) $\int_{t-\tau}^{t} L(s)ds \le q < 1$, *for all* $t \in \boldsymbol{R}$.

Under these conditions the integral equation (1.57) *has a unique* ω*–periodic solution in* $C(\boldsymbol{R},[\alpha,\beta])$ space.

II. Data dependence

In paper [56] I. A. Rus gives a result, presented below, of continuous dependence with respect to the function *f*, of an ω - periodic solution of integral equation (1.57).

Consider the perturbed integral equation

$$y(t) = \int_{t-\tau}^{t} g(s, y(s))ds \ , \quad t \in \boldsymbol{R} \ , \tag{1.58}$$

where $g : \boldsymbol{R} \times [\alpha,\beta] \to \boldsymbol{R}$. Denote by X_ω the set

$$X_\omega = \{x \in C(R,[\alpha,\beta]) / x(t+\omega) = x(t), t \in R\}$$

and let *d* be a metric on X_ω , defined by the relation:

$$d(x,y) = \max_{t \in [0,\omega)} \left| x(t) - y(t) \right| \ , \quad \text{for all } x, y \in X_\omega.$$

The following theorem is true.

Theorem 1.8.9. (I. A. Rus [56]) *Suppose that*:

(i) *the conditions of theorem* 1.8.8 *of existence and uniqueness of an* ω*– periodic solution of integral equation* (1.57) *are satisfied and denote by* x^* *the unique* ω*– periodic solution of this equation*;

(ii) *the function* $g \in C(\boldsymbol{R} \times [\alpha,\beta])$ *is periodic, of period* ω, *i.e.*

$$g(t+\omega,u) = g(t,u), \quad \textit{for all } t \in \boldsymbol{R} \textit{ and } u \in [\alpha,\beta]$$

and

$$m \le g(t,u) \le M, \quad \textit{for all } t \in \boldsymbol{R} \textit{ and } u \in [\alpha,\beta];$$

(iii) *there exists* $\eta > 0$ *such that*

$$| f(t, u) - g(t, u)| \le \eta \ , \quad \textit{for all } t \in \boldsymbol{R} \textit{ and } u \in [\alpha,\beta];$$

Under these conditions, if y^* *is a solution of equation* (1.58), *then*

$$d(x^*, y^*) \leq \frac{\eta\tau}{1-q} . \tag{1.59}$$

When the number of infected individuals, $\phi(t)$ for $t\in[-\tau,0]$, is known, i.e.

$$x(t) = \phi(t) \qquad \text{for } t\in[-\tau,0] , \tag{1.60}$$

then can be studied the existence of the positive and continuous solutions of the integral equation (1.57) ([47]).

Suppose that $\phi(t)$ is a positive continuous function on interval $[-\tau,0]$ and satisfies

$$\phi(0) = b \equiv \int_{-\tau}^{0} f(s,\phi(s))ds . \tag{1.61}$$

Then the problem (1.57)+(1.60) is equivalent with the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) - f(t-\tau, x(t-\tau)), & t\in[0,T] \\ x(t) = \phi(t), & t\in[-\tau,0]. \end{cases} \tag{1.62}$$

An important result is the following theorem.

Theorem 1.8.10. (R. Precup [47]) *If* :

(*i*) *$f(t,x)$ is non-negative and continuous for $t \in [-\tau,T]$ and $x \geq 0$* ;

(*ii*) *$\phi(t)$ is continuous, $\phi(t) \geq a > 0$ for $t \in [-\tau,0]$ and satisfies the condition* (1.61);

(*iii*) *there exists an integrable function g such that*

$$f(t,x) \geq g(t) \quad \text{for } t \in [-\tau,T] \text{ and } x \geq a$$

and

$$\int_{t-\tau}^{t} g(s)ds \geq a \quad \text{for } t \in [0, T] ;$$

(*iv*) *there exists a positive function $h(x)$, such that $\frac{1}{h(x)}$ is locally integrable on* $(a,+\infty)$

$$f(t,x) \leq h(x) \quad \text{for } t\in[0,T] \text{ and } x \geq a$$

and

$$T < \int_{b}^{\infty} \frac{1}{h(x)}dx ,$$

then the integral equation (1.57) *has at least one continuous solution x, and $x(t) \geq a$, for $t\in[-\tau,T]$, that satisfies the condition* (1.60).

If the condition (*iv*) of theorem 1.8.10 is replaced by the following more restrictive condition ([47]):

(*iv'*) *there exists $L > 0$ such that*

$$| f(t,x) - f(t,y)| \leq L|x - y|, \quad \text{for all } t\in[-\tau,T] \text{ and } x, y > a ,$$

then the following theorem is true.

Theorem 1.8.11. (R. Precup [47]) *If the conditions* (*i*), (*ii*), (*iii*) *of theorem* 1.8.10 *and the condition* (*iv'*) *are satisfied, then the integral equation* (1.57) *has a unique continuous solution* x, $x(t) \geq a$, *for* $t \in [-\tau,T]$, *that satisfies the condition* (1.60).

III. Aproximation of the solution

Knowing that the integral equation (1.57) admits a unique solution one can study its approximate solution by numerical methods.

Assume that the conditions of theorems 1.8.10 and.1.8.11 are satisfied and therefore the integral equation (1.57) has a unique continuous solution on interval $[-\tau,T]$, denoted by φ, solution that can be obtained by the successive approximations method. We have the sequence of successive approximations:

$$\varphi(t) = \phi(t) \quad \text{for } t\in[-\tau, 0)$$

$$\varphi_0(t) = \phi(0) = b \equiv \int_{-\tau}^{0} f(s,\phi(s))ds$$

$$\varphi_1(t) = \int_{t-\tau}^{t} f(s,\varphi_0(s))ds \tag{1.63}$$

. . . . .

$$\varphi_m(t) = \int_{t-\tau}^{t} f(s,\varphi_{m-1}(s))ds$$

. . . . . , for $t\in[0,T]$.

The study of this algorithm of approximating the solution by using the trapezoids method for calculating the integrals from the terms of successive approximations sequence was made by C. Iancu in [32], [33]. Also, in [33], C. Iancu gives another method for approximating the solution of integral equation (1.57), using the spline functions method.

I. A. Rus, M. A. Şerban and D. Trif published in [71] an interesting study of the equation (1.57) from biomathematics, proving that the sequence of successive approximations generated by the steps method converges to the solution of integral equation (1.57).

Using the Picard operators technique, I. A. Rus, M. A. Şerban and M. Dobriţoiu studied the existence and uniqueness, and upper and lower solutions, data dependence and differentiability with respect to a parameter of the solution of integral equation (1.57). The results of this study are published in paper [25] and are presented in chapter 6.

1.9 References

[1] Ambro, M., Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat, *Studia Univ. Babeş-Bolyai, Mathematica*, 2(1978), 26-32

[2] András, Sz., Gronwall type inequalities via subconvex sequences, *Seminar on Fixed Point Theory*, 3(2002), 183-188

[3] András, Sz., Fiber φ-contractions on generalized metric spaces and applications, *Mathematica*, 45(68)(2003):1, 3-8

[4] András, Sz., *Ecuaţii integrale Fredholm-Volterra*, Didactică şi Pedagogică, Bucureşti, 2005

[5] Bainov, D., Simeonov, P., *Integral inequalities and applications*, Kluwer, Dordrecht, 1992

[6] Beesak, P. R., Gronwall inequalities, *Carleton Math. Lecture Notes*, 1975, Nr. 11

[7] Bica, A., *Modele matematice în biologie guvernate de ecuaţii diferenţiale*, Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică, Teză de doctorat, 2004

[8] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997

[9] Buică, A., Gronwall-type nonlinear integral inequalities, *Mathematica*, 44(2002), Nr. 1, 19-23

[10] Coman, Gh., Rus, I., Pavel, G., Rus, I. A., *Introduction in the operational equations theory* (Romanian), Dacia, Cluj-Napoca, 1976

[11] Coman, Gh., *Analiză numerică*, Libris, Cluj-Napoca, 1995

[12] Constantin, A., A Gronwall-like inequality and its applications, *Rend. Mat. Acc. Lincei* 1(1990), 111-115

[13] Cooke, K. L., Kaplan, J. L., A periodicity thereshald theorem for epidemics and population growth, *Math. Biosciences*, 31(1976), no.1-2, 87-104

[14] Crăciun, C., On some Gronwall inequalities, *Seminar on Fixed Point Theory*, 1 (2000), 31-34

[15] Dobriţoiu, M., Aproximări ale soluţiei unei ecuaţii integrale Fredholm cu argument modificat, *Analele Univ. Aurel Vlaicu din Arad, seria Matematică, fasc. „Matematică-Informatică"*, Arad, 28-30 nov. 2002, ISSN 1582-344X, 51-56

[16] Dobriţoiu, M., *Formule de cuadratură utilizate în aproximarea soluţiei unei ecuaţii integrale cu argument modificat*, Lucr. Şt. ale Simp. Int. "Universitaria ROPET 2003", Petroşani, 16-18 oct. 2003, Universitas, Petroşani, fasc. „Matematică-Informatică-Fizică", ISBN 973-8260-37-X, 53-56

[17] Dobriţoiu, M., *The rectangle method for approximating the solution to a Fredholm integral equation with a modified argument*, Lucr. Şt. a celei de *a XXX-a Sesiuni de comunicări ştiinţifice cu participare internaţională* "TEHNOLOGII MODERNE ÎN SECOLUL XXI", Academia Tehnică Militară Bucureşti, secţ. 16, "Matematică", Bucureşti, 6-7 nov. 2003, ISBN 973-640-012-3, 36-39

[18] Dobriţoiu, M., A Fredholm integral equation - numerical methods, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM-CVI / 2004, Nr. 2188, ISSN 0133-3526, 285-292

[19] Dobriţoiu, M., An integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIX, nr. 3/2004, 27-33

[20] Dobriţoiu, M., Existence and continuous dependence on data of the solution of an integral equation, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM-CVI / 2005 , Nr. ISSN 0133-3526

[21] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş-Bolyai Mathematica*, vol. 51, nr. 1/2006, 81-94

[22] Dobriţoiu, M., W. W. Kecs, A. Toma, *The Differentiability of the Solution of a Nonlinear Integral Equation*, Proc. of the 8th WSEAS Int. Conf. on Math. Methods and Comp. Techniques in Electrical Eng., Bucharest, Romania, Oct. 16-17, 2006, ISSN 1790-5117, ISBN 960-8457-54-8, 155-158

[23] Dobriţoiu, M., W. W. Kecs, A. Toma, An Application of the Fiber Generalized Contractions Theorem, *WSEAS Transactions on Mathematics*, Issue 12, Vol. 5, Dec. 2006, ISSN 1109-2769, 1330-1335

[24] Dobriţoiu, M., Properties of the solution of an integral equation with modified argument, *Carpathian Journal of Mathematics*, 23(2007), No. 1-2, 77-80

[25] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[26] Dobriţoiu, M., System of integral equations with modified argument, *Carpathian Journal of Mathematics*, vol. 24 (2008), No. 2, 26-36

[27] Dobriţoiu, M., *Gronwall-type lemmas for an integral equation with modified argument*, Proc. of the Int. Conf. on Theory and Applications of Mathematics and Informatics, Alba Iulia, 2007

[28] Dobriţoiu, M., A class of nonlinear integral equations, *Transylvanian Journal of Mathematics and Mechanics*, 4 (2012), No. 2, 117-123

[29] Dragomir, S. S., The Gronwall type lemmas and applications, *Monografii Matematice*, No. 29, Univ. Timişoara, 1987

[30] Guo, D., Lakshmikantham, V., Positive solutions of nonlinear integral equations arising in infectious diseases, *J.Math.Anal.Appl.* 134 (1988), 1-8

[31] Hirsch, M. W., Pugh, C. C., *Stable manifolds and hyperbolic sets*, Proc. Symp. In Pure Math., 14(1970), AMS, 133-143

[32] Iancu, C., A numerical method for approximating the solution of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIII, 4(1998), 37-45

[33] Iancu, C., *Modelare matematică. Teme speciale*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2002

[34] Ionescu, D. V., *Cuadraturi numerice*, Tehnică, Bucureşti, 1957

[35] Kirk, W. A., Sims, B., *Handbook of metric fixed point theory*, Kluwer, Dordrecht, 2001

[36] Lakshmikantham, V., Leela, S., *Differential and Integral Inequalities*, Acad. Press, New York, 1969

[37] Lungu, N., Rus, I. A., Gronwall inequalities via Picard operators, *Analele Ştiinţifice ale Univ. "Al.I. Cuza" din Iaşi, Matematica*, Tomul LVIII, 2012, f. 2, 269-278

[38] Maia, M. G., Un'osservazione sulle contrazioni metriche, *Rend. Sem. Mat. Univ. Padova*, 40(1968), 139-143

[39] Marinescu, Gh., *Analiză matematică*, vol. I, Didactică şi Pedagogică, Bucureşti, 1980

[40] Mitrinović, D. S., Pečarić, J. E., Fink, A. M., *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991

[41] Mureşan, V., *Ecuaţii diferenţiale cu modificarea afină a argumentului*, Transilvania Press, Cluj-Napoca, 1997

[42] Mureşan, V., A Gronwall type inequality for Fredholm operators, *Mathematica*, 41(1999), Nr. 2, 227-231

[43] Pachpatte, B. G., *Inequalities for differential and integral equations*, Academic Press, New York, 1998

[44] Pavel, P., Rus, I. A., *Ecuaţii diferenţiale şi integrale*, Didactică şi Pedagogică, Bucureşti, 1975

[45] Pavel, G., Rezolvarea aproximativă a problemei lui Dirichlet relativă la un sistem de ecuaţii diferenţiale neliniare de ordinul al doilea, *Studia Univ. Babeş-Bolyai, Mathematica*, 1(1977), 47-52

[46] Petruşel, A., Rus, I. A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. 134 (2005), 411-418

[47] Precup, R., Positive solutions of the initial value problem for an integral equation modelling infectious disease, în *"Seminarul de Ecuaţii Diferenţiale: Preprint Nr.3, 1991"*, I. A. Rus ed., Universitatea Babeş-Bolyai, Cluj, 1991, 25-30

[48] Precup, R., Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle, *Studia Univ. Babeş-Bolyai, Mathematica*, 39, No.1 (1994), 47-58

[49] Precup, R., Monotone technique to the initial values problem for a delay integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, 40, No.2 (1995), 63-73

[50] Precup, R., Kirr, E., *Analysis of a nonlinear integral equation modelling infection diseases*, Proc.

of the Int. Conf., University of the West, Timişoara, 1997, 178-195

[51] Precup, R., Continuation results for mappings of contractive type, *Seminar on Fixed Point Theory Cluj-Napoca*, 2(2001), 23-40

[52] Rus, I. A., On a fixed point theorem of Maia, *Studia Univ. Babeş-Bolyai, Mathematica*, 1(1977), 40-42

[53] Rus, I. A., *Principles and applications of the fixed point theory* (Romanian), Dacia, Cluj-Napoca, 1979

[54] Rus, I. A., *Metrical fixed point theorems*, University of Cluj-Napoca, 1979

[55] Rus, I. A., Generalized contractions, *Seminar on Fixed Point Theory*, Babeş-Bolyai Univ., Preprint Nr.3, 1983, 1-130

[56] Rus, I. A., A delay integral equation from biomathematics, *Babeş-Bolyai University of Cluj-Napoca, Preprint* Nr.3, 1989, 87-90

[57] Rus, I. A., Weakly Picard mappings, *Comment. Math. Univ. Caroline*, **34**, 3(1993), 769-773

[58] Rus, I. A., Picard operators and applications, *Babeş-Bolyai Univ. of Cluj-Napoca, Preprint* Nr. 3, 1996

[59] Rus, I. A., *An abstract point of view for some integral equations from applied mathematics*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 256-270

[60] Rus, I. A., A general functional inequality and its applications, *Revue d'An. Num. Th. l'Approx*. 26(1997), 209-213

[61] Rus, I. A., A Fiber generalized contraction theorem and applications, Académie Roumaine, Filiale de Cluj-Napoca, *Mathematica*, Tome 41(64), Nr.1, 1999, 85-90

[62] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334

[63] Rus, I. A., Fiber Picard operators theorem and applications, *Studia Univ. Babeş-Bolyai, Mathematica*, **44**(1999), No. 3, 89-97

[64] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001

[65] Rus, I. A., Weakly Picard operators and applications, *Seminar on Fixed Point Theory*, Babeş-Bolyai University of Cluj-Napoca, Vol. 2, 2001, 41-58

[66] Rus, I. A., Petruşel, A., Petruşel, G., *Fixed point theory 1950-2000 romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002

[67] Rus, I. A., Picard operators and applications, *Scientiae Mathematicae Japonicae*, 58(2003):1, 191-219

[68] Rus, I. A., Fixed points, upper and lower fixed points: abstract Gronwall lemmas, *Carpathian Journal of Mathematics*, 20(2004), No. 1, 125-134

[69] Rus, I. A., Mureşan, S., Mureşan, V., Weakly Picard operators on a set with two metrics, *Fixed Point Theory*, Vol. 6, No. 2, 2005, 323-331

[70] Rus, I. A., Data dependence of the fixed points in a set with two metrics, *Fixed Point Theory*, Vol. 8, No. 1, 2007, 115-123

[71] Rus, I. A., Şerban, M. A., D. Trif, Step method for some integral equations from biomathematics, *Bull. Math. Soc. Sci. Math. Roumanie*, Tome 54(102)(2011) No. 2, 167-183

[72] Sotomayor, J., Smooth dependence of solution of differential equation on initial data: a simple proof, *Bol. Soc. Brasil*, 4, 1(1973), 55-59

[73] Stancu, D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză numerică şi teoria aproximării*, vol. 1, Presa Univ. Clujeană, Cluj-Napoca, 2001

[74] Stancu, D. D., Coman, Gh., Blaga, P., *Analiză numerică şi teoria aproximării*, vol. 2, Presa Univ. Clujeană, Cluj-Napoca, 2002

[75] Şerban, M. A., Fiber φ-contractions, *Studia Univ. Babeş-Bolyai, Mathematica*, 34(1999), Nr.3, 99-108

[76] Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Univ. Clujeană, Cluj-Napoca, 2002

[77] Şerban, M. A., Application of fiber Picard operators to integral equations, *Bul. Ştiinţific Univ. Baia Mare, Seria B, Matematică-Informatică*, Vol. XVIII(2002), Nr.1, 119-128

[78] Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl*., 13(2010), no. 2, 255-269

[79] Walter, W., *Differential and integral inequalities*, Springer, Berlin, 1979

[80] Zima, M., The abstract Gronwall lemma for some nonlinear operators, *Demonstratio Math.*, 31(1998), 325-332

2 Existence and uniqueness of the solution

Several of the basic treatises which have the integral equations like theme, are the following: T. Lalescu [21], I. G. Petrovskii [26], K. Yosida [59], [60], Gh. Marinescu [22], A. Haimovici [19], C. Corduneanu [7], [8], Gh. Coman, I. Rus, G. Pavel and I. A. Rus [6], W. Walter [58], D. Guo, V. Lakshmikantham and X. Liu [17], W. Hackbusch [18], D. V. Ionescu [20], Şt. Mirică [23], V. Mureşan [24], [25], A. D. Polyanin and A. V. Manzhirov [28], R. Precup [30], [35], I. A. Rus [39], [40], [43], [48], M. A. Şerban [55], Sz. András [5].

The existence and uniqueness of the solutions of some particular integral equations were studied in many papers, of which we mention several: R. Ramalho [36], C. A. Stuart [52], B. Rzepecki [51], I. A. Rus [38], [41], [42], [44], [45], [46], [47], [49], I. A. Rus, S. Mureşan and V. Mureşan [50], R. Precup [29], [31], [32], [34], R. Precup and E. Kirr [33], D. O'Regan and A. Petruşel [37], M. Albu [1], A. Petruşel [27], Sz. András [3], [4], M. A. Şerban [53], [54], [56], M. A. Şerban, I. A. Rus and A. Petruşel [57], M. Dobriţoiu, I. A. Rus and M. A. Şerban [15], M. Dobriţoiu (Ambro) [2], [9], [10], [11], [12], [13], [14], [16].

In this chapter we will present a study of existence and uniqueness of the solution of integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[a,b], \qquad (2.1)$$

in the space $C([a,b],\boldsymbol{B})$ and in the sphere $\overline{B}(f;r)\subset C([a,b],\boldsymbol{B})$, where $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ is a Banach space, in the general case, and in two particular cases for $\boldsymbol{B}$, namely: $\boldsymbol{B}=\boldsymbol{R}^m$ and $\boldsymbol{B}=\boldsymbol{l}^2(\boldsymbol{R})$.

This chapter has five paragraphs. In the paragraphs 1, 2 and 3 one gives the conditions of existence and uniqueness of the solution of the integral equation with modified argument (2.1) in the space $C([a,b],\boldsymbol{B})$ and in the sphere $\overline{B}(f;r)\subset C([a,b],\boldsymbol{B})$, in the general case, and in the particular cases which were mentioned above. To determine these results, the following basic theorems were useful: *the Contraction Principle* 1.3.1 and *the Perov's theorem* 1.3.4.

The paragraph 4 contains three examples, namely: two integral equations with modified argument and one system of integral equations with modified argument, and for each of these, one uses the results obtained and presented in the previous paragraphs in order to establish the conditions of existence and uniqueness of the solution.

In the fifth paragraph, one studies the existence and uniqueness of the solution of an integral equation with modified argument, which is a generalization of the equation (2.1), namely:

$$x(t) = \int_\Omega K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds + f(t), \quad t\in\overline{\Omega},$$

where $\Omega\subset \boldsymbol{R}^m$ is a bounded domain, $K:\overline{\Omega}\times\overline{\Omega}\times R^m\times R^m\times C(\partial\Omega,R^m)\to R^m$, $f:\overline{\Omega}\to R^m$ and $g:\overline{\Omega}\to\overline{\Omega}$.

The results presented in this chapter were obtained by the author and published in the papers [13] and [16]. We present them below.

2.1 The general case

A. Existence and uniqueness of the solution in the space $C([a,b],\boldsymbol{B})$

Let $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ be a Banach space. Consider the integral equation (2.1) and suppose that the following conditions are met:

(a_1) $K\in C([a,b]\times[a,b]\times\boldsymbol{B}^4, \boldsymbol{B})$;

(a_2) $f\in C([a,b],\boldsymbol{B})$;

(a_3) $g\in C([a,b], [a,b])$.

The following theorem is true.

Theorem 2.1.1. *Suppose that the conditions* (a_1)–(a_3) *are satisfied. In addition, suppose that*:

(a_4) *there exists* $L_K > 0$ *such that*

$$|K(t, s, u_1, u_2, u_3, u_4) - K(t, s, v_1, v_2, v_3, v_4)| \leq$$
$$\leq L_K(|u_1-v_1|+|u_2-v_2)|+|u_3-v_3)|+|u_4-v_4)|),$$

for all $t, s\in[a,b]$, $u_i, v_i\in\boldsymbol{B}$, $i=\overline{1,4}$;

(a_5) $4L_K(b-a)<1$. (*contraction condition*)

Under these conditions, it results that the integral equation with modified argument (2.1) *has a unique solution* $x^*\in C([a,b],\boldsymbol{B})$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in C([a,b],\boldsymbol{B})$.

Moreover, if x_n *is the n-th successive approximation, then the following estimation is true*:

$$d(x^*,x_n) \leq \frac{4^n L_K^n (b-a)^n}{1-4L_K(b-a)} d(x_0,x_1) \ . \tag{2.2}$$

Proof. On the space $C([a,b],\boldsymbol{B})$, we consider *the metric of Chebyshev*, denoted by d and defined in chapter 1, by the relation (1.5).

Also, we consider the operator $A : C([a,b],\boldsymbol{B}) \to C([a,b],\boldsymbol{B})$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[a,b]. \tag{2.3}$$

The set of the solutions of integral equation (2.1) in the space $C([a,b],\boldsymbol{B})$ coincides with the fixed points set of the operator A, i.e. with F_A .

From the conditions (a_1)–(a_4) we have

$$|A(x)(t) - A(y)(t)| =$$

$$= \left|\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))\,ds - \int_a^b K(t,s,y(s),y(g(s)),y(a),y(b))\,ds\right| \leq$$

$$\leq \left|\int_a^b |K(t,s,x(s),x(g(s)),x(a),x(b)) - K(t,s,y(s),y(g(s)),y(a),y(b))|\,ds\right| \leq$$

$$\leq \int_a^b L_K \cdot \left[|x(s)-y(s)| + |x(g(s))-y(g(s))| + |x(a)-y(a)| + |x(b)-y(b)| \right] ds \leq$$

$$\leq 4L_K d(x,y) \int_a^b ds = 4L_K (b-a) d(x,y), \quad t \in [a,b] .$$

Therefore, with respect to *the metric of Cebyshev*, the operator A satisfies the Lipschitz condition with the constant $4L_K(b-a)$ and we have:

$$d\big(A(x), A(y)\big) \leq 4L_K(b-a)d(x,y), \quad \text{for all } x, y \in C([a,b],\boldsymbol{B}), \tag{2.4}$$

and from the condition (a_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 4L_K(b-a)$.

The conditions of *the Contraction Principle* 1.3.1. being satisfied, it results that the integral equation (2.1) has a unique solution $x^* \in C([a,b],\boldsymbol{B})$, that can be obtained by the successive approximations method, starting at any element $x_0 \in C([a,b],\boldsymbol{B})$.

If we denote by x_n the n-th successive approximation, then the estimation (2.2) is true and the proof is complete.

B. Existence and uniqueness of the solution in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$

In order to study the existence and uniqueness of the solution of integral equation (2.1) in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$, we consider that the conditions (a_2) and (a_3) are met and we replace the condition (a_1) by the following condition:

(a_1') $K \in C([a,b]\times[a,b]\times J^4, \boldsymbol{B})$, $J \subset \boldsymbol{B}$ compact .

In addition, we denote by M_K a positive constant, such that for the restriction $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \boldsymbol{B}$ compact, we have:

$$|K(t,s,u_1,u_2,u_3,u_4)| \leq M_K, \quad \text{for all } t, s \in [a,b],\ u_1,u_2,u_3,u_4 \in J . \tag{2.5}$$

The following theorem is true.

Theorem 2.1.2. *Suppose that the conditions* (a_1'), (a_2) *and* (a_3) *are met. In addition, suppose that*

(a_4') *there exists* $L_K > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \leq$$
$$\leq L_K\left(|u_1-v_1| + |u_2-v_2)| + |u_3-v_3)| + |u_4-v_4)|\right),$$

for all $t, s \in [a,b]$, $u_i, v_i \in J$, $i = \overline{1,4}$;

(a_5) $4L_K(b-a) < 1$. (*contraction condition*)

If $r > 0$ *is a positive number such that*

$$x \in \overline{B}(f;r) \;\Rightarrow\; x(t) \in J \subset \boldsymbol{B},$$

and

(a_6) $M_K(b-a) \leq r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$)

then the integral equation (2.1) *has a unique solution* $x^* \in \overline{B}(f;r) \subset (C[a,b],\boldsymbol{B})$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r) \subset (C[a,b],\boldsymbol{B})$, *and if we denote by* x_n *the n-th successive approximation*, *then the estimation* (2.2) *is satisfied*.

Proof. We attach to the integral equation (2.1) the operator $A : \overline{B}(f;r) \to (C[a,b],\boldsymbol{B})$, defined by the relation (2.3).

Also, we suppose that there exists at least one positive number r with the property above:

$$x \in \overline{B}(f;r) \quad \Rightarrow \quad x(t) \in J \subset \boldsymbol{B},$$

In order to apply *the Contraction Principle* 1.3.1 we establish the condition of invariance of the sphere $\overline{B}(f;r)$ for the operator A. Thus, we have:

$$\left|A(x)(t) - f(t)\right| = \left|\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds\right| \leq$$

$$\leq \int_a^b \left|K(t,s,x(s),x(g(s)),x(a),x(b))\right| ds,$$

and using (2.5) the following inequality is obtained

$$\left|A(x)(t) - f(t)\right| \leq M_K(b-a), \quad \text{for all } t \in [a,b],$$

and now, from the condition (a_6) it results that the sphere $\overline{B}(f;r)$ is an invariant subset for the operator A, i.e. $\overline{B}(f;r) \in I(A)$. Now, we have the operator $A : \overline{B}(f;r) \to \overline{B}(f;r)$, also denoted by A and defined by the same relation (2.3); the sphere $\overline{B}(f;r)$ is a closed subset of Banach space $(C[a,b],\boldsymbol{B})$.

The set of the solutions of integral equation (2.1) in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$ coincides with the fixed points set of the operator A.

By an analogous reasoning to that of the proof of theorem 2.1.1 and using the condition (a_4') it follows that, with respect to *the metric of Chebyshev*, the operator A satisfies the following Lipschitz condition:

$$d\big(A(x),A(y)\big) \leq 4L_K(b-a)d(x,y), \quad \text{for all } x,y \in \overline{B}(f;r)$$

and according to the condition (a_5), it results that the operator A is an α–contraction with the coefficient $\alpha = 4L_K(b–a)$.

The conditions of *the Contraction Principle* 1.3.1. being satisfied, it results that the integral equation (2.1) has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$, that can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(f;r) \subset (C[a,b],\boldsymbol{B})$.

If x_n is the n-th successive approximation, then the estimation (2.2) is satisfied and the theorem is proved.

2.2 The case $\boldsymbol{B} = \boldsymbol{R}^m$

In the particular case $\boldsymbol{B} = \boldsymbol{R}^m$ we have the following system of Fredholm integral equations with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t)\,, \quad t \in [a,b]\,, \tag{2.6}$$

where $x : [a,b] \to \boldsymbol{R}^m$, $K : [a,b] \times [a,b] \times \boldsymbol{R}^m \times \boldsymbol{R}^m \times \boldsymbol{R}^m \times \boldsymbol{R}^m \to \boldsymbol{R}^m$, $g : [a,b] \to [a,b]$, $f : [a,b] \to \boldsymbol{R}^m$, which has the form:

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \end{cases}, \ t \in [a,b]\,. \tag{2.6'}$$

In order to give some conditions of existence and uniqueness of the solution of the system (2.6) in the space $C([a,b],\boldsymbol{R}^m)$ and in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{R}^m)$ respectively, we will apply the theorems 2.1.1, 2.1.2 and *the Perov's theorem* 1.3.4.

Now, in the particular case $\boldsymbol{B} = \boldsymbol{R}^m$, applying the theorem 2.1.1 we obtain the following theorem of existence and uniqueness of the solution of the system (2.6) in the space $C([a,b],\boldsymbol{R}^m)$.

Theorem 2.2.1. *Suppose that*

(b_1) $K \in C([a,b] \times [a,b] \times \boldsymbol{R}^m \times \boldsymbol{R}^m \times \boldsymbol{R}^m \times \boldsymbol{R}^m, \boldsymbol{R}^m)$;

(b_2) $f \in C([a,b],\boldsymbol{R}^m)$;

(b_3) $g \in C([a,b],[a,b])$.

(b_4) *there exists* $L > 0$ *, such that*

$$|K_i(t,s,u_1,u_2,u_3,u_4) - K_i(t,s,v_1,v_2,v_3,v_4)| \le$$
$$\le L\left(\|u_1 - v_1\|_{R^m} + \|u_2 - v_2\|_{R^m} + \|u_3 - v_3\|_{R^m} + \|u_4 - v_4\|_{R^m}\right)$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in \boldsymbol{R}^m$, $i = \overline{1,m}$;

(b_5) $4L(b-a) < 1$.

Under these conditions, it results that the system of integral equations (2.6) *has a unique solution* $x^* \in C([a,b],\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in C([a,b],\boldsymbol{R}^m)$. *If* $x_0 = (x_{01}, x_{02}, \ldots, x_{0m})$ *is the starting function and* $x_k = (x_{k1}, x_{k2}, \ldots, x_{km})$ *is the k-th successive approximation, then the following estimation is satisfied*:

$$\|x^* - x_k\|_{C([a,b],R^m)} \le \frac{[4L(b-a)]^k}{1 - 4L(b-a)} \cdot \|x_0 - x_1\|_{C([a,b],R^m)}\,. \tag{2.7}$$

Remark 2.2.1. If we consider one of the orms presented in chapter 1: *the Euclidean norm* $\|\cdot\|_E$, *the Minkowski's norm* $\|\cdot\|_M$, or *the norm of Chebyshev* $\|\cdot\|_C$, defined on the space $\boldsymbol{R}^m$, then in the theorem 2.2.1 is amended accordingly the assumptions (b_4) and (b_5) and the estimate (2.7) as follows:

a) *Euclidean norm*

(b_{41}) *there exists* $L > 0$ *, such that*

$$\left|K_i(t,s,u_1,u_2,u_3,u_4) - K_i(t,s,v_1,v_2,v_3,v_4)\right| \leq$$

$$\leq L\left[\left(\sum_{j=1}^{m}\left|u_{1j}-v_{1j}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{m}\left|u_{2j}-v_{2j}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{m}\left|u_{3j}-v_{3j}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{m}\left|u_{4j}-v_{4j}\right|^2\right)^{\frac{1}{2}}\right]$$

for all $t, s\in[a,b]$, $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\in \boldsymbol{R^m}$, $i=\overline{1,m}$;

(b_{51}) $4\sqrt{m}\,L\,(b-a)<1$

and

$$\left\| x^* - x_k \right\|_{C\left([a,b],R^m\right)} \leq \frac{\left[4\sqrt{m}L(b-a)\right]^k}{1-4\sqrt{m}L(b-a)}\cdot\left\| x_0 - x_1 \right\|_{C\left([a,b],R^m\right)} . \tag{2.7$_1$}$$

b) *Minkowski's norm*

(b_{42}) *there exists* $L > 0$ *, such that*

$$\left|K_i(t,s,u_1,u_2,u_3,u_4) - K_i(t,s,v_1,v_2,v_3,v_4)\right| \leq$$

$$\leq L\left(\sum_{j=1}^{m}\left|u_{1j}-v_{1j}\right| + \sum_{j=1}^{m}\left|u_{2j}-v_{2j}\right| + \sum_{j=1}^{m}\left|u_{3j}-v_{3j}\right| + \sum_{j=1}^{m}\left|u_{4j}-v_{4j}\right|\right)$$

for all $t, s\in[a,b]$, $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\in \boldsymbol{R^m}$, $i=\overline{1,m}$;

(b_{52}) $4\,mL(b-a) < 1$

and

$$\left\| x^* - x_k \right\|_{C\left([a,b],R^m\right)} \leq \frac{\left[4mL(b-a)\right]^k}{1-4mL(b-a)}\cdot\left\| x_0 - x_1 \right\|_{C\left([a,b],R^m\right)} . \tag{2.7$_2$}$$

c) *Chebyshev's norm*

(b_{43}) *there exists* $L > 0$ *, such that*

$$\max_{1\leq i\leq m}\left|K_i(t,s,u_1,u_2,u_3,u_4) - K_i(t,s,v_1,v_2,v_3,v_4)\right| \leq$$

$$\leq L\left(\max_{1\leq j\leq m}\left|u_{1j}-v_{1j}\right| + \max_{1\leq j\leq m}\left|u_{2j}-v_{2j}\right| + \max_{1\leq j\leq m}\left|u_{3j}-v_{3j}\right| + \max_{1\leq j\leq m}\left|u_{4j}-v_{4j}\right|\right)$$

for all $t, s\in[a,b]$, $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\in \boldsymbol{R^m}$, $i=\overline{1,m}$;

(b_{53}) $4\,L(b-a) < 1$

and

$$\left\| x^* - x_k \right\|_{C\left([a,b],R^m\right)} \leq \frac{\left[4L(b-a)\right]^k}{1-4L(b-a)}\cdot\left\| x_0 - x_1 \right\|_{C\left([a,b],R^m\right)} . \tag{2.7$_3$}$$

In what follows, using *the Perov's theorem* 1.3.4, we establish another result of existence and uniqueness of the solution of the system (2.6) in the space $C([a,b],\boldsymbol{R^m})$.

In order to apply *the Perov's theorem* 1.3.4, we consider the vectorial norm on the space $C([a,b],\boldsymbol{R}^{\mathrm{m}})$ presented in chapter 1 and defined by the relation (1.7):

$$\|x\|_C := \begin{pmatrix} \|x_1\|_C \\ \dots \\ \|x_m\|_C \end{pmatrix}, \quad \text{for all } x = \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix} \in C([a,b],\boldsymbol{R}^m),$$

where $\| x_k \|_C = \max\limits_{t\in[a,b]} | x_k(t) |, \ k = \overline{1,m}$.

Thus, we obtain a complete generalized metric space. In addition, suppose that the function K satisfies the generalized Lipschitz condition with respect to the last four arguments:

(b_6) *there exists* $Q \in M_{m\times m}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_C \le$$
$$\le Q \cdot \left(\|u_1 - v_1\|_C + \|u_2 - v_2\|_C + \|u_3 - v_3\|_C + \|u_4 - v_4\|_C\right)$$

for all $t, s\in [a,b]$, $u_i, v_i \in \boldsymbol{R}^m$, $i = \overline{1,4}$,

where we denote by $\| w\| = (|w_1|,...,|w_m|)$ the norm of an element $w\in \boldsymbol{R}^m$.

Now, applying *the Perov's theorem* 1.3.4, we obtain the following result.

Theorem 2.2.2. *Suppose that the conditions* (b_1)–(b_3) *and* (b_6) *are satisfied. In addition, suppose that*:

(b_7) $[4(b - a)Q]^n \to 0$ *as* $n \to \infty$.

Then the system of integral equations (2.6) *has a unique solution* $x^*\in C([a,b],\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in C([a,b], R^m)$. *If* x_n *is the n-th successive approximation, then the following estimation is satisfied*:

$$\left\| x^* - x_n \right\| \le [4(b-a)Q]^n \cdot [I_m - 4(b-a)Q]^{-1} \cdot \| x_0 - x_1 \| . \tag{2.8}$$

Proof. We consider the operator $A : C([a,b],\boldsymbol{R}^m) \to C([a,b],\boldsymbol{R}^m)$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in [a,b] , \tag{2.9}$$

i.e.

$$A(x)(t) := \begin{cases} A_1(x)(t) := \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ A_2(x)(t) := \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ A_m(x)(t) := \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \end{cases} \tag{2.9'}$$

From the conditions (b_1), (b_2) and (b_3) we deduce that the operator A is properly defined.

One observes that the set of the solutions of the system (2.6) in the space $C([a,b],\boldsymbol{R}^m)$ coincides with the set of the fixed points of the operator A, defined above.

Next, check the conditions of *the Perov's theorem* 1.3.4.

We have the difference:

$$|A(x)(t)-A(y)(t)| = \begin{pmatrix} |A_1(x)(t)-A_1(y)(t)| \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ |A_m(x)(t)-A_m(y)(t)| \end{pmatrix} =$$

$$= \begin{pmatrix} \left|\int_a^b [K_1(t,s,x(s),x(g(s)),x(a),x(b))-K_1(t,s,y(s),y(g(s)),y(a),y(b))]ds\right| \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \left|\int_a^b [K_m(t,s,x(s),x(g(s)),x(a),x(b))-K_m(t,s,y(s),y(g(s)),y(a),y(b))]ds\right| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_a^b |K_1(t,s,x(s),x(g(s)),x(a),x(b))-K_1(t,s,y(s),y(g(s)),y(a),y(b))|ds \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \int_a^b |K_m(t,s,x(s),x(g(s)),x(a),x(b))-K_m(t,s,y(s),y(g(s)),y(a),y(b))|ds \end{pmatrix},$$

of which under the conditions (b_6) one obtains the estimate:

$$\| A(x)-A(y)\|_{C([a,b],R^m)} \le 4(b-a)Q\,\|x-y\|_{C([a,b],R^m)}$$

and we deduce that the operator A satisfies a generalized Lipschitz condition, with the matrix $4(b-a)Q\in M_{m\times m}(\boldsymbol{R}_+)$, and from the condition (b_7) it results that this operator is a contraction.

The conditions of *the Perov's theorem* 1.3.4 being satisfied, it results that the system of integral equations with modified argument (2.6) has a unique solution in the space $C([a,b],\boldsymbol{R}^m)$ and the theorem is proved.

Now, in order to give conditions of existence and uniqueness of the solution of the system of integral equations (2.6) in the sphere $\overline{B}(f;r)\subset C([a,b],\boldsymbol{R}^m)$,

$$\overline{B}(f;r)=\{x\in C([a,b],R^m) \mid \|x-f\|_C\le r,\ r\in M_{m1}(R_+)\}\subset C([a,b],\boldsymbol{R}^m),$$

we apply the theorem 2.1.2 in the particular case $\boldsymbol{B}=\boldsymbol{R}^m$ and we obtain the following result.

Theorem 2.2.3. *Suppose that*

(b_1') $K\in C([a,b]\times[a,b]\times J^4, \boldsymbol{R}^m)$, $J\subset \boldsymbol{R}^m$ *compact* ;

(b_2) $f\in C([a,b],\boldsymbol{R}^m)$;

(b_3) $g\in C([a,b],[a,b])$;

(b_4') *there exists* $L>0$ *, such that*

$$|K_i(t,s,u_1,u_2,u_3,u_4)-K_i(t,s,v_1,v_2,v_3,v_4)| \le$$

$$\le L\left(\|u_1-v_1\|_{R^m}+\|u_2-v_2\|_{R^m}+\|u_3-v_3\|_{R^m}+\|u_4-v_4\|_{R^m}\right)$$

for all $t, s\in[a,b]$, $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\in J\subset\boldsymbol{R}^m$, $i=\overline{1,m}$;

(b_5) $4L(b-a)<1$.

In addition, suppose that there exists at least one positive number r, such that

$$x \in \overline{B}(f;r) \quad \Rightarrow \quad x(t) \in J \subset \boldsymbol{R}^m ,$$

and the following condition is satisfied:

(b_8) $M(b-a) \le r$ (*condition of invariance of the sphere* $\overline{B}(f;r)$),

where we denote by M a positive constant, such that for the restriction $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \boldsymbol{R}^m$ *compact, we have*

$$|K(t,s,u_1,u_2,u_3,u_4)| \le M, \quad \text{for all } t,s \in [a,b],\ u_1,u_2,u_3,u_4 \in J \subset \boldsymbol{R}^m . \tag{2.10}$$

Then the system of the integral equations (2.6) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b],\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element of* $\overline{B}(f;r) \subset C([a,b],\boldsymbol{R}^m)$. *If* x_0 *is the starting function and* x_k *is the k-th successive approximation, then the estimate* (2.7) *is true.*

Applying *the Perov's theorem* 1.3.4 we obtain another result of existence and uniqueness of the solution of the system of integral equations (2.6) in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{R}^m)$ which is presented below.

Theorem 2.2.4. *Suppose that the conditions* (b_1'), (b_2), (b_3) and (b_7) *are satisfied. In addition, suppose that*

(b_6') *there exists* $Q \in M_{m\times m}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_C \le$$

$$\le Q \cdot \left(\|u_1 - v_1\|_C + \|u_2 - v_2\|_C + \|u_3 - v_3\|_C + \|u_4 - v_4\|_C\right)$$

for all $t, s \in [a,b]$, $u_i, v_i \in \boldsymbol{J} \subset \boldsymbol{R}^m$, $i = \overline{1,4}$.

If $r \in M_{m\times 1}(\boldsymbol{R}_+)$ *such that*

$$x \in \overline{B}(f;r) \quad \Rightarrow \quad x(t) \in J \subset \boldsymbol{R}^m ,$$

and the following condition is satisfied:

(b_8') $M_K(b-a) \le r$ (*condition of invariance of the sphere* $\overline{B}(f;r)$),

where we denote by $M_K = \begin{pmatrix} M_K^1 \\ \ldots \\ M_K^m \end{pmatrix} \in M_{m\times 1}(\boldsymbol{R}_+)$ *a matrix with positive constants as elements, such that for the restriction* $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \boldsymbol{R}^m$ *compact, we have*:

$$\| K(t,s,u_1,u_2,u_3,u_4) \|_C \le M_K , \quad \text{for all } t,s \in [a,b],\ u_1,u_2,u_3,u_4 \in J \subset \boldsymbol{R}^m. \tag{2.11}$$

Then the system of integral equations (2.6) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b],\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r)$, *and if the n-th successive approximation is* x_n, *then the estimate* (2.8) *is satisfied.*

Proof. We consider the operator $A : \overline{B}(f;r) \to C([a,b],\boldsymbol{R^m})$, defined by the relation (2.9) and we establish under what conditions the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{R^m})$ is an invariant subset for the operator A. We have:

$$\| A(x)(t) - f(t) \|_C = \left\| \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds \right\|_C \leq$$

$$\leq \int_a^b \| K(t,s,x(s),x(g(s)),x(a),x(b)) \|_C \, ds$$

and using the relation (2.11) it results

$$\| A(x)(t) - f(t) \|_C \leq M_K(b-a), \quad \text{for all } t \in [a,b],$$

Now, from the condition (b_8') it results that the sphere $\overline{B}(f;r)$ is an invariant subset for the operator A, i.e. $\overline{B}(f;r) \in I(A)$ and we have the operator $A : \overline{B}(f;r) \to \overline{B}(f;r)$, denoted also by A and defined by the same relation (2.9), where $\overline{B}(f;r)$ is a closed subset of the Banach space $(C[a,b],\boldsymbol{R^m})$.

The set of the solutions of the system of inttegral equations (2.6) in the sphere $\overline{B}(f;r) \subset (C[a,b],\boldsymbol{R^m})$ coincides with the fixed points set of the operator A, defined by the relation (2.9).

By an analogous reasoning to that of the proof of theorem 2.2.2 and using the conditions (b_6') and (b_7) it results that the operator A is a contraction. Therefore, applying *the Perov's theorem* 1.3.4, it results the conclusion of this theorem and the proof is complete.

2.3 The case $\boldsymbol{B} = l^2(\boldsymbol{R})$

In the particular case $\boldsymbol{B} = l^2(\boldsymbol{R})$ we consider the Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [a,b], \tag{2.12}$$

where $x : [a,b] \to l^2(\boldsymbol{R})$, $K : [a,b]\times[a,b]\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R}) \to l^2(\boldsymbol{R})$, $g : [a,b] \to [a,b]$ and $f : [a,b] \to l^2(\boldsymbol{R})$, which has the form:

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}, \quad t \in [a,b] \tag{2.12'}$$

and we apply the theorems 2.1.1 and 2.1.2 to give some conditions of existence and uniqueness of the solution of integral equation (2.12) in the space $C([a,b],l^2(\boldsymbol{R}))$ and in the sphere $\overline{B}(f;r) \subset C([a,b], l^2(\boldsymbol{R}))$.

Applying the theorem 2.1.1 in the particular case $\boldsymbol{B} = l^2(\boldsymbol{R})$, we obtain the following theorem of existence and uniqueness of the solution of integral equation (2.12) in the space $C([a,b],l^2(\boldsymbol{R}))$.

Theorem 2.3.1. *Suppose that*:

(c_1) $K \in C([a,b]\times[a,b]\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R}), l^2(\boldsymbol{R}))$;

(c_2) $f \in C([a,b], l^2(\boldsymbol{R}))$;

(c_3) $g \in C([a,b], [a,b])$;

(c_4) *there exists* $L_K > 0$, *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_{l^2(R)} \le$$

$$\le L_K\left(\|u_1 - v_1\|_{l^2(R)} + \|u_2 - v_2\|_{l^2(R)} + \|u_3 - v_3\|_{l^2(R)} + \|u_4 - v_4\|_{l^2(R)}\right)$$

for all $t, s\in [a,b]$, $u_j, v_j\in l^2(\boldsymbol{R})$, $j = \overline{1,4}$;

(c_5) $4L_K(b-a) < 1$.

Under these conditions, it results that the integral equation with modified argument (2.12) *has a unique solution* $x^*\in C([a,b],l^2(\boldsymbol{R}))$, *that can be obtained by the successive approximations method starting at any element* $x_0\in C([a,b],l^2(\boldsymbol{R}))$. *Moreover, if* x_n *is the n-th successive approximation, then the following estimate is met:*

$$\| x^* - x_n \|_{l^2(R)} \le \frac{(4L_K(b-a))^n}{1-4L_K(b-a)}\| x_0 - x_1 \|_{l^2(R)} , \tag{2.13}$$

In order to give the conditions of existence and uniqueness of the solution of integral equation (2.12) in the sphere $\overline{B}(f;r) \subset C([a,b],l^2(\boldsymbol{R}))$,

$$\overline{B}(f;r) = \left\{x \in C([a,b],l^2(R)) \mid \| x - f \|_{l^2(R)} \le r,\ r > 0 \right\} \subset C([a,b],l^2(\boldsymbol{R})) ,$$

we apply the theorem 2.1.2 in the particular case $\boldsymbol{B} = l^2(\boldsymbol{R})$ and we obtain:

Theorem 2.3.2. *Suppose that:*

(c_1') $K \in C([a,b]\times[a,b] \times J^4, l^2(\boldsymbol{R}))$, $J \subset l^2(\boldsymbol{R})$ *compact* ;

(c_2) $f \in C([a,b], l^2(\boldsymbol{R}))$;

(c_3) $g \in C([a,b], [a,b])$;

(c_4') *there exists* $L_K > 0$, *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_{l^2(R)} \le$$

$$\le L_K\left(\|u_1 - v_1\|_{l^2(R)} + \|u_2 - v_2\|_{l^2(R)} + \|u_3 - v_3\|_{l^2(R)} + \|u_4 - v_4\|_{l^2(R)}\right)$$

for all $t, s\in [a,b]$, $u_j, v_j\in J$, $j = \overline{1,4}$;

(c_5) $4L_K(b-a) < 1$.

If r is a positive constant, such that

$$x \in \overline{B}(f;r) \;\Rightarrow\; x(t) \in J \subset l^2(\mathbf{R}),$$

and the following condition is fulfilled:

(c_6) $M_K(b-a) \le r$ (*condition of invariance of the sphere* $\overline{B}(f;r)$),

where we denote by M_K *a positive constant, such that for the restriction* $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset l^2(\mathbf{R})$ *compact, we have*:

$$\| K(t,s,u_1,u_2,u_3,u_4) \|_{l^2(R)} \le M_K, \quad \textit{for all } t,s \in [a,b],\; u_1,u_2,u_3,u_4 \in J, \tag{2.14}$$

then the integral equation with modified argument (2.12) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b],l^2(\mathbf{R}))$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r) \subset C([a,b],l^2(\mathbf{R}))$. *Moreover, if* x_n *is the n-th successive approximation, then the estimate* (2.13) *is satisfied.*

2.4 Examples

We consider some examples of integral equations with modified argument and systems of integral equations with modified argument and using the results that were obtained in the previous paragraphs, we will establish some conditions for existence and uniqueness of the solution.

I. Integral equations with modified argument

Example 2.4.1. We consider the integral equation with modified argument

$$x(t) = \int_0^1 \left(\frac{x(s)+x(s/2)}{t+s+3} + \frac{t\,x(0)+s\,x(1)}{9} \right) ds + 1, \quad t \in [0,1] \tag{2.15}$$

where $K \in C([0,1]\times[0,1]\times\mathbf{R}^4)$, $K(t,s,u_1,u_2,u_3,u_4) = \dfrac{u_1+u_2}{t+s+3} + \dfrac{t u_3 + s u_4}{9}$,

$f \in C[0,1]$, $f(t) = 1$, $g \in C([0,1],[0,1])$, $g(s) = s/2$ and $x \in C[0,1]$.

We check the conditions of the theorem 2.1.1 of existence and uniqueness of the solution of equation (2.15) in the space $C[0,1]$.

To study the existence and uniqueness of the solution of integral equation (2.15) in the space $C[0,1]$, we attach to this equation the operator $A : C[0,1] \to C[0,1]$ defined by the relation:

$$A(x)(t) = \int_0^1 \left(\frac{x(s)+x(s/2)}{t+s+3} + \frac{t\,x(0)+s\,x(1)}{9} \right) ds + 1, \quad t \in [0,1]. \tag{2.16}$$

The set of the solutions of integral equation (2.15) in the space $C[0,1]$ coincides with the fixed points set of the operator A, i.e. with F_A.

We have:

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| =$$

$$=\left|\frac{u_1+u_2}{t+s+3}+\frac{tu_3+su_4}{9}-\frac{v_1+v_2}{t+s+3}-\frac{tv_3+sv_4}{9}\right|\le$$

$$\le\left|\frac{1}{t+s+3}\right|\cdot|u_1-v_1|+\left|\frac{1}{t+s+3}\right|\cdot|u_2-v_2|+\frac{1}{9}|t|\cdot|u_3-v_3|+\frac{1}{9}|s|\cdot|u_4-v_4|\le$$

$$\le\frac{1}{3}|u_1-v_1|+\frac{1}{3}|u_2-v_2|+\frac{1}{9}|u_3-v_3|+\frac{1}{9}|u_4-v_4|\ ,$$

for all $t, s\in[0,1]$, $u_i, v_i\in \boldsymbol{R}$, $i=\overline{1,4}$,

hence, the function K satisfies the Lipschitz condition with the constant $\frac{1}{3}$ relative to the third and the fourth argument, and with the constant $\frac{1}{9}$ relative to the fifth and the sixth argument.

From the estimation of the difference:

$$|A(x)(t)-A(y)(t)|=\left|\int_0^1\left(\frac{x(s)+x(s/2)}{t+s+3}+\frac{t\,x(0)+s\,x(1)}{9}-\frac{y(s)+y(s/2)}{t+s+3}-\frac{t\,y(0)+s\,y(1)}{9}\right)ds\right|\le$$

$$\le\int_0^1\left|\frac{x(s)+x(s/2)}{t+s+3}+\frac{t\,x(0)+s\,x(1)}{9}-\frac{y(s)+y(s/2)}{t+s+3}-\frac{t\,y(0)+s\,y(1)}{9}\right|ds\le$$

$$\le\int_0^1\left(\frac{1}{3}|x(s)-y(s)|+\frac{1}{3}|x(s/2)-y(s/2)|+\frac{1}{9}|x(0)-y(0)|++\frac{1}{9}|x(1)-y(1)|\right)ds$$

and using *the Chebyshev norm*, one obtains

$$\|A(x)-A(y)\|_{C[0,1]}\le\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{9}+\frac{1}{9}\right)\cdot\|x-y\|_{C[0,1]}\cdot\int_0^1 ds=\frac{8}{9}\cdot\|x-y\|_{C[0,1]}\ ,$$

hence, it results that the operator A is a contraction with the coefficient $\frac{8}{9}<1$.

The conditions of the theorem 2.1.1 being satisfied, it results that the integral equation (2.15) has a unique solution $x^*\in C[0,1]$, that can be obtained by the successive approximations method, starting at any element $x_0\in C[0,1]$ and if the n-th successive approximation is x_n , then the following estimate is satisfied:

$$d(x^*,x_n)\le\frac{8^n}{9^{n-1}}d(x_0,x_1)\ .$$

Next we determine the conditions of existence and uniqueness of the solution of integral equation (2.15) in the sphere $\overline{B}(1;r)$

$$\overline{B}(1;r)=\{x\in C[0,1]\,|\,\|x-1\|_{C[0,1]}\le r,\ r\in R_+\}$$

from the space $C[0,1]$.

We consider the integral equation (2.15) where $K\in C([0,1]\times[0,1]\times J^4)$, $J\subset \boldsymbol{R}$ is compact, $f\in C[0,1]$ and $g\in C([0,1],[0,1])$ and we check the conditions of theorem 2.1.2 of existence and uniqueness of the solution of equation (2.15) in the sphere $\overline{B}(1;r)\subset C[0,1]$.

We attach to the integral equation (2.15) the operator $A:\overline{B}(1;r)\rightarrow C[0,1]$, defined by the relation (2.16), where r is a positive real number which meets the following condition:

$$x \in \overline{B}(1;r) \;\Rightarrow\; x(t) \in J \subset \mathbf{R}$$

and we show that there exists at least one number $r > 0$ with this property. Thus we have:

$$x \in \overline{B}(1;r) \;\Rightarrow\; |x(t)-1| \le r, \quad t \in [0,1] \;\Rightarrow\; |x(t)| \le r+1, \quad t \in [0,1]$$

and therefore

$$x \in \overline{B}(1;r) \;\Rightarrow\; \|x\|_{C[0,1]} \le r+1 \,. \tag{2.17}$$

In what follows, we determine the conditions which ensure that the sphere $\overline{B}(1;r)$ is an invariant subset for the operator *A*. We have:

$$|A(x)(t)-1| = \left| \int_0^1 \left(\frac{x(s)+x(s/2)}{t+s+3} + \frac{t\,x(0)+s\,x(1)}{9} \right) ds \right| \le \int_0^1 \left| \frac{x(s)+x(s/2)}{t+s+3} + \frac{t\,x(0)+s\,x(1)}{9} \right| ds$$

Also, for the function *K* we have

$$|K(t,s,u_1,u_2,u_3,u_4)| = \left| \frac{u_1+u_2}{t+s+3} + \frac{t\,u_3+s\,u_4}{9} \right| \le$$

$$\le \left|\frac{1}{t+s+3}\right| \cdot |u_1| + \left|\frac{1}{t+s+3}\right| \cdot |u_2| + \frac{1}{9}|t| \cdot |u_3| + \frac{1}{9}|s| \cdot |u_4| \le$$

$$\le \frac{1}{3}|u_1| + \frac{1}{3}|u_2| + \frac{1}{9}|u_3| + \frac{1}{9}|u_4| \,, \text{ for all } t, s \in [0,1],\; u_i, v_i \in J,\; i = \overline{1,4}\,.$$

So, we have

$$|A(x)(t)-1| \le \int_0^1 \left(\frac{1}{3}|x(s)| + \frac{1}{3}|x(s/2)| + \frac{1}{9}|x(0)| + \frac{1}{9}|x(1)| \right) ds$$

and using *the Chebyshev norm* we obtain

$$\|A(x)-1\|_{C[0,1]} \le \frac{8}{9} \cdot \|x\|_{C[0,1]} \cdot \int_0^1 ds = \frac{8}{9} \cdot \|x\|_{C[0,1]} \,,$$

where according to (2.17) we deduce that

$$\|A(x)-1\|_{C[0,1]} \le \frac{8}{9}(r+1)$$

and the condition of invariance of the sphere $\overline{B}(1;r) \subset C[0,1]$ is $\frac{8}{9}(r+1) \le r$.

Therefore, it results that if $r \ge 8$, then the sphere $\overline{B}(1;r)$ is an invariant subset for the operator *A*, i.e. $\overline{B}(1;r) \in I(A)$.

Now, we consider the operator $A : \overline{B}(1;r) \to \overline{B}(1;r)$, denoted also by A and defined by the same relation (2.16); $\overline{B}(1;r)$ is a closed subset of the Banach space $C[0,1]$.

The set of the solutions of integral equation (2.15), in the sphere $\overline{B}(1;r)$, coincides with the fixed points set of the operator *A* such defined.

By an analogous reasoning to that of the existence and uniqueness of the solution of integral equation (2.15) in the space $C[0,1]$, it results that the operator A is a contraction with the coefficient $\frac{8}{9}<1$.

The conditions of the theorem 2.1.2 being satisfied, it results that the integral equation (2.15) has a unique solution $x^*\in \overline{B}(1;r)\subset C[0,1]$, that can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(1;r)\subset C[0,1]$, and if x_n is the n-th successive approximation, then the following estimate is satisfied:

$$d(x^*,x_n) \le \frac{8^n}{9^{n-1}} d(x_0,x_1) \ .$$

Example 2.4.2. We consider the integral equation with modified argument

$$x(t)=\int_0^1\left(\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right)ds+\cos t\,,\qquad t\in[0,1] \tag{2.18}$$

where $K\in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$, $K(t,s,u_1,u_2,u_3,u_4)=\dfrac{\sin u_1+\cos u_2}{7}+\dfrac{u_3+u_4}{5}$,

$f\in C[0,1]$, $f(t)=\cos t$, $g\in C([0,1],[0,1])$, $g(s)=s/2$, $x\in C[0,1]$

and we check the conditions of the theorem 2.1.2 of existence and uniqueness of the solution of integral equation (2.18) in the space $C[0,1]$.

To study the existence and uniqueness of the solution integral equation (2.18) in the space $C[0,1]$, we attach to this equation the operator $A : C[0,1] \to C[0,1]$, defined by the relation:

$$A(x)(t)=\int_0^1\left(\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right)ds+\cos t\,,\ t\in[0,1]. \tag{2.19}$$

The set of the solutions of integral equation (2.18), in the space $C[0,1]$, coincides with the fixed points set of the operator A, i.e. with F_A .

We have:

$$\begin{aligned}
&|K(t,s,u_1,u_2,u_3,u_4)-K(t,s,v_1,v_2,v_3,v_4)|=\\
&\quad=\left|\frac{\sin u_1+\cos u_2}{7}+\frac{u_3+u_4}{5}-\frac{\sin v_1+\cos v_2}{7}-\frac{v_3+v_4}{5}\right|\le\\
&\quad\le\frac{1}{7}|\sin u_1-\sin v_1|+\frac{1}{7}|\sin u_2-\sin v_2|+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\le\\
&\quad\le\frac{1}{7}2\left|\sin\frac{u_1-v_1}{2}\right|\cdot\left|\cos\frac{u_1+v_1}{2}\right|+\frac{1}{7}2\left|\sin\frac{u_2-v_2}{2}\right|\cdot\left|\cos\frac{u_2+v_2}{2}\right|+\\
&\qquad+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\le\\
&\quad\le\frac{1}{7}2\left|\sin\frac{u_1-v_1}{2}\right|+\frac{1}{7}2\left|\sin\frac{u_2-v_2}{2}\right|+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\le\\
&\quad\le\frac{1}{7}|u_1-v_1|+\frac{1}{7}|u_2-v_2|+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\ ,
\end{aligned}$$

for all $t, s\in[0,1]$, $u_i, v_i\in \boldsymbol{R}$, $i=\overline{1,4}$, and it results that the function K satisfies the condition of Lipschitz with the constant $\frac{1}{7}$ relative to the third and the fourth argument, and with the constant $\frac{1}{5}$ relative to the fifth and the sixth argument.

From the estimation of the difference:

$$|A(x)(t)-A(y)(t)| =$$

$$=\left|\int_0^1\left(\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}-\frac{\sin(y(s))+\cos(y(s/2))}{7}-\frac{y(0)+y(1)}{5}\right)ds\right| \le$$

$$\le\int_0^1\left|\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}-\frac{\sin(y(s))+\cos(y(s/2))}{7}-\frac{y(0)+y(1)}{5}\right|ds \le$$

$$\le\int_0^1\left(\frac{1}{7}|x(s)-y(s)|+\frac{1}{7}|x(s/2)-y(s/2)|+\frac{1}{5}|x(0)-y(0)|+\frac{1}{5}|x(1)-y(1)|\right)ds$$

and using *the Chebyshev norm*, one obtains

$$\|A(x)-A(y)\|_{C[0,1]} \le \left(\frac{1}{7}+\frac{1}{7}+\frac{1}{5}+\frac{1}{5}\right)\cdot\|x-y\|_{C[0,1]}\cdot\int_0^1 ds=\frac{24}{35}\cdot\|x-y\|_{C[0,1]} ,$$

hence, it results that the operator A is a contraction with the coefficient $\frac{24}{35}<1$.

The conditions of the theorem 2.1.1 being satisfied, it results that the integral equation (2.18) has a unique solution $x^*\in C[0,1]$, that can be obtained by the successive approximations method, starting at any element $x_0\in C[0,1]$, and if the n-th successive approximation is x_n , then the following estimate is satisfied:

$$d(x^*,x_n) \le \frac{24^n}{35^{n-1}\cdot 11}d(x_0,x_1) .$$

Next we determine the conditions of existence and uniqueness of the solution of integral equation (2.18) in the sphere $\overline{B}(\cos t;r)$,

$$\overline{B}(\cos t;r)=\{x\in C[0,1] \mid \|x-\cos t\|_{C[0,1]} \le r,\ r\in R_+\}$$

from the space $C[0,1]$.

We consider the integral equation (2.18), where $K\in C([0,1]\times[0,1]\times J^4)$, $J\subset \boldsymbol{R}$ is compact, $f\in C[0,1]$ and $g\in C([0,1],[0,1])$ and we check the conditions of the theorem 2.1.2 of existence and uniqueness of the solution of integral equation (2.18) in the sphere $\overline{B}(\cos t;r)\subset C[0,1]$.

We attach to the integral equation (2.18) the operator A : $\overline{B}(\cos t;r)\to C[0,1]$, defined by the relation (2.19), where r is a positive real number which meets the condition:

$$x\in \overline{B}(\cos t;r) \;\Rightarrow\; x(t)\in J\subset \boldsymbol{R} ,$$

and show that there exists at least one number $r > 0$ with this property. Thus we have:

$$x\in \overline{B}(\cos t;r) \;\Rightarrow\; |x(t)-\cos t|\le r,\quad t\in[0,1] \;\Rightarrow\; |x(t)|\le r+1,\quad t\in[0,1]$$

and therefore

$$x \in \overline{B}(\cos t; r) \quad \Rightarrow \quad \| x \|_{C[0,1]} \le r+1 \; . \tag{2.20}$$

In what follows, we determine the conditions which ensure that the sphere $\overline{B}(\cos t; r)$ is an invariant subset for the operator *A*. We have:

$$| A(x)(t) - \cos t | = \left| \int_0^1 \left(\frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right) ds \right| \le$$

$$\le \int_0^1 \left| \frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right| ds \; .$$

Also, for the function *K* we have:

$$| K(t,s,u_1,u_2,u_3,u_4) | = \left| \frac{\sin u_1 + \cos u_2}{7} + \frac{u_3 + u_4}{5} \right| \le \frac{1}{7} | \sin u_1 | + \frac{1}{7} | \cos u_2 | + \frac{1}{5} | u_3 | + \frac{1}{5} | u_4 | \le$$

$$\le \frac{1}{7} | u_1 | + \frac{1}{7} | u_2 | + \frac{1}{5} | u_3 | + \frac{1}{5} | u_4 | \; ,$$

for all $t, s \in [0,1]$, $u_i, v_i \in J$, $i = \overline{1,4}$. So, we have:

$$| A(x)(t) - \cos t | \le \int_0^1 \left(\frac{1}{7} |x(s)| + \frac{1}{7} |x(s/2)| + \frac{1}{5} |x(0)| + \frac{1}{5} |x(1)| \right) ds$$

and using *the Chebyshev norm* we obtain

$$\| A(x) - \cos t \|_{C[0,1]} \le \frac{24}{35} \cdot \| x \|_{C[0,1]} \cdot \int_0^1 ds = \frac{24}{35} \cdot \| x \|_{C[0,1]} \; ,$$

where according to (2.20) we deduce that

$$\| A(x) - \cos t \|_{C[0,1]} \le \frac{24}{35}(r+1)$$

and the condition of invariance of the sphere $\overline{B}(\cos t; r) \subset C[0,1]$ is $\frac{24}{35}(r+1) \le r$.

Therefore, it results that if $r \ge \frac{24}{11}$, then the sphere $\overline{B}(\cos t; r)$ is an invariant subset for the operator *A*, i.e. $\overline{B}(\cos t; r) \in I(A)$.

Now, we consider the operator $A : \overline{B}(\cos t; r) \to \overline{B}(\cos t; r)$, which we denote also by A and is defined by the same relation (2.19); $\overline{B}(\cos t; r)$ is a closed subset of the Banach space $C[0,1]$.

The set of the solutions of integral equation (2.18), in the sphere $\overline{B}(\cos t; r) \subset C[0,1]$, coincides with the set of the fixed points of the operator *A* such defined.

By an analogous reasoning to that of the existence and uniqueness of the solution of integral equation (2.18) in the space $C[0,1]$, it results that the operator *A* is a contraction with the coefficient $\frac{24}{35} < 1$.

The conditions of the theorem 2.1.2 being satisfied, it results that the integral equation (2.18) has a unique solution $x^* \in \overline{B}(\cos t; r) \subset C[0,1]$, that can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(\cos t; r) \subset C[0,1]$, and if x_n is the *n*-th successive approximation, then the following estimate is satisfied:

$$d(x^*,x_n) \le \frac{24^n}{35^{n-1}\cdot 11} d(x_0,x_1) \ .$$

II. System of integral equations with modified argument

Example 2.4.3. We consider the system of integral equations with modified argument

$$\begin{cases} x_1(t) = \int_0^1 \left(\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right) ds + 2t + 1 \\ \\ x_2(t) = \int_0^1 \left(\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right) ds + \sin t \end{cases} , \tag{2.21}$$

where $K \in C([0,1]\times[0,1]\times \boldsymbol{R}^2\times\boldsymbol{R}^2\times\boldsymbol{R}^2\times\boldsymbol{R}^2,\boldsymbol{R}^2)$,

$$K(t,s,u_1,u_2,u_3,u_4) = \left(K_1(t,s,u_1,u_2,u_3,u_4),\, K_2(t,s,u_1,u_2,u_3,u_4)\right),$$

$$K_1(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{15} u_{11} + \frac{2t+1}{15} u_{21} + \frac{1}{5} u_{31} + \frac{1}{5} u_{41} \ ,$$

$$K_2(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{21} u_{12} + \frac{2t+1}{21} u_{22} + \frac{1}{7} u_{32} + \frac{1}{7} u_{42} \ ,$$

$f \in C([0,1],R^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 2t+1$, $f_2(t) = \sin t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$ and $x \in C([0,1], \boldsymbol{R}^2)$,

and we check the conditions of the theorem 2.2.2 of existence and uniqueness of the solution of the system of integral equations (2.21) in the space $C([0,1],\boldsymbol{R}^2)$.

In order to study the existence and uniqueness of the solution of the system of integral equations (2.21) in the space $C([0,1],\boldsymbol{R}^2)$, we attach to this system the operator $A:C([0,1],\boldsymbol{R}^2)\to C([0,1],\boldsymbol{R}^2)$ defined by the relation:

$$A(x)(t) = \begin{cases} A_1(x)(t) = \int_0^1 \left(\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right) ds + 2t + 1 \\ \\ A_2(x)(t) = \int_0^1 \left(\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right) ds + \sin t \ . \end{cases} \tag{2.22}$$

The set of the solutions of the system of integral equations (2.21), in the space $C([0,1],\boldsymbol{R}^2)$, coincides with the the fixed points set of the operator A, defined by the relation (2.22). Thus, we have:

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ |K_2(t,s,u_1,u_2,u_3,u_4) - K_2(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \le$$

$$\le \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix} \cdot \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| + |u_{41}-v_{41}| \\ |u_{12}-v_{12}| + |u_{22}-v_{22}| + |u_{32}-v_{32}| + |u_{42}-v_{42}| \end{pmatrix} , \tag{2.23}$$

for all $t, s\in[0,1]$, $u_i, v_i\in \boldsymbol{R}^2$, $i=\overline{1,4}$,

and it results that the function K satisfies a generalized Lipschitz condition with respect to the last four arguments, with the matrix

$$Q=\begin{pmatrix}1/5 & 0\\ 0 & 1/7\end{pmatrix},\quad Q\in M_{2\times 2}(\boldsymbol{R}_+)$$

and therefore, it results that the condition (b_6) of the theorem 2.2.2 is met.

From the estimation of the difference:

$$|A(x)(t)-A(y)(t)| = \begin{pmatrix}|A_1(x)(t)-A_1(y)(t)|\\ |A_2(x)(t)-A_2(y)(t)|\end{pmatrix} \le$$

$$\le \begin{pmatrix}\left|\int_0^1 (K_1(t,s,x(s),x(s/2),x(0),x(1))-K_1(t,s,y(s),y(s/2),y(0),y(1)))ds\right|\\ \left|\int_0^1 (K_2(t,s,x(s),x(s/2),x(0),x(1))-K_2(t,s,y(s),y(s/2),y(0),y(1)))ds\right|\end{pmatrix} \le$$

$$\le \begin{pmatrix}\int_0^1 |K_1(t,s,x(s),x(s/2),x(0),x(1))-K_1(t,s,y(s),y(s/2),y(0),y(1))|\,ds\\ \int_0^1 |K_2(t,s,x(s),x(s/2),x(0),x(1))-K_2(t,s,y(s),y(s/2),y(0),y(1))|\,ds\end{pmatrix},$$

and using the relation (2.23) and *the generalized Chebyshev norm*, one obtains

$$\| A(x)-A(y)\|_{C([0,1],R^2)} \le \begin{pmatrix}4/5 & 0\\ 0 & 4/7\end{pmatrix}\cdot \| x-y\|_{C([0,1],R^2)},$$

hence, the operator A satisfies a generalized Lipschitz condition with respect to the last four arguments, with the matrix $\begin{pmatrix}4/5 & 0\\ 0 & 4/7\end{pmatrix}$, which converges to zero according to theorem 1.3.3. Therefore, the condition (b_7) of the theorem 2.2.2 is met, i.e. the operator A is a generalized contraction.

The conditions of the theorem 2.2.2 being satisfied, it results that the system of integral equations with modified argument (2.21) has a unique solution $x^*\in C([0,1],\boldsymbol{R}^2)$, that can be obtained by the method of successive approximations, starting any element $x_0\in C([0,1], \boldsymbol{R}^2)$, and if x_n is the n-th successive approximation, then the following estimation is true:

$$\| x^*-x_n\|_C \le \begin{pmatrix}4/5 & 0\\ 0 & 4/7\end{pmatrix}^n\cdot\begin{pmatrix}1/5 & 0\\ 0 & 3/7\end{pmatrix}^{-1}\cdot \| x_0-x_1\|_C$$

i.e.

$$\| x^*-x_n\|_C \le 4^n\cdot\begin{pmatrix}\frac{1}{5^{n-1}} & 0\\ 0 & \frac{1}{3\cdot 7^{n-1}}\end{pmatrix}\cdot \| x_0-x_1\|_C .$$

Next, we determine the conditions of existence and uniqueness of the solution of the system of integral equations (2.21) in the sphere $\overline{B}(f;r)$,

$$\overline{B}(f;r)=\{x\in C([0,1],R^2)\mid \|x-f\|_{C([0,1],R^2)}\le r,\ r\in R_+^2\}$$

from the space $C([0,1],\boldsymbol{R}^2)$.

We consider the system of integral equations (2.21), where $K\in C([0,1]\times[0,1]\times J^4,\boldsymbol{R}^2)$, $J\subset\boldsymbol{R}^2$ compact, $f\in C([0,1],\boldsymbol{R}^2)$ and $g\in C([0,1],[0,1])$.

Now, we check the conditions of the theorem 2.2.4 of existence and uniqueness of the solution of the system of integral equations (2.21) in the sphere $\overline{B}(f;r)\subset C([0,1],\boldsymbol{R}^2)$.

We attach to the system of integral equations (2.21), the operator A: $\overline{B}(f;r)\to C([0,1],\boldsymbol{R}^2)$, defined by the relation (2.22), where $r\in R_+^2$, satisfies the condition:

$$x\in\overline{B}(f;r)\ \Rightarrow\ x(t)\in J\subset\boldsymbol{R}^2,$$

and we show that there exists at least one r with this property. We have:

$$x\in\overline{B}(f;r)\ \Rightarrow\ |x(t)-f(t)|\le r\ \Rightarrow\ \begin{pmatrix}|x_1(t)-(2t+1)|\\ \\ |x_2(t)-\sin t|\end{pmatrix}\le\begin{pmatrix}r_1\\ r_2\end{pmatrix}$$

and therefore, it results that

$$x\in\overline{B}(f;r)\ \Rightarrow\ \begin{pmatrix}|x_1(t)|\\ |x_2(t)|\end{pmatrix}\le\begin{pmatrix}r_1+3\\ r_2+1\end{pmatrix},\quad t\in[0,1]. \tag{2.24}$$

In what follows, we determine the conditions which ensure that the sphere $\overline{B}(f;r)\subset C([0,1],\boldsymbol{R}^2)$ is an invariant subset for the operator A.

We have:

$$|A(x)(t)-f(t)|=\begin{pmatrix}\left|\int_0^1\left(\frac{t+2}{15}x_1(s)+\frac{2t+1}{15}x_1(s/2)+\frac{1}{5}x_1(0)+\frac{1}{5}x_1(1)\right)ds\right|\\ \\ \left|\int_0^1\left(\frac{t+2}{21}x_2(s)+\frac{2t+1}{21}x_2(s/2)+\frac{1}{7}x_2(0)+\frac{1}{7}x_2(1)\right)ds\right|\end{pmatrix}\le$$

$$\le\begin{pmatrix}\int_0^1\left|\frac{t+2}{15}x_2(s)+\frac{2t+1}{15}x_2(s/2)+\frac{1}{5}x_2(0)+\frac{1}{5}x_2(1)\right|ds\\ \\ \int_0^1\left|\frac{t+2}{21}x_2(s)+\frac{2t+1}{21}x_2(s/2)+\frac{1}{7}x_2(0)+\frac{1}{7}x_2(1)\right|ds\end{pmatrix}.$$

Also, for the function K we have

$$\begin{pmatrix}|K_1(t,s,u_1,u_2,u_3,u_4)|\\ |K_2(t,s,u_1,u_2,u_3,u_4)|\end{pmatrix}\le\begin{pmatrix}1/5 & 0\\ 0 & 1/7\end{pmatrix}\cdot\begin{pmatrix}|u_{11}|+|u_{21}|+|u_{31}|+|u_{41}|\\ |u_{12}|+|u_{22}|+|u_{32}|+|u_{42}|\end{pmatrix},$$

for all $t,s\in[0,1]$, $u_i,v_i\in J$, $i=\overline{1,4}$.

So, we have

$$|A(x)(t)-f(t)| \le \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix} \cdot \begin{pmatrix} \int_0^1 \left(|x_1(s)|+|x_1(s/2)|+|x_1(0)|+|x_1(1)|\right)ds \\ \int_0^1 \left(|x_2(s)|+|x_2(s/2)|+|x_2(0)|+|x_2(1)|\right)ds \end{pmatrix}$$

and using *the generalized Chebyshev norm*, one obtains

$$\| A(x)-f \|_{C([0,1],R^2)} \le \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix} \cdot \| x \|_{C([0,1],R^2)} ,$$

hence, according to (2.24) we deduce that

$$\| A(x)-f \|_{C([0,1],R^2)} \le \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix} \cdot \begin{pmatrix} r_1+3 \\ r_2+1 \end{pmatrix} = \begin{pmatrix} \dfrac{4r_1+12}{5} \\ \dfrac{4r_2+4}{7} \end{pmatrix}$$

and now, it results the following condition of invariance of the sphere $\overline{B}(f;r) \subset C([0,1],\boldsymbol{R}^2)$:

$$\begin{pmatrix} \dfrac{4r_1+12}{5} \\ \dfrac{4r_2+4}{7} \end{pmatrix} \le \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} .$$

Therefore, if

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \ge \begin{pmatrix} 12 \\ 4/3 \end{pmatrix},$$

then the sphere $\overline{B}(f;r) \subset C([0,1],\boldsymbol{R}^2)$ is an invariant subset for the operator A, i.e. $\overline{B}(f;r) \in I(A)$.

Now, we consider the operator $A : \overline{B}(f;r) \to \overline{B}(f;r)$, which we denote also by A and is defined by the same relation (2.22); the sphere $\overline{B}(f;r)$ is a closed subset of the Banach space $(C[0,1],\boldsymbol{R}^2)$.

The set of the solutions of the system of integral equations (2.21), in the sphere $\overline{B}(f;r) \subset C([0,1],\boldsymbol{R}^2)$, coincides with the fixed points set of the operator A such defined.

By an analogous reasoning to that of the existence and uniqueness of the solution of the system of integral equations (2.21) in the space $C([0,1],\boldsymbol{R}^2)$, it results that the operator A is a generalized contraction.

The conditions of the theorem 2.2.4 being satisfied, it results that the system of integral equations (2.21) has a unique solution $x^* \in \overline{B}(f;r) \subset C([0,1],\boldsymbol{R}^2)$, that can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(f;r) \subset C([0,1],\boldsymbol{R}^2)$, and if x_n is the n-th successive approximation, then the following estimate is satisfied:

$$\left\| x^* - x_n \right\|_C \le 4^n \cdot \begin{pmatrix} \dfrac{1}{5^{n-1}} & 0 \\ 0 & \dfrac{1}{3 \cdot 7^{n-1}} \end{pmatrix} \cdot \left\| x_0 - x_1 \right\|_C .$$

2.5 Generalization

We consider the integral equation with modified argument

$$x(t) = \int_{\Omega} K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds + f(t), \quad t \in \overline{\Omega}, \tag{2.25}$$

where $\Omega \subset \boldsymbol{R}^m$ is a bounded domain, $K : \overline{\Omega}\times\overline{\Omega}\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times C(\partial\Omega,\boldsymbol{R}^m) \to \boldsymbol{R}^m$, $f: \overline{\Omega} \to \boldsymbol{R}^m$ and $g : \overline{\Omega} \to \overline{\Omega}$.

This equation is a generalization of the integral equation with modified argument (2.1) and we intend to apply *the Contraction Principle* 1.3.1 and *the Perov's theorem* 1.3.4, to obtain the existence and uniqueness theorems of the solution in the space $C(\overline{\Omega},\boldsymbol{R}^m)$ and in the sphere $\overline{B}(f;r) \subset C(\overline{\Omega},\boldsymbol{R}^m)$. In establishing of these results will be useful also, the results given by I.A. Rus in the paper [39].

Let $\overline{\Omega} \subset \boldsymbol{R}^m$, be a bounded domain.

To establish the existence and uniqueness theorems of the solution of equation (2.25) in the space $C(\overline{\Omega},\boldsymbol{R}^m)$ we reduce the problem of determining these solutions to a fixed point problem.

To this end we consider the operator $A : C(\overline{\Omega},\boldsymbol{R}^m) \to C(\overline{\Omega},\boldsymbol{R}^m)$, defined by the relation:

$$A(x)(t) = \int_{\Omega} K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds + f(t) . \tag{2.26}$$

Observe that the set of the solutions of integral equation (2.25), in the space $C(\overline{\Omega},\boldsymbol{R}^m)$, coincides with the fixed points set of the operator A defined by the relation (2.26).

Applying *the Contraction Principle* 1.3.1 for the operator A, we obtain the following theorem of existence and uniqueness:

Theorem 2.5.1. *Suppose that*

(i) $K \in C(\overline{\Omega}\times\overline{\Omega}\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times C(\partial\Omega,\boldsymbol{R}^m), \boldsymbol{R}^m)$, $f \in C(\overline{\Omega},\boldsymbol{R}^m)$, $g \in C(\overline{\Omega},\overline{\Omega})$;

(ii) *there exists* $L > 0$, *such that*

$$\left|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)\right| \le$$

$$\le L\left(\|u_1 - v_1\|_{R^m} + \|u_2 - v_2\|_{R^m} + \|u_3 - v_3\|_{C(\partial\Omega,R^m)}\right)$$

for all $t, s \in \overline{\Omega}$, $u_1, u_2, v_1, v_2 \in \boldsymbol{R}^m$, $u_3, v_3 \in C(\partial\Omega,\boldsymbol{R}^m)$, $i = \overline{1,m}$;

(iii) $3\cdot L\cdot mes(\Omega) < 1$.

Under these conditions, it results that the integral equation (2.25) *has a unique solution* $x^* \in C(\overline{\Omega},\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element from* $C(\overline{\Omega},\boldsymbol{R}^m)$. *Moreover, if* x_0 *is the starting function and* x_k *is the k-th successive approximation*, *then the following estimation is satisfied*:

$$\|x^* - x_k\|_{C(\overline{\Omega},R^m)} \le \frac{[3\cdot L\cdot mes(\Omega)]^k}{1 - 3\cdot L\cdot mes(\Omega)}\cdot\|x_0 - x_1\|_{C(\overline{\Omega},R^m)} . \tag{2.27}$$

Proof. Using the condition (i) we deduce that the operator A is properly defined.

We check the conditions of *the Contraction Principle* 1.3.1. First, we show that the operator A is a contraction.

According to condition (*ii*) we have:

$$|A(x)(t)-A(y)(t)| \le \left| \int_{\Omega} \left[K(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K(t,s,y(s),y(g(s)),y|_{\partial\Omega})\right] ds \right| \le$$

$$\le \; 3\cdot L\cdot mes(\Omega)\cdot \| x-y\|_{C(\overline{\Omega},R^m)}$$

and using *the generalized Chebyshev norm*, it results that

$$\| A(x)-A(y)\|_{C(\overline{\Omega},R^m)} \le 3\cdot L\cdot mes(\Omega)\cdot \| x-y\|_{C(\overline{\Omega},R^m)} .$$

Therefore, the operator A satisfies the Lipschitz condition with the constant $3{\cdot}L{\cdot}mes(\Omega)$. The condition (*iii*) allows us to apply *the Contraction Principle* 1.3.1 and so the proof is complete.

Theorem 2.5.2. *Suppose that the following conditions are met*:

(*i*) $K\in C(\overline{\Omega}\times\overline{\Omega}\times(J_1\times\ldots\times J_m)\times(J_1\times\ldots\times J_m)\times C(\partial\Omega,\mathbf{R}^m),\mathbf{R}^m)$, $f\in C(\overline{\Omega},\mathbf{R}^m)$, $g\in C(\overline{\Omega},\overline{\Omega})$, *where* $J_1,\ldots,J_m\subset\mathbf{R}$ *are closed and finite intervals* ;

(*ii*) *there exists* $L>0$, *such that*

$$|K_i(t,s,u_1,u_2,u_3)-K_i(t,s,v_1,v_2,v_3)| \le L\left(\| u_1-v_1\|_{R^m}+\| u_2-v_2\|_{R^m}+\| u_3-v_3\|_{C(\partial\Omega,R^m)}\right)$$

for all $t,s\in\overline{\Omega}$, $u_1,u_2,v_1,v_2\in J_1\times\ldots\times J_m$, $u_3,v_3\in C(\partial\Omega,\mathbf{R}^m)$, $i=\overline{1,m}$;

(*iii*) $3{\cdot}L{\cdot}mes(\Omega) < 1$.

If r *is a positive number such that*

$$x\in\overline{B}(f;r) \;\Rightarrow\; x(t)\in J_1\times\ldots\times J_m , \qquad (2.28)$$

and the following condition is met:

(*iv*) $M_K\, mes(\Omega) \le r$, (*condition of invariance of the sphere* $\overline{B}(f;r)$),

where we denote by M_K *a positive constant, such that the function K verifies the inequality*:

$$|K_i(t,s,u,v,w)| \le M_K \;, \quad \textit{for all } t,s\in\overline{\Omega},\; u,v\in J_1\times\ldots\times J_m,\; w\in C(\partial\Omega,\mathbf{R}^m), \qquad (2.29)$$

then the integral equation (2.25) *has a unique solution* $x^*\in\overline{B}(f;r)\subset C(\overline{\Omega},\mathbf{R}^m)$, *that can be obtained by the successive approximations method, starting at any element from* $\overline{B}(f;r)$, *and if* x_0 *is the starting function and* x_k *is the k-th successive approximation, then the estimation* (2.27) *is satisfied.*

Proof. According to the condition (*i*) we deduce that the operator A: $\overline{B}(f;r)\to C(\overline{\Omega},\mathbf{R}^m)$ defined by the relation (2.26) is properly defined, and from the condition (*iv*) and using the relation (2.29) it results that $A(\overline{B}(f;r))\subset\overline{B}(f;r)$, i.e. $\overline{B}(f;r)\in I(A)$. Now, we will consider the operator $A:\overline{B}(f;r)\to\overline{B}(f;r)$, also denoted by A and defined by the same relation (2.26); the sphere $\overline{B}(f;r)$ is a closed subset of the Banach space $C(\overline{\Omega},\mathbf{R}^m)$.

The set of the solutions of integral equation (2.25), in the sphere $\overline{B}(f;r)\subset C(\overline{\Omega},\mathbf{R}^m)$, coincides with the fixed points set of the operator A such defined.

Using the conditions (*ii*) and (*iii*) we deduce that the operator $A:\overline{B}(f;r)\to\overline{B}(f;r)$, defined by the relation (2.26), satisfies the contraction condition with the coefficient $3\cdot L\cdot mes(\Omega)$.

Now, applying *the Contraction Principle* 1.3.1, it results the conclusion of theorem and the proof is complete.

To ensure the conditions for applying *the Perov's theorem* 1.3.4, we consider *the generalized Chebyshev norm* on the space $C(\overline{\Omega},\boldsymbol{R}^m)$, defined in chapter 1, by the relation (1.7)

$$\|x\|_C := \begin{pmatrix} \|x_1\|_C \\ \dots \\ \|x_m\|_C \end{pmatrix}, \quad \text{for every } x = \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix} \in C(\overline{\Omega},\boldsymbol{R}^m)$$

and such, we obtain a complete generalized Banach space.

Applying *the Perov's fixed point theorem* 1.3.4, we obtain:

Theorem 2.5.3. *Such that*

(*i*) $K\in C(\overline{\Omega}\times\overline{\Omega}\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times C(\partial\Omega,\boldsymbol{R}^m),\boldsymbol{R}^m)$, $f\in C(\overline{\Omega},\boldsymbol{R}^m)$, $g\in C(\overline{\Omega},\overline{\Omega})$;

(*ii*) *there exists* $Q\in M_{m\times m}(\boldsymbol{R}_+)$ *such that*

$$\| K(t,s,u_1,u_2,u_3)-K(t,s,v_1,v_2,v_3)\|_C \le Q\left(\| u_1-v_1\|_C+\| u_2-v_2\|_C+\| u_3-v_3\|_{C(\partial\Omega,R^m)}\right)$$

for all $t, s\in\overline{\Omega}$, $u_1, u_2, v_1, v_2\in\boldsymbol{R}^m$, $u_3, v_3\in C(\partial\Omega,\boldsymbol{R}^m)$;

(*iii*) $3\cdot mes(\Omega)\cdot Q$ *is a matrix which converges to null matrix.*

Under these condition, it results that the integral equation (2.25) *has a unique solution* $x^*\in C(\overline{\Omega},\boldsymbol{R}^m)$, *that can be obtained by the successive approximations method, starting at any element from* $C(\overline{\Omega},\boldsymbol{R}^m)$. *Moreover, if* x_0 *is the starting element and* x_k *is the k-th successive approximation, then the following estimation is true:*

$$\left\| x^*-x_k\right\|_{C\left(\overline{\Omega},R^m\right)} \le [3\cdot mes(\Omega)\cdot Q]^k\cdot\left[I_m-3\cdot mes(\Omega)\cdot Q\right]^{-1}\cdot\| x_0-x_1\|_{C\left(\overline{\Omega},R^m\right)} \qquad (2.30)$$

Proof. We consider the operator A: $C(\overline{\Omega},\boldsymbol{R}^m)\to C(\overline{\Omega},\boldsymbol{R}^m)$, defined by the relation (2.26).

From the condition (*i*) we deduce that the operator A is properly defined.

Note that the set of the solutions of equation (2.25), in the space $C(\overline{\Omega},\boldsymbol{R}^m)$, coincides with the fixed points set of the operator A, defined above.

We check the conditions of *the Perov's theorem* 1.3.4. We show that the operator A is a contraction. According to condition (*ii*), the function K satisfies a generalized Lipschitz condition with respect to the last three arguments, with a matrix $Q\in M_{m\times m}(\boldsymbol{R}_+)$ and therefore we have:

$$| A(x)(t)-A(y)(t)| = \begin{pmatrix} |A_1(x)(t)-A_1(y)(t)| \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ |A_m(x)(t)-A_m(y)(t)| \end{pmatrix} =$$

$$= \begin{pmatrix} \left| \int_a^b \left[K_1(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_1(t,s,y(s),y(g(s)),y|_{\partial\Omega}) \right] ds \right| \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \left| \int_a^b \left[K_m(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_m(t,s,y(s),y(g(s)),y|_{\partial\Omega}) \right] ds \right| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_a^b \left| K_1(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_1(t,s,y(s),y(g(s)),y|_{\partial\Omega}) \right| ds \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \int_a^b \left| K_m(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_m(t,s,y(s),y(g(s)),y|_{\partial\Omega}) \right| ds \end{pmatrix},$$

and according to *the Chebyshev norm* on $C(\overline{\Omega},\boldsymbol{R}^m)$, defined in chapter 1, by the relation (1.7), the following estimate is obtained:

$$\| A(x) - A(y) \|_{C(\overline{\Omega},R^m)} \le 3 \cdot mes(\Omega) \cdot Q \cdot \| x - y \|_{C(\overline{\Omega},R^m)} .$$

Hence we deduce that the operator A satisfies a generalized Lipschitz condition with respect to the last three arguments, with the matrix $3\cdot mes(\Omega)\cdot Q \in M_{m\times m}(\boldsymbol{R}_+)$. From the condition (*iii*) it results that the operator A is a generalized contraction.

Now, the conditions of *the Perov's theorem* 1.3.4 being satisfied, it results that the integral equation with modified argument (2.25) has a unique solution in the space $C(\overline{\Omega},\boldsymbol{R}^m)$ and the proof is complete.

Theorem 2.5.4. *Suppose that the following conditions are met:*

(*i*) $K \in C(\overline{\Omega}\times\overline{\Omega}\times(J_1\times \ldots \times J_m)\times(J_1\times \ldots \times J_m)\times C(\partial\Omega,\boldsymbol{R}^m), \boldsymbol{R}^m)$, $f \in C(\overline{\Omega},\boldsymbol{R}^m)$, $g \in C(\overline{\Omega},\overline{\Omega})$, *where* $J_1, \ldots, J_m \subset \boldsymbol{R}$ *are closed and finite intervals* ;

(*ii*) *there exists* $Q \in M_{m\times m}(\boldsymbol{R}_+)$ *such that*

$$\| K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3) \|_C \le Q \left(\| u_1 - v_1 \|_C + \| u_2 - v_2 \|_C + \| u_3 - v_3 \|_{C(\partial\Omega,R^m)} \right)$$

for all $t, s \in \overline{\Omega}$, $u_1, u_2, v_1, v_2 \in \boldsymbol{R}^m$, $u_3, v_3 \in C(\partial\Omega,\boldsymbol{R}^m)$;

(*iii*) $3\cdot mes(\Omega)\cdot Q$ *is a matrix which converges to the null matrix*.

If $r \in M_{m\times 1}(\boldsymbol{R}_+)$ *is a matrix such that*

$$x \in \overline{B}(f;r) \ \Rightarrow\ x(t) \in J_1\times \ldots \times J_m , \tag{2.31}$$

and the following condition is met:

(*iv*) $mes(\Omega)\cdot M_K \le r$ (*condition of invariance of the sphere* $\overline{B}(f;r)$) ,

where we denote by $M_K = \begin{pmatrix} M_K^1 \\ \ldots \\ M_K^m \end{pmatrix} \in M_{m\times 1}(\boldsymbol{R}_+)$ *a matrix with positive constants as elements, such that the function K verifies the inequality*:

$$\| K(t,s,u,v,w) \|_C \le M_K \ , \quad \text{for all } t,s \in \overline{\Omega}\ ,\ u,v \in J_1 \times \ldots \times J_m,\ w \in C(\partial\Omega, \mathbf{R}^m)\ , \tag{2.32}$$

then the integral equation (2.25) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C(\overline{\Omega}, \mathbf{R}^m)$, *that can be obtained by the successive approximations method starting at any element from* $\overline{B}(f;r)$. *Moreover, if* x_0 *is the starting element and* x_k *is the k-th successive approximation, then the estimation* (2.30) *is satisfied.*

Proof. We consider the operator A: $\overline{B}(f;r) \to C(\overline{\Omega}, \mathbf{R}^m)$ defined by the relation (2.26). Using the condition (*i*) we deduce that the operator A is properly defined.

The condition (*iv*) together with the relation (2.29) assures us that the sphere $\overline{B}(f;r) \subset C(\overline{\Omega}, \mathbf{R}^m)$ is an invariant subset for the operator A, i.e. $\overline{B}(f;r) \in I(A)$. Now, we consider the operator A : $\overline{B}(f;r) \to \overline{B}(f;r)$, also, denoted by A and defined by the same relation (2.26); $\overline{B}(f;r)$ is a closed subset of the Banach space $C(\overline{\Omega}, \mathbf{R}^m)$.

The set of the solutions of integral equation (2.25), in the sphere $\overline{B}(f;r) \subset C(\overline{\Omega}, \mathbf{R}^m)$, coincides with the fixed points set of the operator A such defined.

From the conditions (*ii*) and (*iii*) we deduce that the operator A : $\overline{B}(f;r) \to \overline{B}(f;r)$, defined by the relation (2.26), is a contraction with the coefficient $3 \cdot L \cdot mes(\Omega)$.

The conditions of *the Contraction Principle* 1.3.1 being satisfied, it results that the integral equation (2.25) has a unique solution $x^* \in \overline{B}(f;r) \subset C(\overline{\Omega}, \mathbf{R}^m)$, and the proof is complete.

2.6 References

[1] Albu, M., A fixed point theorem of Maia-Perov type, *Studia Univ. Babeş-Bolyai, Mathematica*, 3(1978), 76-79

[2] Ambro, M., Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat, *Studia Univ. Babeş-Bolyai, Mathematica*, 2(1978), 26-32

[3] András, Sz., Fredholm-Volterra integral equations, *Pure Math. Appl.*, 13(2002)1:2, 21-30

[4] András, Sz., Fiber φ-contractions on generalized metric spaces and applications, *Mathematica*, 45(68)(2003):1, 3-8, Cluj-Napoca

[5] András, Sz., *Ecuaţii integrale Fredholm-Volterra*, Ed. Didactică şi Pedagogică, Bucureşti, 2005

[6] Coman, Gh., Rus, I., Pavel, G., Rus, I. A., *Introducere în teoria ecuaţiilor operatoriale*, Ed. Dacia, Cluj-Napoca, 1976

[7] Corduneanu, C., *Ecuaţii diferenţiale şi integrale*, Univ. Iaşi, 1971

[8] Corduneanu, C., *Integral Equations and Applications*, Cambridge University Press, 1991

[9] Dobriţoiu, M., The solution to a Fredholm implicit integral equation in the $\overline{B}(0;R)$ sphere, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM CV/2003, Nr.2162, , ISSN 0133-3526, 27-32

[10] Dobriţoiu, M., *The generalization of an integral equation*, Proc. of the 9th National Conf. of the Romanian Math. Soc., Ed. Universităţii de Vest Timisoara, 2005, Seria Alef, ISBN 973-7608-37-2, 392-396

[11] Dobriţoiu, M., Existence and continuous dependence on data of the solution of an integral equation, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM - CVI / 2005 , Nr.

ISSN 0133-3526

[12] Dobriţoiu, M., On an integral equation with modified argument, *Acta Univ. Apulensis, Mathematics-Informatics*, No.11/2006, ISSN 1582-5329, 387-391

[13] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. 51, nr. 1/2006, 81-94

[14] Dobriţoiu, M., A Fredholm-Volterra integral equation with modified argument, *Analele Univ. din Oradea, Fasc. Matematica*, tom XIII, 2006, ISSN 1221-1265, 133-138

[15] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[16] Dobriţoiu, M., System of integral equations with modified argument, *Carpathian Journal of Mathematics*, vol. 24 (2008), No. 2, 26-36

[17] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear Integral Equation in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996

[18] Hackbusch, W., *Integral equations*, Birkhäuser, Berlin, 1995

[19] Haimovici, A., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1965

[20] Ionescu, D. V., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti,1972

[21] Lalescu, T., *Introducere în teoria ecuaţiilor integrale*, Ed. Academiei, Bucureşti, 1956

[22] Marinescu, Gh., *Teoria ecuaţiilor diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1963

[23] Mirică, Şt., *Ecuaţii diferenţiale şi integrale*, vol. I, II, III, Ed. Universităţii Bucureşti, 1999

[24] Mureşan, V., *Ecuaţii diferenţiale cu modificarea afină a argumentului*, Transilvania Press, Cluj-Napoca, 1997

[25] Mureşan, V., *Functional-Integral Equations*, Ed. Mediamira, Cluj-Napoca, 2003

[26] Petrovskii, I. G., *Lectures on the theory of integral equations*, Graylock Press, Rochester, 1957

[27] Petruşel, A., Fredholm-Volterra integral equations and Maia's theorem, *Preprint* Nr.3, 1988, *Univ. Babeş-Bolyai*, 79-82

[28] Polyanin, A. D., Manzhirov, A. V., *Handbook of integral equations*, CRC Press, London, 1998

[29] Precup, R., Positive solutions of the initial value problem for an integral equation modelling infectious disease, în *"Seminarul de Ecuaţii Diferenţiale: Preprint* Nr.3, 1991", I. A. Rus ed., Univ. Babeş-Bolyai, Cluj-Napoca, 1991, 25-30

[30] Precup, R., *Ecuaţii integrale neliniare*, Univ. Babeş-Bolyai Cluj-Napoca, 1993

[31] Precup, R., Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle, *Studia Univ. Babeş-Bolyai, Mathematica*, 39, No.1 (1994), 47-58

[32] Precup, R., Existence and approximation of positive fixed points of nonexpansive maps, *Rev. Anal. Numer. Theor. Approx.*, 26(1997), 203-208

[33] Precup, R., Kirr, E., *Analysis of a nonlinear integral equation modelling infection diseases*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 178-195

[34] Precup, R., Some existence results for differential equations with both retarded and advanced arguments, *Mathematica*, Tome 44(67), Nr. 1, 2002, 31-38

[35] Precup, R., *Methods in nonlinear integral equations*, Kluwer Academic Publishers, 2002

[36] Ramalho, R., Existence and uniqueness theorems for nonlinear integral equation, *Univ. Federal de Pernambuco, Notas e Comun. de Matematica*, 40(1972), 1-42

[37] O'Regan, D., Petruşel, A., Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal. Appl.*, 2008

[38] Rus, I. A., On a fixed point theorem of Maia, *Studia Univ. Babeş-Bolyai, Mathematica*, 1(1977), 40-42

[39] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[40] Rus, I. A., *Metrical fixed point theorems*, University of Cluj-Napoca, 1979

[41] Rus, I. A., A delay integral equation from biomathematics, *Babeş-Bolyai Univ. of Cluj-Napoca*, *Preprint* Nr.3, 1989, 87-90

[42] Rus, I. A., Weakly Picard mappings, *Comment. Math. Univ. Caroline*, **34**, 3(1993), 769-773

[43] Rus, I. A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Casa de editură Transilvania Press, Cluj-Napoca, 1996

[44] Rus, I. A., Picard operators and applications, *Babeş-Bolyai Univ. of Cluj-Napoca, Preprint* Nr. 3, 1996

[45] Rus, I. A., *An abstract point of view for some integral equations from applied mathematics*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 256-270

[46] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334

[47] Rus, I. A., On a class of functional-integral equations, *Sem. On Best Approximation Theory*, Cluj-Napoca, 2000

[48] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001

[49] Rus, I. A., Picard operators and applications, *Scientiae Mathematicae Japonicae*, 58(2003):1, 191-219

[50] Rus, I. A., Mureşan, S., Mureşan, V., Weakly Picard operators on a set with two metrics, *Fixed Point Theory*, Vol. 6, No. 2, 2005, 323-331

[51] Rzepecki, B., On the existence of exactly one solution of integral equations in the space L^p weith a mixed norm, *Ann. Polon. Math*., XXX, 3, 1975

[52] Stuart, C. A., Existence theorems for a class of nonlinear integral equations, *Math. Z.* 137(1974), 49-64

[53] Şerban, M. A., Existence and uniqueness theorems for the Chandrasekhar's equation, Académie Roumaine Filiale de Cluj-Napoca*, Mathematica*, Tome **41**(64), Nr. 1, 1999, 91-103

[54] Şerban, M. A., Fiber φ-contractions, *Studia Univ. Babeş-Bolyai, Mathematica*, 34(1999), Nr.3, 99-108

[55] Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj-Napoca, 2002

[56] Şerban, M. A., Application of fiber Picard operators to integral equations, *Bul. Ştiinţific Univ. Baia Mare, Seria B, Matematică-Informatică*, Vol. XVIII(2002), Nr.1, 119-128

[57] Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl*., 13(2010), no. 2, 255-269

[58] Walter, W., *Differential and integral inequalities*, Springer, Berlin, 1979

[59] Yosida, K., *Équations différentielles et integrales*, DUNOD, Paris, 1971

[60] Yosida, K., *Lectures on differential and integral equations*, Interscience, 1960

3 Gronwall lemmas and comparison theorems

The integral inequalities have been studied using both classical theory and *the abstract Gronwall's lemma*.

From the basic treatises which have as their object of study the integral inequalities based on the classical theory, we mention: D. Bainov and P. Simeonov [3], D. Guo, V. Lakshmikantham and X. Liu [13], V. Lakshmikantham and S. Leela [14], D. S. Mitrinović, J. E. Pečarić and A. M. Fink [16], B. G. Pachpatte [19], R. Precup [21], J. Schröder [36], W. Walter [40] and from the basic treatises, having the integral inequalities like theme, obtained using *the abstract Gronwall's lemma*, we mention: Sz. András [2], Gh. Coman, I. Rus, G. Pavel and I. A. Rus [6], S. S. Dragomir [12], V. Mureşan [18], I. A. Rus [22], [23], [27], [32], M. A. Şerban [37].

Also, we mention some of the articles that contain Gronwall type integral inequalities: Sz. András [1], P. R. Beesak [4], A. Buică [5], A. Constantin [7], C. Crăciun [8], N. Lungu and I. A. Rus [15], I. A. Rus [24], [25], [26], [28], [29], [30], [31], [33], [34], [35], V. Mureşan [17], A. Petruşel and I. A. Rus [20], M. A. Şerban [38], M. A. Şerban, I. A. Rus and A. Petruşel [39], M. Dobriţoiu [9], [11], M. Dobriţoiu, I. A. Rus and M. A. Şerban [10], M. Zima [41].

In this chapter, divided into three paragraphs, we use the Picard operators technique for integral equations, *the abstract Gronwall's lemma* 1.4.1 and *the abstract comparison lemma* 1.4.5 to establish some comparison theorems and integral inequalities concerning the solution of the integral equation with modified argument (2.1). These results are given in the first two paragraphs of this chapter.

Note that in establishing these integral inequalities and comparison theorems, some results from the following treatises, were useful: Gh. Coman, I. Rus, G. Pavel and I. A. Rus [6], I. A. Rus [22], [27], [32], M. A. Şerban [37], Sz. András [2]. Also, the results given by I. A.Rus in the paper [33] were usefull.

In the third paragraph three applications of the theorems 3.1.1, 3.1.2 and 3.2.2, established in the first two paragraphs, are given.

The results presented in this chapter have been obtained by the author, and published in [9] and [11]. We present them below.

3.1 Gronwall lemmas

Let $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ be an ordered Banach space. We consider the integral equation with modified argument (2.1)

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t)$$

where $K : [a,b]\times[a,b]\times \boldsymbol{B}^4 \to \boldsymbol{B}$, $f : [a,b] \to \boldsymbol{B}$, $g : [a,b] \to [a,b]$.

Theorem 3.1.1. *Suppose that the following conditions are met*:

(*i*) $K\in C([a,b]\times[a,b]\times\boldsymbol{B}^4, \boldsymbol{B})$, $f\in C([a,b],\boldsymbol{B})$, $g\in C([a,b], [a,b])$;

(*ii*) $K(t, s, \cdot, \cdot, \cdot, \cdot)$ *is increasing for all* $t, s\in [a,b]$;

(*iii*) *there exists* $L_K > 0$ *such that*

$$|K(t, s, u_1, u_2, u_3, u_4) - K(t, s, v_1, v_2, v_3, v_4)| \le$$
$$\le L_K (|u_1-v_1| + |u_2-v_2)| + |u_3-v_3)| + |u_4-v_4)|),$$

for all $t, s\in [a,b]$, $u_i, v_i\in \boldsymbol{B}$, $i = \overline{1,4}$;

(*iv*) $4L_k (b - a) < 1$

and let $x^*\in C([a,b],\boldsymbol{B})$ *be the unique solution of the integral equation with modified argument* (2.1). *Under these conditions it results that*:

(*a*) *if* $x\in C([a,b],\boldsymbol{B})$ *is a lower-solution of the integral equation* (2.1)*, then* $x \le x^*$.

(*b*) *if* $x\in C([a,b],\boldsymbol{B})$ *is an upper-solution of the integral equation* (2.1)*, then* $x \ge x^*$.

Proof. We consider the operator $A : C([a,b],\boldsymbol{B}) \to C([a,b],\boldsymbol{B})$ defined by the relation (2.3)

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in [a,b].$$

The conditions of the theorem 2.1.1 being satisfied (hypothesis (*i*), (*iii*) and (*iv*)), it results that the integral equation (2.1) has a unique solution in the space $C([a,b],\boldsymbol{B})$, that we denote by x^*.

From the condition (*ii*) it results that the operator A is increasing.

Now, we apply *the abstract Gronwall's lemma* 1.4.1 and it results that the following two implications are true:

$$x \le A(x) \quad \Rightarrow \quad x \le x^*$$

$$x \ge A(x) \quad \Rightarrow \quad x \ge x^*$$

or explicitly

$$x \le A(x) \quad \Rightarrow \quad x(t) \le \int_a^b K(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b))ds + f(t), \quad t\in [a,b]$$

$$x \ge A(x) \quad \Rightarrow \quad x(t) \ge \int_a^b K(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b))ds + f(t), \quad t\in [a,b].$$

and the proof is complete.

Remark 3.1.1. The theorem 3.1.1 remains true in the particular cases $\boldsymbol{B} = \boldsymbol{R}$, $\boldsymbol{B} = \boldsymbol{R}^m$ and $\boldsymbol{B} = l^2(\boldsymbol{R})$, if we replace the conditions (*i*), (*iii*) and (*iv*) with certain conditions that ensure the existence and uniqueness of the solution of the integral equation with modified argument (2.1) in the spaces $C[a,b]$, $C([a,b],\boldsymbol{R}^m)$ and $C([a,b],l^2(\boldsymbol{R}))$ respectively. We present these results in the cases $\boldsymbol{B} = \boldsymbol{R}^m$ and $\boldsymbol{B} = l^2(\boldsymbol{R})$.

In the particular case $\boldsymbol{B} = \boldsymbol{R}^m$, for the system of integral equations with modified argument (2.6)

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in [a,b] ,$$

or

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \end{cases}, \quad t\in[a,b],$$

where $x : [a,b] \to \boldsymbol{R}^m$, $K : [a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m \to \boldsymbol{R}^m$, $g : [a,b] \to [a,b]$ and $f : [a,b] \to \boldsymbol{R}^m$, we have:

Theorem 3.1.2. *Suppose that the following conditions are met*:

(*i'*) $K\in C([a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m, \boldsymbol{R}^m)$, $f\in C([a,b], \boldsymbol{R}^m)$, $g\in C([a,b], [a,b])$;

(*ii*) $K(t, s, \cdot, \cdot, \cdot, \cdot)$ *is increasing for all* $t, s\in[a,b]$;

(*iii'*) *there exists* $Q\in M_{m\times m}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_C \le$$
$$\le Q\cdot\left(\|u_1-v_1\|_C + \|u_2-v_2\|_C + \|u_3-v_3\|_C + \|u_4-v_4\|_C\right)$$

for all $t, s\in[a,b]$, $u_i, v_i\in\boldsymbol{R}^m$, $i=\overline{1,4}$;

(*iv'*) $[4(b-a)Q]^n \to 0$ *as* $n\to\infty$

and let $x^*\in C([a,b],\boldsymbol{R}^m)$ *be the unique solution of the system of integral equations with modified argument* (2.6).

Under these conditions it results that:

(*a*) *if* $x\in C([a,b],\boldsymbol{R}^m)$ *is a lower solution of the system of integral equations* (2.6)*, then* $x \le x^*$, i.e.

$$x(t) \le \int_a^b K(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b))ds + f(t).$$

(*b*) *if* $x\in C([a,b],\boldsymbol{R}^m)$ *is an upper solution of the system of integral equations* (2.6)*, then* $x \ge x^*$, i.e.

$$x(t) \ge \int_a^b K(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b))ds + f(t).$$

In the particular case $\boldsymbol{B} = l^2(\boldsymbol{R})$, for the Fredholm integral equation with modified argument (2.12) (or (2.12'))

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[a,b],$$

or

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \text{...} \qquad , \quad t\in[a,b] , \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \\ \text{...} \end{cases}$$

where $x : [a,b] \to l^2(\boldsymbol{R})$, $K : [a,b]\times[a,b]\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R}) \to l^2(\boldsymbol{R})$, $g : [a,b]\to[a,b]$ and $f : [a,b] \to l^2(\boldsymbol{R})$, we have:

Theorem 3.1.3. *Suppose that the following conditions are met*:

(*i'*) $K\in C([a,b]\times[a,b]\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R})\times l^2(\boldsymbol{R}), l^2(\boldsymbol{R}))$, $f\in C([a,b],l^2(\boldsymbol{R}))$, $g\in C([a,b],[a,b])$;

(*ii*) $K(t, s, \cdot, \cdot, \cdot, \cdot)$ *is increasing for all* $t, s\in[a,b]$;

(*iii'*) *there exists* $L_K > 0$, *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)\|_{l^2(R)} \le$$

$$\le L_K\left(\|u_1 - v_1\|_{l^2(R)} + \|u_2 - v_2\|_{l^2(R)} + \|u_3 - v_3\|_{l^2(R)} + \|u_4 - v_4\|_{l^2(R)}\right)$$

for all $t, s\in[a,b]$, $u_j, v_j\in l^2(\boldsymbol{R})$, $j = \overline{1,4}$;

(*iv'*) $4L_K(b-a) < 1$

and let $x^*\in C([a,b],l^2(\boldsymbol{R}))$ *be the unique solution of the integral equation with modified argument* (2.12). *Under these conditions it results that*:

(*a*) *if* $x\in C([a,b],l^2(\boldsymbol{R}))$ *is a lower-solution of the integral equation* (2.12), *then* $x \le x^*$.

(*b*) *if* $x\in C([a,b],l^2(\boldsymbol{R}))$ *is an upper-solution of the integral equation* (2.12), *then* $x \ge x^*$.

Next, we apply the theorem 1.4.7, given by Sz. András in [2], to the operator defined by using the integral equation (2.1) and we establish an integral inequality.

To ensure the conditions for the applicability of theorem 1.4.7 (Sz. András [2]), we assume that the functions $K : [a,b]\times[a,b]\times \boldsymbol{J}^4\to \boldsymbol{R}_+$, $J \subset \boldsymbol{R}$ is a compact interval, $f : [a,b] \to \boldsymbol{R}_+$ and $g : [a,b]\to[a,b]$, are continuous.

Then we obtain the following theorem.

Theorem 3.1.4. *If the following conditions are met*:

(*i*) $K\in C([a,b]\times[a,b]\times J^4, \boldsymbol{R}_+)$, $J\subset\boldsymbol{R}$ *is a compact interval*, $f\in C([a,b],\boldsymbol{R}_+)$, $g\in C([a,b],[a,b])$;

(*ii*) $M_K (b-a) \le r$, *where* M_K *is a positive constant, such that for the restriction* $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \boldsymbol{R}$ *compact interval, we have*:

$$|K(t,s,u_1,u_2,u_3,u_4)| \le M_K, \quad \text{for all } t,s\in[a,b],\ u_1,u_2,u_3,u_4\in J; \tag{3.1}$$

(iii) *there exists* $L_K > 0$ *such that*

$$|K(t, s, u_1, u_2, u_3, u_4) - K(t, s, v_1, v_2, v_3, v_4)| \leq$$

$$\leq L_K (|u_1 - v_1| + |u_2 - v_2)| + |u_3 - v_3)| + |u_4 - v_4)|),$$

for all $t, s \in [a,b]$, $u_i, v_i \in J$, $i = \overline{1,4}$;

(iv) $4L_K(b-a) < 1$,

then the following inequality

$$x(t) \leq r + M_f + \beta \| A(x) - x \|_{C[a,b]}, \quad t \in [a,b], \ \beta \in (0,1),$$

implies

$$x(t) \leq x^*(t), \quad \text{for all } t \in [a,b],$$

where x^* *is the unique solution of the integral equation* (2.1)), *in the sphere* $\overline{B}(f;r) \subset C[a,b]$, *and* M_f *is a positive constant such that*

$$|f(t)| \leq M_f, \quad t \in [a,b]. \tag{3.2}$$

Proof. We consider the operator $A: \overline{B}(f;r) \to \overline{B}(f;r)$, $\overline{B}(f;r) \subset C[a,b]$, defined by the relation:

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [a,b].$$

From the conditions (i), (ii), (iii), (iv) and since the functions K and f are positive, it results that the operator A is an increasing Picard operator and we have:

$$\alpha A(x)(t) + \beta A^2(x)(t) = (1-\beta)A(x)(t) + \beta A^2(x)(t) =$$

$$= A(x)(t) + \beta\left[A^2(x)(t) - A(x)(t)\right] = A(x)(t) + \beta\left[A(A(x))(t) - A(x)(t)\right] =$$

$$= \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) +$$

$$+ \beta\left[\int_a^b K(t,s,A(x)(s),A(x)(g(s)),A(x)(a),A(x)(b))ds + f(t) -\right.$$

$$\left. - \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds - f(t)\right] =$$

$$= \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) +$$

$$+ \beta\int_a^b \left[K(t,s,A(x)(s),A(x)(g(s)),A(x)(a),A(x)(b)) - K(t,s,x(s),x(g(s)),x(a),x(b))\right]ds \leq$$

$$\leq \int_a^b \left|K(t,s,x(s),x(g(s)),x(a),x(b))\right|ds + |f(t)| +$$

$$+\beta\int_a^b \left| K(t,s,A(x)(s),A(x)(g(s)),A(x)(a),A(x)(b)) - K(t,s,x(s),x(g(s)),x(a),x(b)) \right| ds \le$$

where $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$. According to relations (3.1), (3.2) and to condition (*iii*) it results that

$$\alpha A(x)(t) + \beta A^2(x)(t) \le M_K(b-a) + M_f + \beta\int_a^b L_K\Big(\left|A(x)(s)-x(s)\right| +$$

$$+\left|A(x)(g(s))-x(g(s))\right| + \left|A(x)(a)-x(a)\right| + \left|A(x)(b)-x(b)\right|\Big)ds\,.$$

From the condition (*ii*) of invariance of the sphere $\overline{B}(f;r) \subset C[a,b]$ and using *the Chebyshev norm* in the right side, we obtain:

$$\alpha A(x)(t) + \beta A^2(x)(t) \le r + M_f + 4L_K(b-a)\beta\left\|A(x)-x\right\|_{C[a,b]}\,,$$

and according to condition (*iv*) of contraction of the operator *A*, it results that

$$\alpha A(x)(t) + \beta A^2(x)(t) \le r + M_f + \beta\left\|A(x)-x\right\|_{C[a,b]}\,.$$

Now, applying the theorem 1.4.7 (Sz. Andras [2]) it results that $x(t) \le x^*(t)$, for any $t \in [a,b]$ and the proof is complete.

3.2 Comparison theorems

Consider the integral equation with modified argument (2.1) corresponding to functions K_i and f_i, $i = 1, 2$

$$x(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t)\,, \quad t \in [a,b] \tag{3.3}$$

and

$$x(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t)\,, \quad t \in [a,b]\,, \tag{3.4}$$

where $K_1, K_2 : [a,b]\times[a,b]\times \boldsymbol{B}^4 \to \boldsymbol{B}$, $f_1, f_2 : [a,b] \to \boldsymbol{B}$, $g : [a,b] \to [a,b]$.

Theorem 3.2.1. *If the following conditions are met*:

(*i*) $K_i \in C([a,b]\times[a,b]\times\boldsymbol{B}^4, \boldsymbol{B})$, $f_i \in C([a,b],\boldsymbol{B})$, $i = 1, 2$ *and* $g \in C([a,b], [a,b])$;

(*ii*) $u_1 \le v_1$, $u_2 \le v_2$, $u_3 \le v_3$, $u_4 \le v_4$ $\Rightarrow$ $K_1(t,s,u_1,u_2,u_3,u_4) \le K_2(t,s,v_1,v_2,v_3,v_4)$;

(*iii*) *there exists* $L_i > 0$, $i = 1,2$ *such that*

$$\left|K_i(t, s, u_1, u_2, u_3, u_4) - K_i(t, s, v_1, v_2, v_3, v_4)\right| \le$$
$$\le L_i\left(\left|u_1 - v_1\right| + \left|u_2 - v_2\right| + \left|u_3 - v_3\right| + \left|u_4 - v_4\right|\right),$$

for all $t, s \in [a,b]$, $u_j, v_j \in \boldsymbol{B}$, $j = \overline{1,4}$;

(*iv*) $4L_i(b-a) < 1$, $i = 1, 2$

and denote by x^*, $\overline{x^*}$ *respectively, the unique solution of the integral equation* (3.3), *and* (3.4) respectively, *then the following inequality is true:*

$$x^*(t) \le \overline{x^*}(t), \quad \textit{for all } t \in [a,b].$$

Proof. We consider the operators $A_i : C([a,b],\boldsymbol{B}) \to C([a,b],\boldsymbol{B})$, $i = 1,2$, defined by the relations:

$$A_i(x)(t) := \int_a^b K_i(t,s,x(s),x(g(s)),x(a),x(b))ds + f_i(t), \quad t \in [a,b],\ i = 1,2$$

and the functions x_0 and $\overline{x}_0 \in C([a,b],\boldsymbol{B})$, $x_0(t) \le \overline{x}_0(t)$, for any $t \in [a,b]$.

The successive approximations sequences corresponding to the operators A_1 and A_2 will be

$$x_{n+1} = A_1(x_n), \quad \overline{x}_{n+1} = A_2(\overline{x}_n), \text{ for } n \in \boldsymbol{N}.$$

From the conditions (*i*), (*iii*) and (*iv*) it results that the sequences $(x_n)_{n\in N}$ and $(\overline{x}_n)_{n\in N}$ respectively, converge to x^* and to $\overline{x^*}$ respectively, and from the condition (*ii*) it results that the following inequality is true:

$$x_n(t) \le \overline{x}_n(t), \quad \text{for all } t \in [a,b] \text{ and } n \in \boldsymbol{N}.$$

The inequality from the conclusion of the theorem is obtained when $n \to \infty$.
The proof is complete.

Now, we consider the integral equation with modified argument (2.1), corresponding to the functions K_i and f_i, $i = 1, 2, 3$

$$x(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t), \quad t \in [a,b], \tag{3.5}$$

$$x(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t), \quad t \in [a,b], \tag{3.6}$$

$$x(t) = \int_a^b K_3(t,s,x(s),x(g(s)),x(a),x(b))ds + f_3(t), \quad t \in [a,b]. \tag{3.7}$$

Theorem 3.2.2. *Suppose that the functions* K_i, f_i, $i = 1, 2, 3$ *and* g *satisfy the following conditions:*

(*i*) $K_i \in C([a,b]\times[a,b]\times\boldsymbol{B}^4, \boldsymbol{B})$, $f_i \in C([a,b],\boldsymbol{B})$, $i = 1, 2, 3$ *and* $g \in C([a,b], [a,b])$;

(*ii*) $K_2(t, s, \cdot, \cdot, \cdot, \cdot)$ *is increasing for all* $t, s \in [a,b]$;

(*iii*) $K_1 \le K_2 \le K_3$ *and* $f_1 \le f_2 \le f_3$;

(*iv*) *there exists* $L_i > 0$, $i = 1, 2, 3$ *such that*

$$|K_i(t, s, u_1, u_2, u_3, u_4) - K_i(t, s, v_1, v_2, v_3, v_4)| \le$$
$$\le L_i\,(|u_1 - v_1| + |u_2 - v_2)| + |u_3 - v_3)| + |u_4 - v_4)|),$$

for all $t, s \in [a,b]$, $u_j, v_j \in \boldsymbol{B}$, $j = \overline{1,4}$;

(*v*) $4L_i\,(b - a) < 1$, $i = 1, 2, 3$.

If we denote by x_1^*, x_2^* and x_3^* *respectively, the unique solution of the integral equation* (3.5), (3.6) *and* (3.7) *respectively, then*

$$x_1^* \leq x_2^* \leq x_3^* .$$

Proof. We consider the operators $A_i : C([a,b],\boldsymbol{B}) \to C([a,b],\boldsymbol{B})$, $i = 1, 2, 3$, defined by the relations:

$$A_i(x)(t) := \int_a^b K_i(t,s,x(s),x(g(s)),x(a),x(b))ds + f_i(t), \quad t \in [a,b],\ i = 1, 2, 3.$$

From the conditions (*i*) (*iv*) and (*v*) it results that the operators A_i are α_i–contractions with the coefficients $\alpha_i = 4L_i \cdot (b-a)$, i=1, 2, 3 and therefore they are Picard operators. According to *the Contraction Principle* 1.3.1, it results that every of the integral equations (3.5), (3.6) and (3.7) has a unique solution in the space $C([a,b],\boldsymbol{B})$ and we denote these solutions by x_i^*, $i = 1, 2, 3$.

From the conditions (*ii*) it results that A_2 is an increasing operator, and from the condition (*iii*) it results that $A_1 \leq A_2 \leq A_3$.

The conditions of *the abstract comparison lemma* 1.4.5 being satisfied, it results that the following implication is true:

$$x_1 \leq x_2 \leq x_3 \quad \Rightarrow \quad A_1^\infty(x_1) \leq A_2^\infty(x_2) \leq A_3^\infty(x_3) ,$$

and A_1 , A_2 , A_3 are Picard operators and according to the remark 1.4.2 we obtain

$$x_1^* \leq x_2^* \leq x_3^*$$

and the proof is complete.

3.3 Examples

In this paragraph we give three examples: two integral equations with modified argument and a system of integral equations with modified argument, and we will verify the conditions of some Gronwall type lemmas and comparison theorems respectively, of the first two paragraphs.

Example 3.3.1. We consider the integral equation with modified argument (the case $\boldsymbol{B} = \boldsymbol{R}$)

$$x(t) = \int_0^1 \left[\frac{t}{7}x(s) + \frac{s}{5}x\left(\frac{s+1}{2}\right) + \frac{1}{7}x(0) + \frac{1}{5}x(1) \right] ds + \frac{13}{14}t - \frac{17}{60}, \quad t \in [0,1] \tag{3.8}$$

where $K \in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$, $K(t,s,u_1,u_2,u_3,u_4) = \frac{t}{7}u_1 + \frac{s}{5}u_2 + \frac{1}{7}u_3 + \frac{1}{5}u_4$,

$f \in C[0,1]$, $f(t) = \frac{13}{14}t - \frac{17}{60}$, $g \in C([0,1],[0,1])$, $g(s) = \frac{s+1}{2}$ and $x \in C[0,1]$.

The solution of this integral equation is $x^*(t) = t$, $t \in [0,1]$.

We attach to this equation, the operator $A : C[0,1] \to C[0,1]$, defined by the relation:

$$A(x)(t)=\int_0^1\left[\frac{t}{7}x(s)+\frac{s}{5}x\left(\frac{s+1}{2}\right)+\frac{1}{7}x(0)+\frac{1}{5}x(1)\right]ds+\frac{13}{14}t-\frac{17}{60},\ t\in[0,1]. \tag{3.9}$$

The set of the solutions of integral equation (3.8), in the space $C[0,1]$, coincides with the fixed points set of the operator A, defined above.

Since the function K satisfies the Lipschitz condition with the constant $\frac{1}{7}$ relative to the third and the fifth argument respectively, and with the constant $\frac{1}{5}$ relative to the fourth and the sixth argument respectively, it results that the operator A is a contraction with the coefficient $\alpha=\frac{24}{35}$ and therefore A is a Picard operator.

According to theorem 2.1.1, in the particular case $\boldsymbol{B}=\boldsymbol{R}$, it results that the integral equation (3.8) has a unique solution $x^*\in C[0,1]$. This solution is $x^*(t)=t,\ t\in[0,1]$.

Since the function $K(t, s, \cdot, \cdot, \cdot, \cdot)$ is increasing for any $t, s\in[a,b]$, it results that the conditions of the theorem 3.1.1 are met (the case $\boldsymbol{B}=\boldsymbol{R}$) and the following integral inequalities are true:

- if $x\in C[0,1]$ is a lower-solution of the integral equation (3.8), then

$$x(t)\le\int_0^1\left[\frac{t}{7}x^*(s)+\frac{s}{5}x^*\left(\frac{s+1}{2}\right)+\frac{1}{7}x^*(0)+\frac{1}{5}x^*(1)\right]ds+\frac{13}{14}t-\frac{17}{60},\quad t\in[0,1];$$

- if $x\in C[0,1]$ is an upper-solution of the integral equation (3.8), then

$$x(t)\ge\int_0^1\left[\frac{t}{7}x^*(s)+\frac{s}{5}x^*\left(\frac{s+1}{2}\right)+\frac{1}{7}x^*(0)+\frac{1}{5}x^*(1)\right]ds+\frac{13}{14}t-\frac{17}{60},\quad t\in[0,1].$$

Example 3.3.2. We consider the system of integral equations with modified argument (the case $\boldsymbol{B}=\boldsymbol{R}^2$)

$$\begin{cases} x_1(t)=\int_0^1\left[\frac{t+2}{15}x_1(s)+\frac{2t+1}{15}x_1(s/2)+\frac{1}{5}x_1(0)+\frac{1}{5}x_1(1)\right]ds+2t+1 \\ x_2(t)=\int_0^1\left[\frac{t+2}{21}x_2(s)+\frac{2t+1}{21}x_2(s/2)+\frac{1}{7}x_2(0)+\frac{1}{7}x_2(1)\right]ds+t \end{cases} \tag{3.10}$$

where $K\in C([0,1]\times[0,1]\times\boldsymbol{R}^2\times\boldsymbol{R}^2\times\boldsymbol{R}^2\times\boldsymbol{R}^2, \boldsymbol{R}^2)$,

$$K(t,s,u_1,u_2,u_3,u_4)=\left(K_1(t,s,u_1,u_2,u_3,u_4),\ K_2(t,s,u_1,u_2,u_3,u_4)\right),$$

$$K_1(t,s,u_1,u_2,u_3,u_4)=\frac{t+2}{15}u_{11}+\frac{2t+1}{15}u_{21}+\frac{1}{5}u_{31}+\frac{1}{5}u_{41},$$

$$K_2(t,s,u_1,u_2,u_3,u_4)=\frac{t+2}{21}u_{12}+\frac{2t+1}{21}u_{22}+\frac{1}{7}u_{32}+\frac{1}{7}u_{42},$$

$f\in C([0,1],\boldsymbol{R}^2)$, $f(t)=(f_1(t), f_2(t))$, $f_1(t)=2t+1$, $f_2(t)=t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$ and $x \in C([0,1],\boldsymbol{R}^2)$.

We attach to this system of integral equations the operator $A : C([0,1],\boldsymbol{R}^2) \to C([0,1],\boldsymbol{R}^2)$, defined by the relation:

$$\begin{cases} A_1(x)(t) = \int_0^1 \left[\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right] ds + 2t + 1 \\ \\ A_2(x)(t) = \int_0^1 \left[\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right] ds + t\,. \end{cases} \tag{3.11}$$

The set of the solutions of the system of integral equations (3.10), in the space $C([0,1],\boldsymbol{R}^2)$, coincides with the fixed points set of the operator A, i.e. with F_A .

The operator A satisfies a generalized Lipschitz condition with the matrix

$$Q = \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix}, \; Q \in M_{2\times 2}(\boldsymbol{R}_+),$$

which according to theorem 1.3.3, converges to zero and therefore, it results that the operator A is a generalized contraction with the matrix Q.

Now, the conditions of the theorem 2.2.2 being satisfied it results that the system of integral equations (3.10) has a unique solution $x^* \in C([0,1],\boldsymbol{R}^2)$.

Since the function $K(t,s,\cdot,\cdot,\cdot)$ is increasing for any t, $s \in [0,1]$, it results that the conditions of the theorem 3.1.2 are met and the following integral inequalities are true:

– if $x \in C([0,1],\boldsymbol{R}^2)$ is a lower solution of the system of integral equations (3.10), then

$$\begin{pmatrix} x_1(t) \\ \\ x_2(t) \end{pmatrix} \le \begin{pmatrix} \int_0^1 \left[\frac{t+2}{15} x_1^*(s) + \frac{2t+1}{15} x_1^*(s/2) + \frac{1}{5} x_1^*(0) + \frac{1}{5} x_1^*(1) \right] ds + 2t + 1 \\ \\ \int_0^1 \left[\frac{t+2}{21} x_2^*(s) + \frac{2t+1}{21} x_2^*(s/2) + \frac{1}{7} x_2^*(0) + \frac{1}{7} x_2^*(1) \right] ds + t \end{pmatrix};$$

– if $x \in C([0,1],\boldsymbol{R}^2)$ is an upper-solution of the system of integral equations (3.10), then

$$\begin{pmatrix} x_1(t) \\ \\ x_2(t) \end{pmatrix} \ge \begin{pmatrix} \int_0^1 \left[\frac{t+2}{15} x_1^*(s) + \frac{2t+1}{15} x_1^*(s/2) + \frac{1}{5} x_1^*(0) + \frac{1}{5} x_1^*(1) \right] ds + 2t + 1 \\ \\ \int_0^1 \left[\frac{t+2}{21} x_2^*(s) + \frac{2t+1}{21} x_2^*(s/2) + \frac{1}{7} x_2^*(0) + \frac{1}{7} x_2^*(1) \right] ds + t \end{pmatrix}.$$

Example 3.3.3. We consider the integral equations with modified argument (the case $\boldsymbol{B} = \boldsymbol{R}$)

$$x(t) = \int_0^1 \left[\frac{t}{8} x(s) + \frac{t}{8} x\left(\frac{s+1}{2}\right) - \frac{1}{5} x(0) - \frac{1}{5} x(1) \right] ds + \frac{27}{32} t + \frac{1}{5}, \quad t \in [0,1], \tag{3.12}$$

$$x(t)=\int_0^1\left[\frac{1}{6}x(s)+\frac{t}{6}x\left(\frac{s+1}{2}\right)-\frac{1}{7}x(0)-\frac{1}{7}x(1)\right]ds+\frac{17}{24}t+\frac{33}{28},\quad t\in[0,1], \tag{3.13}$$

$$x(t)=\int_0^1\left[\frac{1}{4}x(s)+\frac{1}{4}x\left(\frac{s+1}{2}\right)-\frac{1}{9}x(0)-\frac{1}{9}x(1)\right]ds+t+\frac{179}{144},\quad t\in[0,1], \tag{3.14}$$

where K_1, K_2, $K_3 \in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$,

$$K_1(t,s,u_1,u_2,u_3,u_4)=\frac{t}{8}u_1+\frac{t}{8}u_2-\frac{1}{5}u_3-\frac{1}{5}u_4\ ,$$

$$K_2(t,s,u_1,u_2,u_3,u_4)=\frac{1}{6}u_1+\frac{t}{6}u_2-\frac{1}{7}u_3-\frac{1}{7}u_4\ ,$$

$$K_3(t,s,u_1,u_2,u_3,u_4)=\frac{1}{4}u_1+\frac{1}{4}u_2-\frac{1}{9}u_3-\frac{1}{9}u_4,$$

f_1, f_2, $f_3 \in C[0,1]$, $\quad f_1(t)=\dfrac{27}{32}t+\dfrac{1}{5}$, $\quad f_2(t)=\dfrac{17}{24}t+\dfrac{33}{28}$, $\quad f_3(t)=t+\dfrac{179}{144}$,

$g\in C([0,1],[0,1])$, $g(s)=\dfrac{s+1}{2}$ and $x\in C[0,1]$.

The solutions of the integral equations (3.12), (3.13) and (3.14) respectively, are $x_1^*(t)=t$, $x_2^*(t)=t+1$ and $x_3^*(t)=t+2$ respectively, for $t\in[0,1]$.

We attach to these integral equations, the operators A_1, A_2, A_3 : $C[0,1]\to C[0,1]$, defined by the relations:

$$A_1(x)(t)=\int_0^1\left[\frac{t}{8}x(s)+\frac{t}{8}x\left(\frac{s+1}{2}\right)-\frac{1}{5}x(0)-\frac{1}{5}x(1)\right]ds+\frac{27}{32}t+\frac{1}{5},\quad t\in[0,1], \tag{3.15}$$

$$A_2(x)(t)=\int_0^1\left[\frac{1}{6}x(s)+\frac{t}{6}x\left(\frac{s+1}{2}\right)-\frac{1}{7}x(0)-\frac{1}{7}x(1)\right]ds+\frac{17}{24}t+\frac{33}{28},\quad t\in[0,1], \tag{3.16}$$

$$A_3(x)(t)=\int_0^1\left[\frac{1}{4}x(s)+\frac{1}{4}x\left(\frac{s+1}{2}\right)-\frac{1}{9}x(0)-\frac{1}{9}x(1)\right]ds+t+\frac{179}{144},\quad t\in[0,1]. \tag{3.17}$$

The function K_1 satisfies the Lipschitz condition with the constant $\frac{1}{8}$ relative to the third and the fourth argument respectively, and with the constant $\frac{1}{5}$ relative to the fifth and the sixth argument respectively, and therefore the operator A_1 is a contraction with the coefficient $\alpha_1=\frac{13}{20}$.

The function K_2 satisfies the Lipschitz condition with the constant $\frac{1}{6}$ relative to the third and the fourth argument respectively, and with the constant $\frac{1}{7}$ relative to the fifth and the sixth argument respectively, and therefore the operator A_2 is a contraction with the coefficient $\alpha_2 = \frac{13}{21}$.

The function K_3 satisfies the Lipschitz condition with the constant $\frac{1}{4}$ relative to the third and the fourth argument respectively, and with the constant $\frac{1}{9}$ relative to the fifth and the sixth argument respectively, and therefore the operator A_3 is a contraction with the coefficient $\alpha_3 = \frac{13}{18}$.

Therefore, A_1 , A_2 , A_3 are Picard operators. According to theorem 2.1.1, in the particular case $\boldsymbol{B} = \boldsymbol{R}$, it results that the integral equations (3.12), (3.13) and (3.14) have the unique solutions x_1^*, x_2^* and $x_3^* \in C[0,1]$ respectively, and therefore the solutions of these equations are

$$x_1^*(t) = t\,, \quad x_2^*(t) = t+1 \quad \text{and} \quad x_3^*(t) = t+2 \ \text{ respectively, } \ t \in [0,1].$$

Since the function $K_2(t, s, \cdot, \cdot, \cdot, \cdot)$ is increasing for any $t, s \in [0,1]$, it results that A_2 is an increasing operator.

Also, between the functions K_1 , K_2 , K_3 and f_1 , f_2 , f_3 respectively, there are the relations

$$K_1 \le K_2 \le K_3 \quad \text{and} \quad f_1 \le f_2 \le f_3 \ \text{ respectively.}$$

Since the conditions of the theorem 3.2.2. are met, it results that

$$x_1^* \le x_2^* \le x_3^*\,,$$

what is observed, also, in

$$t \le t+1 \le t+2\,, \quad \text{for all } t \in [0,1]\,.$$

3.4 References

[1] András, Sz., Gronwall type inequalities via subconvex sequences, *Seminar on Fixed Point Theory*, 3(2002), 183-188

[2] András, Sz., *Ecuații integrale Fredholm-Volterra*, Ed. Didactică şi Pedagogică, Bucureşti, 2005

[3] Bainov, D., Simeonov, P., *Integral inequalities and applications*, Kluwer, Dordrecht, 1992

[4] Beesak, P. R., Gronwall inequalities, *Carleton Math. Lecture Notes*, 1975, Nr. 11

[5] Buică, A., Gronwall-type nonlinear integral inequalities, *Mathematica*, 44(2002), Nr. 1, 19-23

[6] Coman, Gh., Rus, I., Pavel, G., Rus, I. A., *Introducere în teoria ecuațiilor operatoriale*, Ed. Dacia, Cluj-Napoca, 1976

[7] Constantin, A., A Gronwall-like inequality and its applications, *Rend. Mat. Acc. Lincei* 1(1990), 111-115

[8] Crăciun, C., On some Gronwall inequalities, *Seminar on Fixed Point Theory*, 1 (2000), 31-34

[9] Dobriţoiu, M., Properties of the solution of an integral equation with modified argument, *Carpathian Journal of Mathematics*, 23(2007), No. 1-2, 77-80

[10] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[11] Dobriţoiu, M., *Gronwall-type lemmas for an integral equation with modified argument*, Proc. of the Int. Conf. on Theory and Applications of Mathematics and Informatics, Alba Iulia, 2007

[12] Dragomir, S. S., The Gronwall type lemmas and applications, *Monografii Matematice*, No. 29, Univ. Timişoara, 1987

[13] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear Integral Equation in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996

[14] Lakshmikantham, V., Leela, S., *Differential and Integral Inequalities*, Acad. Press, New York, 1969

[15] Lungu, N., Rus, I. A., Gronwall inequalities via Picard operators, *Analele Ştiinţifice ale Univ. "Al.I. Cuza" din Iaşi, Matematica*, Tomul LVIII, 2012, f. 2, 269-278

[16] Mitrinović, D. S., Pečarić, J. E., Fink, A. M., *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991

[17] Mureşan, V., A Gronwall type inequality for Fredholm operators, *Mathematica*, 41(1999), Nr. 2, 227-231

[18] Mureşan, V., *Functional-Integral Equations*, Ed. Mediamira, Cluj-Napoca, 2003

[19] Pachpatte, B. G., *Inequalities for differential and integral equations*, Academic Press, New York, 1998

[20] Petruşel, A., Rus, I. A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. 134 (2005), 411-418

[21] Precup, R., *Methods in nonlinear integral equations*, Kluwer Academic Publishers, 2002

[22] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[23] Rus, I. A., *Metrical fixed point theorems*, University of Cluj-Napoca, 1979

[24] Rus, I. A., Generalized contractions, *Seminar on Fixed Point Theory*, Babeş-Bolyai University, *Preprint* Nr.3, 1983, 1-130

[25] Rus, I. A., Picard mappings, *Studia Univ. Babeş-Bolyai, Mathematica*, 33(1988), 70-73

[26] Rus, I. A., Weakly Picard mappings, *Comment. Math. Univ. Caroline*, **34**, 3(1993), 769-773

[27] Rus, I. A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Casa de editură Transilvania Press, Cluj-Napoca, 1996

[28] Rus, I. A., Picard operators and applications, *Babeş-Bolyai Univ. of Cluj-Napoca, Preprint* Nr. 3, 1996

[29] Rus, I. A., *An abstract point of view for some integral equations from applied mathematics*, Proc. of the Int. Conf. University of the West, Timişoara, 1997, 256-270

[30] Rus, I. A., A general functional inequality and its applications, *Revue d'An. Num. Th. l'Approx*. 26(1997), 209-213

[31] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334

[32] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001

[33] Rus, I. A., Weakly Picard operators and applications, *Seminar on Fixed Point Theory*, Babeş-Bolyai University of Cluj-Napoca, Vol. 2, 2001, 41-58

[34] Rus, I. A., Picard operators and applications, *Scientiae Mathematicae Japonicae*, 58(2003):1, 191-219

[35] Rus, I. A., Fixed points, upper and lower fixed points: abstract Gronwall lemmas, *Carpathian Journal of Mathematics*, 20(2004), No. 1, 125-134

[36] Schröder, J., *Operator Inequalities*, Acad. Press, New York, 1980

[37] Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj-Napoca, 2002

[38] Şerban, M. A., Application of fiber Picard operators to integral equations, *Bul. Ştiinţific Univ. Baia Mare, Seria B, Matematică-Informatică*, Vol. XVIII(2002), Nr.1, 119-128

[39] Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl*., 13(2010), no. 2, 255-269

[40] Walter, W., *Differential and integral inequalities*, Springer, Berlin, 1979

[41] Zima, M., The abstract Gronwall lemma for some nonlinear operators, *Demonstratio Math*., 31(1998), 325-332

4 Data dependence

The data dependence was studied both by direct methods and abstract methods. We mention several of the basic treatises which have the data dependence like theme, studied by direct methods: D. V. Ionescu [13], Gh. Marinescu [15], C. Corduneanu [4], A. Haimovici [12], Şt. Mirică [16], and by abstract methods: P. Pavel and I. A. Rus [19], V. Mureşan [17], [18], I. A. Rus [22], [25], [32], I. A. Rus, A. Petruşel and G. Petruşel [34], M. A. Şerban [40], V. Berinde [3], Sz. András [2].

We mention, also, several of the articles which contain the data dependence results: I. A. Rus [23], [24], [26], [27], [29], [30], [31], [33], [35], [37], I. A. Rus and S. Mureşan [28], I. A. Rus, S. Mureşan and V. Mureşan [36], R. Precup [21], R. Precup and E. Kirr [20], E. Kirr [14], M. Dobriţoiu, I. A. Rus and M. A. Şerban [10], M. A. Şerban [39], [41], M. A. Şerban, I. A. Rus and A. Petruşel [42], J. Sotomayor [38], Sz. András [1], M. Dobriţoiu [5], [6].

In this chapter, divided into four paragraphs, we study the data dependence of the solution of the integral equation (2.1) and the differentiability of the solution of this equation with respect to a and b and with respect to a parameter, respectively.

In the first three paragraphs theorems of the data dependence of the solution and theorems of the differentiability of the solution with respect to a parameter, are given. It also gives a theorem of data dependence of the solution of the system of integral equations with modified argument (2.6). For establishing of these results were useful the following theorems: *the abstract data dependence theorem* 1.3.5 and some of the results of I. A. Rus in the papers [29], [31], [33] and [35].

In the paragraph 4 three examples are treated, two integral equations with modified argument and a system of integral equations with modified argument; the first two examples are applications of the theorems 4.1.1 and 4.1.3 respectively, and the third example is an application of the theorem 4.2.1.

The results presented in this chapter were obtained by the author and they were published in the papers [7], [8], [9] and [11].

4.1 Continuous data dependence

A. Data dependence of the solution of the integral equation with modified argument

Consider the integral equation with modified argument (2.1)

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[a,b],$$

and the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t), \quad t\in[a,b], \tag{4.1}$$

where $K, H : [a,b]\times[a,b]\times\boldsymbol{B}^4 \to \boldsymbol{B}$, $f, h : [a,b] \to \boldsymbol{B}$, $g : [a,b] \to [a,b]$, and $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ is a Banach space.

We have the following theorem of continuous data dependence of the solution of integral equation (2.1).

Theorem 4.1.1. *Suppose that*

(i) *the conditions of the theorem* 2.1.1 *of existence and uniqueness of the solution of the integral equation* (2.1) *in the space* $C([a,b],\mathbf{B})$ *are fulfilled and we denote by* $x^* \in C([a,b],\mathbf{B})$ *the unique solution of this equation*;

(ii) $H \in C([a,b]\times[a,b]\times\mathbf{B}^4,\mathbf{B})$ *and* $h \in C([a,b],\mathbf{B})$;

(iii) *there exists* $\eta_1,\ \eta_2 > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)| \le \eta_1,$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3, u_4 \in \mathbf{B}$

and

$$|f(t) - h(t)| \le \eta_2, \quad \text{for all } t \in [a,b] .$$

Under these conditions, if $y^* \in C([a,b],\mathbf{B})$ *is a solution of the integral equation* (4.1), *then the following estimate is true*:

$$\left\|x^* - y^*\right\|_{(C[a,b],B)} \le \frac{\eta_1(b-a)+\eta_2}{1-4L_K(b-a)} . \tag{4.2}$$

Proof. We consider the operator from the proof of theorem 2.1.1, A: $C([a,b],\mathbf{B}) \to C([a,b],\mathbf{B})$, attached to integral equation (2.1) and defined by the relation (2.3):

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [a,b] .$$

Now we attach to the perturbed integral equation (4.1) the operator D: $C([a,b],\mathbf{B}) \to C([a,b],\mathbf{B})$, defined by the relation:

$$D(y)(t) := \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t), \quad t \in [a,b] . \tag{4.3}$$

Using the condition (ii) and the condition (a_3) from the theorem 2.1.1, we deduce that the operator D is correctly defined.

The set of the solutions of the perturbed integral equation (4.1), in the space $C([a,b],\mathbf{B})$, coincides with the fixed points set of the operator D defined by the relation (4.3). We have:

$$|A(x)(t) - D(x)(t)| = \left|\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) - \right.$$

$$\left. - \int_a^b H(t,s,x(s),x(g(s)),x(a),x(b))ds - h(t)\right| \le$$

$$\le \left|\int_a^b [K(t,s,x(s),x(g(s)),x(a),x(b)) - H(t,s,x(s),x(g(s)),x(a),x(b))]ds\right| +$$

$$+ |f(t) - h(t)| \le$$

$$\leq \int_a^b \left| K(t,s,x(s),x(g(s)),x(a),x(b)) - H(t,s,x(s),x(g(s)),x(a),x(b)) \right| ds \ +$$

$$+ \ \left| f(t) - h(t) \right| \ ,$$

and according to condition (*iii*) it results that:

$$\left| A(x)(t) - D(x)(t) \right| \leq \eta_1 (b-a) + \eta_2 \,, \quad \text{for all } t \in [a,b].$$

Now, using *the Chebyshev norm* we obtain:

$$\left\| A(x) - D(x) \right\|_{(C[a,b],B)} \leq \eta_1 (b-a) + \eta_2 \ . \tag{4.4}$$

We apply *the abstract theorem of data dependence* 1.3.5, and it results the estimate (4.2). The proof is complete.

Next we consider the perturbed integral equation:

$$y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + f(t) \,, \quad t \in [a,b] \ , \tag{4.5}$$

where $H : [a,b]\times[a,b]\times J^4 \to \boldsymbol{B}$, $f : [a,b] \to \boldsymbol{B}$, $g : [a,b] \to [a,b]$, $(\boldsymbol{B},+,\boldsymbol{R},|\cdot|)$ is a Banach space and $J \subset \boldsymbol{B}$ is compact.

We denote by M_H a positive constant such that for the restriction $H \,|_{[a,\,b]\times[a,\,b]\times J^4}$, $J \subset \boldsymbol{B}$ compact, we have:

$$|H(t,s,u_1,u_2,u_3,u_4)| \leq M_H \,, \quad \text{for all } t, s \in [a,b], \ u_1, u_2, u_3, u_4 \in J. \tag{4.6}$$

Theorem 4.1.2. *Suppose that*

(*i*) *the conditions of the theorem* 2.1.2 *of existence and uniqueness of the solution of the integral equation* (2.1) *in the sphere* $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$ *are fulfilled and we denote by* $x^* \in \overline{B}(f;r)$ *the unique solution of this equation* ;

(*ii*) $H \in C([a,b]\times[a,b]\times J^4, \boldsymbol{B})$, $J \subset \boldsymbol{B}$ *compact* ;

(*iii*) $M_H (b-a) \leq r$;

(*iv*) *there exists* $\eta > 0$ *such that*

$$\left| K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4) \right| \leq \eta \,,$$
for all $t, s \in [a,b]$, $u_1, u_2, u_3, u_4 \in J$.

Under these conditions, if $y^* \in \overline{B}(f;r) \subset (C[a,b],\boldsymbol{B})$ *is a solution of the integral equation* (4.5), *then the following estimate is true*:

$$\left\| x^* - y^* \right\|_C \leq \frac{\eta\,(b-a)}{1 - 4L_K (b-a)} \ . \tag{4.7}$$

Proof. We consider the operator from the proof of theorem 2.1.2, A: $\overline{B}(f;r) \to \overline{B}(f;r)$, attached to integral equation (2.1) and defined by the relation (2.3):

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t \in [a,b].$$

We attach to the perturbed integral equation (4.5) the operator D: $\overline{B}(f;r) \to C([a,b],\boldsymbol{B})$, defined by the relation:

$$D(y)(t) := \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + f(t), \quad t \in [a,b]. \tag{4.8}$$

Using the condition (*iii*) it results that the sphere $\overline{B}(f;r)$ is an invariant subset for the operator D, i.e. $\overline{B}(f;r) \in I(D)$, and now, we can consider the operator, also denoted by D, D : $\overline{B}(f;r) \to \overline{B}(f;r)$ and defined by the same relation (4.8).

Using the condition (*ii*) and the conditions (a_2) and (a_3) from the theorem 2.1.2, we deduce that the operator D is correctly defined.

The set of the solutions of the perturbed integral equation (4.5), in the sphere $\overline{B}(f;r) \subset C([a,b],\boldsymbol{B})$, coincides with the fixed points set of the operator D defined by the relation (4.8). We have:

$$|A(x)(t) - D(x)(t)| = \left|\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) - \right.$$

$$\left. - \int_a^b H(t,s,x(s),x(g(s)),x(a),x(b))ds - f(t)\right| =$$

$$= \left|\int_a^b [K(t,s,x(s),x(g(s)),x(a),x(b)) - H(t,s,x(s),x(g(s)),x(a),x(b))]ds\right| \leq$$

$$\leq \int_a^b |K(t,s,x(s),x(g(s)),x(a),x(b)) - H(t,s,x(s),x(g(s)),x(a),x(b))|ds \ ,$$

and according to condition (*iv*) we deduce that:

$$|A(x)(t) - D(x)(t)| \leq \eta(b-a), \quad \text{for all } t \in [a,b]$$

Using *the Chebyshev norm* we obtain:

$$\|A(x) - D(x)\|_C \leq \eta(b-a) \ . \tag{4.9}$$

We apply *the abstract theorem of data dependence* 1.3.5, and it results the estimate (4.7).

B. Data dependence of the solution of a system of integral equations with modified argument

In the particular case $\boldsymbol{B} = \boldsymbol{R}^m$, we consider the system of integral equations with modified argument (2.6) or (2.6’):

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[a,b] ,$$

or

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \end{cases}, \quad t\in[a,b] ,$$

where $K : [a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m \to \boldsymbol{R}^m$, $f : [a,b] \to \boldsymbol{R}^m$, $g : [a,b] \to [a,b]$.

To study the dependence of the solution of the system of integral equations (2.6) with respect to the functions K and f, we consider the perturbed system:

$$y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t), \quad t\in[a,b] \tag{4.10}$$

or

$$\begin{cases} y_1(t) = \int_a^b H_1(t,s,y(s),y(g(s)),y(a),y(b))ds + h_1(t) \\ y_2(t) = \int_a^b H_2(t,s,y(s),y(g(s)),y(a),y(b))ds + h_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_m(t) = \int_a^b H_m(t,s,y(s),y(g(s)),y(a),y(b))ds + h_m(t) \end{cases}, \quad t\in[a,b] \tag{4.10'}$$

where $H : [a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m \to \boldsymbol{R}^m$, $h : [a,b] \to \boldsymbol{R}^m$, $g : [a,b] \to [a,b]$.

We have the following theorem of continuous data dependence for the solution of of the system of integral equations (2.6).

Theorem 4.1.3. *Suppose that*

(*i*) *the conditions of theorem* 2.2.2 *of existence and uniqueness of the solution of the system of integral equations* (2.6) *in the space* $C([a,b],\boldsymbol{R}^m)$ *are fulfilled and we denote by* $x^*\in C([a,b],\boldsymbol{R}^m)$ *the unique solution of this system* ;

(*ii*) $H \in C([a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m, \boldsymbol{R}^m)$ *and* $h \in C([a,b],\boldsymbol{R}^m)$;

(*iii*) *there exists* $T_1, T_2 \in M_{m\times 1}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)\|_C \le T_1 ,$$

for all $t, s\in[a,b]$, $u_1, u_2, u_3, u_4\in\boldsymbol{R}^m$

and

$$\| f(t) - h(t) \|_C \quad \le \quad T_2 \ , \quad \textit{for all} \ \ t \in [a,b] \ .$$

Under these conditions, if $y^* \in C([a,b],\boldsymbol{R^m})$ *is a solution of the system of integral equations* (4.10), *then the following estimate is true*:

$$\left\| x^* - y^* \right\|_C \quad \le \quad \left[I_m - 4(b-a)Q \right]^{-1} \left[(b-a)T_1 + T_2 \right] \ . \tag{4.11}$$

Proof. We consider the operator from the proof of theorem 2.2.2, A : C([a,b],$\boldsymbol{R^m}$) → C([a,b],$\boldsymbol{R^m}$), attached to the system (2.6), defined by the relation (2.9) (or (2.9')):

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t) \ , \qquad t \in [a,b] \ ,$$

or

$$A(x)(t) := \begin{cases} A_1(x)(t) := \int_a^b K_1(t,s,x(s),x(g(s)),x(a),x(b))ds + f_1(t) \\ A_2(x)(t) := \int_a^b K_2(t,s,x(s),x(g(s)),x(a),x(b))ds + f_2(t) \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_m(x)(t) := \int_a^b K_m(t,s,x(s),x(g(s)),x(a),x(b))ds + f_m(t) \ . \end{cases}$$

Also, we consider the operator D : C([a,b],$\boldsymbol{R^m}$) → C([a,b],$\boldsymbol{R^m}$), attached to the perturbed system (4.10), defined by the relation:

$$D(y)(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t) \ , \qquad t \in [a,b]. \tag{4.12}$$

Using the condition (*ii*) and the condition (b_3) from the theorem 2.2.2, we deduce that the operator D is correctly defined. The set of the solutions of the perturbed system (4.10), in the space C([a,b],$\boldsymbol{R^m}$), coincides with the fixed points set of the operator D, defined by the relation (4.12). We have:

$$\left\| x^* - y^* \right\|_C = \left\| A(x^*) - D(y^*) \right\|_C \le \left\| A(x^*) - A(y^*) \right\|_C + \left\| A(y^*) - D(y^*) \right\|_C$$

and

$$\begin{pmatrix} \left| A_1(x^*)(t) - A_1(y^*)(t) \right| \\ . \ \ . \ \ . \\ \left| A_m(x^*)(t) - A_m(y^*)(t) \right| \end{pmatrix} + \begin{pmatrix} \left| A_1(y^*)(t) - D_1(y^*)(t) \right| \\ . \ \ . \ \ . \\ \left| A_m(y^*)(t) - D_m(y^*)(t) \right| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_a^b \left| K_1(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b)) - K_1(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b)) \right| ds \\ . \\ \int_a^b \left| K_m(t,s,x^*(s),x^*(g(s)),x^*(a),x^*(b)) - K_m(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b)) \right| ds \end{pmatrix} +$$

$$+\begin{pmatrix} \int_a^b \left|K_1(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b)) - H_1(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b))\right| ds \\ \cdots \\ \int_a^b \left|K_m(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b)) - H_m(t,s,y^*(s),y^*(g(s)),y^*(a),y^*(b))\right| ds \end{pmatrix} +$$

$$+\begin{pmatrix} |f_1(t)-h_1(t)| \\ \cdots \\ |f_m(t)-h_m(t)| \end{pmatrix}.$$

Since the function K satisfies a generalized Lipschitz condition with respect to the last four arguments, with the matrix Q (condition (b_6) of the theorem 2.2.2, from the chapter 2) and according to condition (*iii*) and to generalized norm, given in chapter 1, by the relation (1.7), we obtain:

$$\left\|x^* - y^*\right\|_C \leq 4(b-a)Q\left\|x^* - y^*\right\|_C + (b-a)T_1 + T_2 ,$$

and now it results the estimate (4.11). The proof is complete.

4.2 The differentiability of the solution with respect to *a* and *b*

We consider the Fredholm-type integral equation with modified argument (2.1)

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \quad t\in[\alpha,\beta] \tag{4.13}$$

where $\alpha,\beta \in \boldsymbol{R}$, $\alpha \leq \beta$, $a, b \in [\alpha,\beta]$ and $K\in C([\alpha,\beta]\times[\alpha,\beta]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m, \boldsymbol{R}^m)$, $f\in C([\alpha,\beta],\boldsymbol{R}^m)$,

$g\in C[\alpha,\beta]$, $a \leq g(s) \leq b$, $s\in[a,b]$ and $x\in C([\alpha,\beta],\boldsymbol{R}^m)$.

We have:

Theorem 4.2.1. *Suppose that there exists a matrix* $Q\in M_{m\times m}(\boldsymbol{R}_+)$ *such that*:

(*i*) $[4(\beta-\alpha)Q]^n \to 0$ *as* $n\to\infty$;

$$(ii)\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ \cdots \\ |K_m(t,s,u_1,u_2,u_3,u_4) - K_m(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \leq$$

$$\leq Q\begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| + |u_{41}-v_{41}| \\ \cdots \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| + |u_{4m}-v_{4m}| \end{pmatrix}$$

for all $t, s\in[\alpha,\beta]$, $u_i, v_i\in\boldsymbol{R}^m$, $i=\overline{1,4}$.

Then

(*a*) *the integral equation* (4.13) *has a unique solution,* $x^*(\cdot,a,b)\in C([\alpha,\beta],\boldsymbol{R}^m)$;

(b) *for all* $x_0 \in C([\alpha,\beta],\boldsymbol{R}^m)$, *the sequence* $(x^n)_{n\in \boldsymbol{N}}$, *defined by the relation*:

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t) \ ,$$

converges uniformly to x^*, *for all t, a,* $b\in[\alpha,\beta]$ *and*

$$\begin{pmatrix} \|x_1^n - x_1^*\|_C \\ . \ . \ . \\ \|x_m^n - x_m^*\|_C \end{pmatrix} \le [I_m - 4(\beta-\alpha)Q]^{-1}[4(\beta-\alpha)Q]^n \begin{pmatrix} \|x_1^1 - x_1^0\|_C \\ . \ . \ . \\ \|x_m^1 - x_m^0\|_C \end{pmatrix};$$

(c) *the function* $x^*:[\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta] \to \boldsymbol{R}^m$, $(t, a, b)\mapsto x^*(t; a, b)$ *is continuous*;

(d) *if* $K(t,s, \cdot, \cdot, \cdot, \cdot) \in C^1(\boldsymbol{R}^m \times\boldsymbol{R}^m \times\boldsymbol{R}^m \times\boldsymbol{R}^m, \boldsymbol{R}^m)$ *for all t,* $s\in[\alpha,\beta]$, *then*

$$x^*(t; \cdot, \cdot)\in C^1([\alpha,\beta]\times[\alpha,\beta],\boldsymbol{R}^m) \quad \textit{for all } t\in[\alpha,\beta] \ .$$

Proof. Denote $X := C([\alpha,\beta]^3,\boldsymbol{R}^m)$. We consider on X *the generalized norm* defined in the chapter 1 by the relation (1.7):

$$\|x\|_C := \begin{pmatrix} \|x_1\|_C \\ ... \\ \|x_m\|_C \end{pmatrix}, \quad \text{for all } x = \begin{pmatrix} x_1 \\ ... \\ x_m \end{pmatrix} \in C([a,b],\boldsymbol{R}^m) \ ,$$

where $\| x_k \|_C = \max_{t\in[a,b]} | x_k(t) |$, $k=\overline{1,m}$.

Also, we consider the operator $B : X \to X$ defined by the relation:

$$B(x)(t;a,b) := \int_a^b K(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))ds \ , \tag{4.14}$$

for all t, a, $b\in[\alpha,\beta]$.

Using the conditions (i), (ii) and applying *the Perov's theorem* 1.3.4, it results that the conclusions (a), (b) and (c) are fulfilled.

(d) We prove that there exists $\frac{\partial x^*}{\partial a}, \frac{\partial x^*}{\partial b} \in X$.

If we assume that there exists $\frac{\partial x^*}{\partial a}$, then from (4.13) it results that:

$$\frac{\partial x^*(t;a,b)}{\partial a} = -K(t,a,x^*(a;a,b),x^*(g(a);a,b),x^*(a;a,b),x^*(b;a,b)) +$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(s;a,b)} \right) \cdot \left(\frac{\partial x^*(s;a,b)}{\partial a} \right) + \right.$$

$$+ \left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(g(s);a,b)} \right) \cdot \left(\frac{\partial x^*(g(s);a,b)}{\partial a} \right) +$$

$$+\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(a;a,b)}\right)\cdot\left(\frac{\partial x^*(a;a,b)}{\partial a}\right)+$$

$$+\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(g(s);a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(b;a,b)}\right)\cdot\left(\frac{\partial x^*(b;a,b)}{\partial a}\right)\Bigg]ds\ .$$

This relation leads us to consider the operator $C: X\times X\to X$ defined by the relation:

$$C(x,y)(t;a,b):=-K(t,a,x(a;a,b),x(g(a);a,b),x(a;a,b),x(b;a,b))+$$

$$+\int_a^b\Bigg[\left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i(s;a,b)}\right)\cdot y(s;a,b)+$$

$$+\left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i(g(s);a,b)}\right)\cdot y(g(s);a,b)+$$

$$+\left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i(a;a,b)}\right)\cdot y(a;a,b)+$$

$$+\left(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i(b;a,b)}\right)\cdot y(b;a,b)\Bigg]ds\ . \qquad (4.15)$$

Using the condition (*ii*) we obtain:

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{1i}}\right|\right)_{i,j=1}^m\le Q,\qquad \left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{2i}}\right|\right)_{i,j=1}^m\le Q,$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{3i}}\right|\right)_{i,j=1}^m\le Q,\qquad \left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{4i}}\right|\right)_{i,j=1}^m\le Q, \qquad (4.16)$$

for all $t, s\in[\alpha,\beta]$, $u_1, u_2, u_3, u_4\in \boldsymbol{R}^m$.

Using (4.15) and (4.16) it results that:

$$\|C(x,y_1)-C(x,y_2)\|\le 4(\beta-\alpha)Q\cdot\|y_1-y_2\|\ ,\ \text{ for all } x, y_1, y_2\in X.$$

Now, if we consider the operator $A: X\times X\to X\times X$, $A=(B,C)$ then we observe that the conditions of *the fiber generalized contractions theorem* 1.5.2 are fulfilled and therefore it results that A is a Picard operator and the sequence $\left(x^{n+1}(t;a,b),\ y^{n+1}(t;a,b)\right)$, defined by the relations:

$$x^{n+1}(t;a,b):=\int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds+f(t)\ ,$$

$$y^{n+1}(t;a,b):=-K(t,a,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))+$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{1i}} \right) \cdot y^n(s;a,b) + \right.$$

$$+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{2i}} \right) \cdot y^n(g(s);a,b) +$$

$$+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{3i}} \right) \cdot y^n(a;a,b) +$$

$$\left. +\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{4i}} \right) \cdot y^n(b;a,b) \right] ds$$

converges uniformly (with respect to t, a, $b\in[\alpha,\beta]$) to $(x^*,y^*)\in F_A$, for all $(x^0,y^0)\in X\times X$.

If we take $x^0 = y^0 = 0$, then $y^1 = \frac{\partial x^1}{\partial a}$ and we prove through induction that $y^n = \frac{\partial x^n}{\partial a}$. Thus, we have:

$$x^n \xrightarrow{uniformly} x^* \quad \text{as} \quad n \to \infty$$

$$\frac{\partial x^n}{\partial a} \xrightarrow{uniformly} y^* \quad \text{as} \quad n \to \infty ,$$

and it results that there exists $\frac{\partial x^*}{\partial a}$ (i.e. x^* is differentiable with respect to a) and $\frac{\partial x^*}{\partial a} = y^*$.

By an analogous reasoning we prove that there exists $\frac{\partial x^*}{\partial b}$.

4.3 The differentiability of the solution with respect to a parameter

In what follows we apply *the fiber generalized contractions theorem* 1.5.2, to study the differentiability with respect to a parameter of the solution of the Fredholm-type integral equation with modified argument:

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b);\lambda)ds + f(t), \quad t\in[a,b] \tag{4.17}$$

where $K \in C([a,b]\times[a,b]\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times\boldsymbol{R}^m\times J, \boldsymbol{R}^m)$, $J \subset \boldsymbol{R}$ is a compact interval and

$f\in C([a,b],\boldsymbol{R}^m)$, $g \in C([a,b], [a,b])$ and $x \in C([a,b],\boldsymbol{R}^m)$.

The following theorem of differentiability of the solution is true.

Theorem 4.3.1. *Suppose that there exists a matrix* $Q\in M_{m\times m}(\boldsymbol{R}_+)$ *such that*:

(*i*) $[4(b-a)Q]^n \to 0$ *as* $n \to \infty$;

$$(ii)\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4)-K_1(t,s,v_1,v_2,v_3,v_4)| \\ \cdot\ \cdot\ \cdot \\ |K_m(t,s,u_1,u_2,u_3,u_4)-K_m(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \le$$

$$\le\ Q\begin{pmatrix} |u_{11}-v_{11}|+|u_{21}-v_{21}|+|u_{31}-v_{31}|+|u_{41}-v_{41}| \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ |u_{1m}-v_{1m}|+|u_{2m}-v_{2m}|+|u_{3m}-v_{3m}|+|u_{4m}-v_{4m}| \end{pmatrix}$$

for all $t, s\in[a, b]$, $u_i, v_i\in \mathbf{R}^m$, $i=\overline{1,4}$.

Then

(a) *for all* $\lambda\in J$, *the integral equation* (4.17) *has a unique solution,* $x^*(\cdot,\lambda)\in C([a,b], \mathbf{R}^m)$;

(b) *for all* $x_0\in C([a,b]\times J, \mathbf{R}^m)$, *the sequence* $(x^n)_{n\in \mathbf{N}}$, *defined by the relation*:

$$x^{n+1}(t;\lambda):=\int_a^b K(t,s,x^n(s;\lambda),x^n(g(s);\lambda),x^n(a;\lambda),x^n(b;\lambda))ds+f(t)\ ,$$

converges uniformly to x^*, *for all* $t\in[a,b]$, $\lambda\in J$ *and*

$$\begin{pmatrix} \|x_1^n-x_1^*\|_C \\ \cdot\ \cdot\ \cdot \\ \|x_m^n-x_m^*\|_C \end{pmatrix} \le [I_m-4(b-a)Q]^{-1}[4(b-a)Q]^n \begin{pmatrix} \|x_1^1-x_1^0\|_C \\ \cdot\ \cdot\ \cdot \\ \|x_m^1-x_m^0\|_C \end{pmatrix};$$

(c) *the function* $x^*:[a,b]\times J\to \mathbf{R}^m$, $(t;\lambda)\mapsto x^*(t;\lambda)$ *is continuous*;

(d) *if* $K(t,s,\cdot,\cdot,\cdot,\cdot,\cdot)\in C^1(\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^m\times J, \mathbf{R}^m)$ *for all* $t, s\in[a,b]$,

then $x^*(t;\cdot)\in C^1(J, \mathbf{R}^m)$ *for all* $t\in[a,b]$.

Proof. Denote $X := C([a,b]\times J,\mathbf{R}^m)$. We consider the generalized norm on X, defined in the chapter 1 by the relation (1.7).

Also, we consider the operator $B: X\to X$ defined by the relation:

$$B(x)(t;\lambda):=\int_a^b K(t,s,x(s;\lambda),x(g(s);\lambda),x(a;\lambda),x(b;\lambda))ds+f(t)\ , \tag{4.18}$$

for all $t\in[a,b]$, $\lambda\in J$.

From conditions (i), (ii) and applying *the Perov's theorem* 1.3.4, it results that the conclusions (a), (b) and (c) are fulfilled.

(d) We prove that there exists $\frac{\partial x^*}{\partial\lambda}$ and $\frac{\partial x^*}{\partial\lambda}\in X$.

We assume that there exists $\frac{\partial x^*}{\partial\lambda}$. Then using (4.17) we obtain:

$$\frac{\partial x^*(t;\lambda)}{\partial\lambda}=\int_a^b\Bigg[\left(\frac{\partial K_j(t,s,x^*(s;\lambda),x^*(g(s);\lambda),x^*(a;\lambda),x^*(b;\lambda);\lambda)}{\partial x_i^*(s;\lambda)}\right)_{i,j=1}^m\cdot\left(\frac{\partial x^*(s;\lambda)}{\partial\lambda}\right)+$$

$$+\left(\frac{\partial K_j(t,s,x^*(s;\lambda),x^*(g(s);\lambda),x^*(a;\lambda),x^*(b;\lambda);\lambda)}{\partial x_i^*(g(s);\lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(g(s);\lambda)}{\partial \lambda}\right)+$$

$$+\left(\frac{\partial K_j(t,s,x^*(s;\lambda),x^*(g(s);\lambda),x^*(a;\lambda),x^*(b;\lambda);\lambda)}{\partial x_i^*(a;\lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(a;\lambda)}{\partial \lambda}\right)+$$

$$+\left(\frac{\partial K_j(t,s,x^*(s;\lambda),x^*(g(s);\lambda),x^*(a;\lambda),x^*(b;\lambda);\lambda)}{\partial x_i^*(b;\lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(b;\lambda)}{\partial \lambda}\right)+$$

$$+\left.\left(\frac{\partial K_j(t,s,x^*(s;\lambda),x^*(g(s);\lambda),x^*(a;\lambda),x^*(b;\lambda);\lambda)}{\partial \lambda}\right)_{i=1}^m \right] ds \ . \tag{4.19}$$

This relation leads us to consider the operator $C : X \times X \to X$ defined by the relation:

$$C(x,y)(t,\lambda) := \int_a^b \left[\left(\frac{\partial K_j(t,s,x(s;\lambda),x(g(s);\lambda),x(a;\lambda),x(b;\lambda);\lambda)}{\partial x_i(s;\lambda)}\right)_{i,j=1}^m \cdot y(s;\lambda)+\right.$$

$$+\left(\frac{\partial K_j(t,s,x(s,\lambda),x(g(s),\lambda),x(a,\lambda),x(b,\lambda);\lambda)}{\partial x_i(g(s);\lambda)}\right)_{i,j=1}^m \cdot y(g(s),\lambda)+$$

$$+\left(\frac{\partial K_j(t,s,x(s;\lambda),x(g(s);\lambda),x(a;\lambda),x(b;\lambda);\lambda)}{\partial x_i(a;\lambda)}\right)_{i,j=1}^m \cdot y(a,\lambda)+$$

$$+\left(\frac{\partial K_j(t,s,x(s;\lambda),x(g(s);\lambda),x(a;\lambda),x(b;\lambda);\lambda)}{\partial x_i(b;\lambda)}\right)_{i,j=1}^m \cdot y(b;\lambda)+$$

$$+\left.\left(\frac{\partial K_j(t,s,x(s;\lambda),x(g(s);\lambda),x(a;\lambda),x(b;\lambda);\lambda)}{\partial \lambda}\right)_{j=1}^m \right] ds \ , \tag{4.20}$$

for all $x, y \in X$.

From condition (*ii*) we obtain:

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{1i}}\right|\right)_{i,j=1}^m \le Q, \qquad \left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{2i}}\right|\right)_{i,j=1}^m \le Q,$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{3i}}\right|\right)_{i,j=1}^m \le Q, \qquad \left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3,u_4)}{\partial u_{4i}}\right|\right)_{i,j=1}^m \le Q \ , \tag{4.21}$$

for all $t, s \in [a, b]$, $u_1, u_2, u_3, u_4 \in \mathbf{R}^m$.

From (4.20) and (4.21) it results that

$$\|C(x,y_1)-C(x,y_2)\| \le 4(b-a)Q \cdot \|y_1-y_2\| \ , \quad \text{for all } x, y_1, y_2 \in X \ .$$

Now, if we consider the operator A: $X \times X \to X \times X$, $A=(B,C)$, $A(x,y) = (B(x),C(x,y))$, then we observe that the conditions of *the fiber generalized contractions theorem* 1.5.2 are fulfilled and therefore it results that A is a Picard operator and the sequences:

$$x^{n+1}(t,\lambda) := B(x^n(t,\lambda))$$

$$y^{n+1}(t,\lambda) = C(x^n(t,\lambda), y^n(t,\lambda)),$$

converge uniformly (with respect to $t\in[a,b]$ and $\lambda \in J$) to $(x^*,y^*)\in F_A$, for all $(x^0,y^0)\in X\times X$.

If we take $x^0\in X$, $y^0\in X$, such that $y^0 = \frac{\partial x^0}{\partial \lambda}$, then we prove by induction that $y^n = \frac{\partial x^n}{\partial \lambda}$. Thus, we have:

$$x^n \xrightarrow{uniform} x^* \qquad \text{când} \quad n \to \infty$$

$$\frac{\partial x^n}{\partial \lambda} \xrightarrow{uniform} y^* \qquad \text{când} \quad n \to \infty .$$

Using *the Weierstrass's theorem* it results that there exists $\frac{\partial x^*}{\partial \lambda}$ (x^* is differentiable with respect to λ) and $\frac{\partial x^*}{\partial \lambda} = y^*$.

4.4 Examples

Example 4.4.1. We consider the integral equation with modified argument:

$$x(t) = \int_0^1 \left[\frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right] ds + 2\cos t + 1, \quad t \in [0,1] \tag{4.22}$$

where $K\in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$, $K(t,s,u_1,u_2,u_3,u_4) = \frac{\sin(u_1)+\cos(u_2)}{7} + \frac{u_3+u_4}{5}$,

$f \in C[0,1]$, $f(t) = 2\cos t + 1$, $g\in C([0,1],[0,1])$, $g(s) = s/2$, and $x\in C[0,1]$

and the perturbed integral equation:

$$y(t) = \int_0^1 \left[\frac{\sin(y(s)) + \cos(y(s/2))}{7} + \frac{y(0)+y(1)}{5} - t - 2 \right] ds + \cos t, \ t \in [0,1] \tag{4.23}$$

where $H\in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$, $H(t,s,v_1,v_2,v_3,v_4) = \frac{\sin(v_1)+\cos(v_2)}{7} + \frac{v_3+v_4}{5} - t - 2$,

$h\in C[0,1]$, $h(t) = \cos t$, $g\in C([0,1],[0,1])$, $g(s) = s/2$, and $y\in C[0,1]$.

The operator $A : C[0,1] \to C[0,1]$, attached to equation (4.22) and defined by the relation:

$$A(x)(t) = \int_0^1 \left[\frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right] ds + 2\cos t + 1, \ t \in [0,1] \tag{4.24}$$

is an α-contraction with the coefficient $\alpha = \frac{24}{35}$.

Since the conditions of theorem 2.1.1 of existence and uniqueness of the solution in the space $C[0,1]$ are fulfilled (chapter 2, paragraph 2.4, example 2.4.1, II), it results that the integral equation (4.22) has a unique solution $x^* \in C[0,1]$.

We have:

$$|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)| = |t+2| \le 3, \quad \text{for all } t,s \in [0,1]$$

and

$$|f(t) - h(t)| = |\cos t + 1| \le 2, \quad \text{for all } t \in [0,1].$$

The conditions of theorem 4.1.1 are fulfilled and therefore, if $y^* \in C[0,1]$ is a solution of the integral equation (4.23), then the following estimate is true:

$$\|x^* - y^*\|_{C[0,1]} \le \frac{3\cdot(1-0)+2}{1-\frac{24}{35}} = \frac{175}{11}.$$

Example 4.4.2. În what follows we consider the system of integral equations with modified argument:

$$\begin{cases} x_1(t) = \int_0^1 \left[\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{t}{5} x_1(0) + \frac{t}{5} x_1(1) \right] ds + 2t + 1 \\ x_2(t) = \int_0^1 \left[\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{t}{7} x_2(0) + \frac{t}{7} x_2(1) \right] ds + \sin t \end{cases}, \ t \in [0,1], \qquad (4.25)$$

where $K \in C([0,1]\times[0,1]\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2, \mathbf{R}^2)$,

$$K(t,s,u_1,u_2,u_3,u_4) = \left(K_1(t,s,u_1,u_2,u_3,u_4),\ K_2(t,s,u_1,u_2,u_3,u_4)\right),$$

$$K_1(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{15} u_{11} + \frac{2t+1}{15} u_{21} + \frac{1}{5} u_{31} + \frac{1}{5} u_{41},$$

$$K_2(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{21} u_{12} + \frac{2t+1}{21} u_{22} + \frac{1}{7} u_{32} + \frac{1}{7} u_{42},$$

$f \in C([0,1],\mathbf{R}^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 2t+1$, $f_2(t) = \sin t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$ and $x \in C([0,1],\mathbf{R}^2)$

and the perturbed system of integral equations with modified argument:

$$\begin{cases} y_1(t) = \int_0^1 \left[\frac{s+3}{15} y_1(s) + \frac{2s+3}{15} y_1(s/2) + \frac{1}{5} y_1(0) + \frac{1}{5} y_1(1) - 3 \right] ds + 2t - 1 \\ y_2(t) = \int_0^1 \left[\frac{s+3}{21} y_2(s) + \frac{2s+3}{21} y_2(s/2) + \frac{1}{7} y_2(0) + \frac{1}{7} y_2(1) - 1 \right] ds + \cos t \end{cases}, t \in [0,1] \qquad (4.26)$$

where $H \in C([0,1]\times[0,1]\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2, \mathbf{R}^2)$,

$$H(t,s,v_1,v_2,v_3,v_4) = \left(H_1(t,s,v_1,v_2,v_3,v_4),\ H_2(t,s,v_1,v_2,v_3,v_4)\right),$$

$$H_1(t,s,v_1,v_2,v_3,v_4)=\frac{s+3}{15}v_{11}+\frac{2s+3}{15}v_{21}+\frac{1}{5}v_{31}+\frac{1}{5}v_{41}-3\ ,$$

$$H_2(t,s,v_1,v_2,v_3,v_4)=\frac{s+3}{21}v_{12}+\frac{2s+3}{21}v_{22}+\frac{1}{7}v_{32}+\frac{1}{7}v_{42}-1\ ,$$

$h\in C([0,1],\boldsymbol{R}^2)$, $h(t) = (h_1(t), h_2(t))$, $h_1(t)=2t-1$, $h_2(t)=\cos t$,

$g\in C([0,1],[0,1])$, $g(s) = s/2$, and $x\in C([0,1], \boldsymbol{R}^2)$.

The operator $A : C([0,1],\boldsymbol{R}^2) \to C([0,1],\boldsymbol{R}^2)$, $A(x)(t)=(A_1(x)(t),A_2(x)(t))$, attached to system (4.25) and defined by the relation:

$$\begin{aligned}A_1(x)(t)&=\int_0^1\left[\frac{t+2}{15}x_1(s)+\frac{2t+1}{15}x_1(s/2)+\frac{t}{5}x_1(0)+\frac{t}{5}x_1(1)\right]ds+2t+1\\ A_2(x)(t)&=\int_0^1\left[\frac{t+2}{21}x_2(s)+\frac{2t+1}{21}x_2(s/2)+\frac{t}{7}x_2(0)+\frac{t}{7}x_2(1)\right]ds+\sin t\end{aligned}\ ,\quad t\in[0,1], \qquad (4.27)$$

satisfies a generalized Lipschitz condition with the matrix $Q = \begin{pmatrix}1/5 & 0\\ 0 & 1/7\end{pmatrix}$ and according to theorem 1.3.3 it results that the matrix $4(1-0)Q = \begin{pmatrix}4/5 & 0\\ 0 & 4/7\end{pmatrix}$ converges to zero. So, the operator A is a contraction with the matrix $\begin{pmatrix}4/5 & 0\\ 0 & 4/7\end{pmatrix}$.

The conditions of theorem 2.2.2 of existence and uniqueness of the solution of a system of integral equations, being satisfied, it results that the system of integral equations with modified argument (4.25) has a unique solution $x^*\in C([0,1],\boldsymbol{R}^2)$ and the following estimates are true:

$$\| K(t,s,u_1,u_2,u_3,u_4)-H(t,s,u_1,u_2,u_3,u_4))\|_{R^2}\le\begin{pmatrix}3\\1\end{pmatrix},\text{ for all } t, s\in[0,1]$$

and

$$\| f(t)-h(t)\|_{R^2}\le\begin{pmatrix}2\\2\end{pmatrix},\quad\text{for all } t\in[0,1]\ .$$

Under these conditions, if $y^*\in C([0,1], \boldsymbol{R}^2)$ is a solution of the system of integral equations (4.25), then according to theorem 4.1.3, the following estimate is true:

$$\left\|x^*-y^*\right\|_{R^2}\le\begin{pmatrix}1/5 & 0\\ 0 & 3/7\end{pmatrix}^{-1}\cdot\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}5 & 0\\ 0 & 7/3\end{pmatrix}\cdot\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}25\\7\end{pmatrix}.$$

Example 4.4.3. We consider the system of Fredholm-type integral equations:

$$\begin{cases}x_1(t)=\int_a^b\left[\frac{1}{10}(t+s)x_1(s)+\frac{1}{5}x_1(s/2)+\frac{2t+1}{15}x_1(a)+\frac{t+2}{15}x_1(b)\right]ds+1-\cos t\\ x_2(t)=\int_a^b\left[\frac{1}{2}x_1(s)+\frac{2t+s}{24}x_2(s)+\frac{1}{8}x_2(s/2)+\frac{2t+1}{24}x_2(a)+\frac{t+2}{24}x_2(b)\right]ds+\sin t\end{cases}\qquad (4.28)$$

where $t, a, b \in [0,1]$, $K \in C([0,1]\times[0,1]\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2, \mathbf{R}^2)$,

$$K(t,s,u_1,u_2,u_3,u_4) = \left(K_1(t,s,u_1,u_2,u_3,u_4),\ K_2(t,s,u_1,u_2,u_3,u_4)\right),$$

$$K_1(t,s,u_1,u_2,u_3,u_4) = \frac{1}{10}(t+s)u_{11} + \frac{1}{5}u_{21} + \frac{2t+1}{15}u_{31} + \frac{t+2}{15}u_{41},$$

$$K_2(t,s,u_1,u_2,u_3,u_4) = \frac{1}{2}u_{11} + \frac{2t+s}{24}u_{22} + \frac{1}{8}u_{22} + \frac{2t+1}{24}u_{32} + \frac{t+2}{24}u_{42},$$

$f \in C([0,1],\mathbf{R}^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 1-\cos t$, $f_2(t) = \sin t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$ and $x \in C([0,1],\mathbf{R}^2)$

and applying the theorem 4.2.1 we will study the differentiability of the solution of this system with respect to a and b.

From the condition (*ii*) of the theorem 4.2.1, we have:

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ \\ |K_2(t,s,u_1,u_2,u_3,u_4) - K_2(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \le$$

$$\le \begin{pmatrix} 1/5 & 0 \\ 1/2 & 1/8 \end{pmatrix} \cdot \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| + |u_{41}-v_{41}| \\ \\ |u_{12}-v_{12}| + |u_{22}-v_{22}| + |u_{32}-v_{32}| + |u_{42}-v_{42}| \end{pmatrix}, \quad t,s \in [0,1].$$

According to theorem 1.3.3, it results that the matrix

$$4(b-a)Q = (b-a)\begin{pmatrix} 4/5 & 0 \\ 2 & 1/2 \end{pmatrix}, \quad 0 < b-a < 1, \quad Q \in M_{2\times 2}(\mathbf{R}_+)$$

converges to zero.

Hence, the conditions of theorem 4.2.1 being satisfied, it results that:

- the system of integral equations (4.28) has a unique solution $x^*(\cdot, a, b)$ in the space $C([0,1],\mathbf{R}^2)$;
- for all $x^0 \in C([0,1],\mathbf{R}^2)$, the sequence $(x^n)_{n\in N}$, defined by the relation:

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t)$$

converges uniformly to x^*, for all $t, a, b \in [0,1]$, and

$$\begin{pmatrix} \|x_1^n - x_1^*\|_C \\ \|x_2^n - x_2^*\|_C \end{pmatrix} \le \begin{pmatrix} 5 & 0 \\ 20 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4/5 & 0 \\ 2 & 1/2 \end{pmatrix}^n \cdot \begin{pmatrix} \|x_1^1 - x_1^0\|_C \\ \|x_2^1 - x_2^0\|_C \end{pmatrix}$$

- the function $x^* : [0,1]\times[0,1]\times[0,1] \to \mathbf{R}^2$), $(t; a, b) \to x^*(t; a, b)$ is continuous;
- if $K(t,s,\cdot,\cdot,\cdot,\cdot) \in C^1(\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2, \mathbf{R}^2)$ for all $t,s \in [0,1]$, then

  $x^*(t;\cdot,\cdot) \in C^1([0,1]\times[0,1], \mathbf{R}^2)$ for all $t \in [0,1]$.

4.5 References

[1] András, Sz., Fiber φ-contractions on generalized metric spaces and applications, *Mathematica*, 45(68)(2003):1, 3-8, Cluj-Napoca

[2] András, Sz., *Ecuaţii integrale Fredholm-Volterra*, Ed. Didactică şi Pedagogică, Bucureşti, 2005

[3] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997

[4] Corduneanu, C., *Ecuaţii diferenţiale şi integrale*, Univ. Iaşi, 1971

[5] Dobriţoiu, M., An integral equation with modified argument, *Studia Univ. Babeş-Bolyai*, *Mathematica*, vol. XLIX, nr. 3/2004, 27-33

[6] Dobriţoiu, M., Existence and continuous dependence on data of the solution of an integral equation, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM-CVI / 2005 , Nr. ISSN 0133-3526

[7] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş–Bolyai, Mathematica*, vol. 51, nr. 1/2006, 81-94

[8] Dobriţoiu, M., W. W. Kecs, A. Toma, *The Differentiability of the Solution of a Nonlinear Integral Equation*, Proc. of the 8th WSEAS Int. Conf. on Math. Methods and Comp. Techniques in Electrical Eng., Bucharest, Romania, Oct. 16-17, 2006, ISSN 1790-5117, ISBN 960-8457-54-8, 155-158

[9] Dobriţoiu, M., W. W. Kecs, A. Toma, An Application of the Fiber Generalized Contractions Theorem, *WSEAS Transactions on Mathematics*, Issue 12, Vol. 5, Dec. 2006, ISSN 1109-2769, 1330-1335

[10] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[11] Dobriţoiu, M., System of integral equations with modified argument, *Carpathian Journal of Mathematics*, vol. 24 (2008), No. 2, 26-36

[12] Haimovici, A., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1965

[13] Ionescu, D. V., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti,1972

[14] Kirr, E., Existence and continuous dependence on data of the positive solutions of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, 41, No.2 (1996)

[15] Marinescu, Gh., *Teoria ecuaţiilor diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1963

[16] Mirică, Şt., *Ecuaţii diferenţiale şi integrale*, vol. I, II, III, Ed. Universităţii Bucureşti, 1999

[17] Mureşan, V., *Ecuaţii diferenţiale cu modificarea afină a argumentului*, Transilvania Press, Cluj-Napoca, 1997

[18] Mureşan, V., *Functional-Integral Equations*, Ed. Mediamira, Cluj-Napoca, 2003

[19] Pavel, P., Rus, I. A., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1975

[20] Precup, R., Kirr, E., *Analysis of a nonlinear integral equation modelling infection diseases*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 178-195

[21] Precup, R., *Continuation results for mappings of contractive type*, *Seminar on Fixed Point Theory* Cluj-Napoca, 2(2001), 23-40

[22] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[23] Rus, I. A., Picard mappings, *Studia Univ. Babeş-Bolyai, Mathematica*, 33(1988), 70-73

[24] Rus, I. A., Weakly Picard mappings, *Comment. Math. Univ. Caroline*, **34**, 3(1993), 769-773

[25] Rus, I. A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Casa de editură Transilvania Press, Cluj-Napoca, 1996

[26] Rus, I. A., Picard operators and applications, *Babeş-Bolyai Univ. of Cluj-Napoca, Preprint* Nr. 3,

1996
[27] Rus, I. A., *An abstract point of view for some integral equations from applied mathematics*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 256-270
[28] Rus, I. A., Mureşan, S., Data dependence of the fixed point set of weakly Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, 43(1998):1, 79-83
[29] Rus, I. A., A Fiber generalized contraction theorem and applications, Académie Roumaine, Filiale de Cluj-Napoca, *Mathematica*, Tome 41(64), Nr.1, 1999, 85-90
[30] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334
[31] Rus, I. A., Fiber Picard operators theorem and applications, *Studia Univ. Babeş-Bolyai*, *Mathematica*, **44**(1999), No. 3, 89-97
[32] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001
[33] Rus, I. A., Weakly Picard operators and applications, *Seminar on Fixed Point Theory*, Babeş-Bolyai University of Cluj-Napoca, Vol. 2, 2001, 41-58
[34] Rus, I. A., Petruşel, A., Petruşel, G., *Fixed point theory 1950-2000 romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002
[35] Rus, I. A., Picard operators and applications, *Scientiae Mathematicae Japonicae*, 58(2003):1, 191-219
[36] Rus, I. A., Mureşan, S., Mureşan, V., Weakly Picard operators on a set with two metrics, *Fixed Point Theory*, Vol. 6, No. 2, 2005, 323-331
[37] Rus, I. A., Data dependence of the fixed points in a set with two metrics, *Fixed Point Theory*, Vol. 8, No. 1, 2007, 115-123
[38] Sotomayor, J., Smooth dependence of solution of differential equation on initial data: a simple proof, *Bol. Soc. Brasil*, 4, 1(1973), 55-59
[39] Şerban, M. A., Fiber φ-contractions, *Studia Univ. Babeş-Bolyai, Mathematica*, 34(1999), Nr.3, 99-108
[40] Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj-Napoca, 2002
[41] Şerban, M. A., Application of fiber Picard operators to integral equations, *Bul. Ştiinţific Univ. Baia Mare, Seria B, Matematică-Informatică*, Vol. XVIII(2002), Nr.1, 119-128
[42] Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl.*, 13(2010), no. 2, 255-269 Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl.*, 13(2010), no. 2, 255-269

5 Numerical analysis of the Fredholm integral equation with modified argument (2.1)

The study of an integral equation represents the development of a fixed point theory, which contains the results on existence and uniqueness of the solution, the integral inequalities, the theorems of comparison, the theorems of data dependence of the solution and an algorithm of approximation of the solution.

The numerical analysis of an integral equation consists in establishment of a method for approximating the solution of this equation.

The references used to establish a method for approximating the solution of the integral equation (2.1) includes treatises of numerical analysis of integral equations (W. Hackbusch [10]), treatises which have chapters with this subject of study (Gh. Coman, I. Rus, G. Pavel and I. A. Rus [3], D. D. Stancu, Gh. Coman, O. Agratini and R. Trîmbiţaş [21], D. D. Stancu, Gh. Coman, P. Blaga [22]), papers with this subject of study (M. Ambro [1], G. Pavel [15], C. Iancu [11], R. Precup [17], M. Dobriţoiu [5], [6], [7], [8] and [9]) and other results which are used in this book on this topic (D.V. Ionescu [13], Gh. Coman [4], P. Cerone and S. S. Dragomir [2], C. Iancu [12], Gh. Marinescu [14], A. D. Polyanin and A. V. Manzhirov [16], I. A. Rus [18], [19], I. A. Rus, M. A. Şerban and D. Trif [20]).

In this chapter, divided into five paragraphs, a procedure for approximating the solution of the integral equation with modified argument (2.1) is given. For this, we assume that the conditions of one of the theorems of existence and uniqueness, established in chapter 2, are fulfilled.

The first paragraph contains the problem statement, specifying the conditions under which the method for approximating the solution of the integral equation with modified argument (2.1) is given.

In the following three paragraphs, the successive approximations method is used to determine a method for approximating the solution, and for the approximate calculation of the integrals that arise in the terms of the successive approximations sequence is used the trapezoids formula, the rectangle formula and the Simpson's quadrature formula, respectively.

In the paragraph 5 we use the results presented in the first four paragraphs, to establish a method for approximating the solution of the integral equation with modified argument that has been considered as example.

The results presented in this chapter were published in the paper [9].

5.1 The statement of the problem

To establish the procedure for approximating the solution of the integral equation with the argument modified (2.1) were used the results given by Gh Coman, I. Rus, G. Pavel and I. A. Rus [3], D.V. Ionescu [13], I. A. Rus [19] and Gheorghe Marinescu [14].

We suppose that the integral equation with modified argument (2.1):

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t), \quad t\in[a,b],$$

has a unique solution in the sphere $\overline{B}(f;r) \subset C[a,b]$. Hence, in the particular case $\boldsymbol{B} = \boldsymbol{R}$, the conditions of the theorem 2.1.2 are fulfilled, i.e.:

(h_1) $K \in C([a,b]\times[a,b]\times J^4)$, $J \subset \boldsymbol{R}$ compact ;

(h_2) $f \in C[a,b]$;

(h_3) $g \in C([a,b],[a,b])$;

(h_4) $M_K\,(b-a) \le r$ *(condition of invariance of the sphere* $\overline{B}(f;r)$*)*,

where we denote by M_K a positive constant such that for the restriction $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \boldsymbol{R}$ compact, we have:

$$|K(t,s,u_1,u_2,u_3,u_4)| \le M_K\,,\ \text{ for all } t,\, s \in [a,b],\ u_1, u_2, u_3, u_4 \in J\,; \tag{5.1}$$

(h_5) there exists $L_K > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \le$$
$$\le L_K\,(|u_1 - v_1| + |u_2 - v_2)| + |u_3 - v_3)| + |u_4 - v_4)|),$$

for all $t,\, s \in [a,b]$, $u_i, v_i \in J$, $i = \overline{1,4}$;

(h_6) $4L_K(b-a) < 1$. *(contraction condition)*

We denote this solution by $x^* \in \overline{B}(f;r) \subset C[a,b]$. According to theorem 2.1.2 ($\boldsymbol{B} = \boldsymbol{R}$) this solution can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(f;r) \subset C[a,b]$. Moreover, if x_n is the n-th successive approximation, then the following estimation is true:

$$|x_n - x^*| \le \frac{4^n L_K^n (b-a)^n}{1 - 4L_K(b-a)} |x_1 - x_0|\ . \tag{5.2}$$

Therefore, for the determination of x^* we apply the successive approximations method.
The sequence of the successive approximations is:

$$
\begin{aligned}
&x_0(t) = f(t), \quad x_0 \in \overline{B}(f;R) \subset C[a,b]\\
&x_1(t) = \int_a^b K(t,s,x_0(s),x_0(g(s)),x_0(a),x_0(b))ds + f(t)\\
&x_2(t) = \int_a^b K(t,s,x_1(s),x_1(g(s)),x_1(a),x_1(b))ds + f(t)\\
&\ldots\ldots\\
&x_m(t) = \int_a^b K(t,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))ds + f(t)\\
&\ldots\ldots
\end{aligned}
\tag{5.3}
$$

To get a better result, it is considered an equidistant division Δ of the interval $[a,b]$ through the points $a = t_0 < t_1 < \ldots < t_n = b$ and the successive approximations sequence will be:

$$x_0(t_k) = f(t_k)$$

$$x_0(g(s)) = f(g(s))$$

$$x_0(a) = f(a)$$

$$x_0(b) = f(b)$$

$$x_1(t_k) = \int_a^b K(t_k, s, f(s), f(g(s)), f(a), f(b))ds + f(t_k)$$

$$x_1(g(s)) = \int_a^b K(g(s), s, f(s), f(g(s)), f(a), f(b))ds + f(g(s))$$

$$x_1(a) = \int_a^b K(a, s, f(s), f(g(s)), f(a), f(b))ds + f(a)$$

$$x_1(b) = \int_a^b K(b, s, f(s), f(g(s)), f(a), f(b))ds + f(b) \qquad (5.3')$$

. . . . .

$$x_{m-1}(t_k) = \int_a^b K(t_k, s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))ds + f(t_k)$$

$$x_{m-1}(g(s)) = \int_a^b K(g(s), s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))ds + f(g(s))$$

$$x_{m-1}(a) = \int_a^b K(a, s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))ds + f(a)$$

$$x_{m-1}(b) = \int_a^b K(b, s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))ds + f(b)$$

$$x_m(t_k) = \int_a^b K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))ds + f(t_k)$$

$$x_m(g(s)) = \int_a^b K(g(s), s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))ds + f(g(s))$$

$$x_m(a) = \int_a^b K(a, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))ds + f(a)$$

$$x_m(b) = \int_a^b K(b, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))ds + f(b)$$

. . . . .

In the following three paragraphs we present the method for approximating the solution of integral equation (2.1), obtained by applying the successive approximations method and using, also, the trapezoids formula, the rectangles formula and the Simpson's formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence.

5.2 The approximation of the solution using the trapezoids formula

We suppose that the following conditions are fulfilled:

(h_{11}) $K \in C^2\left([a,b]\times[a,b]\times J^4\right)$, $J \subset \boldsymbol{R}$ is closed interval ;

(h_{12}) $f \in C^2[a,b]$;

(h_{13}) $g \in C^2\left([a,b],[a,b]\right)$

and using the trapezoids formula (1.14) for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence (5.3’), with the estimate of the rest given by (1.15), we will approximate the terms of this sequence.

In the general case for $x_m(t_k)$ we obtain:

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b)) + \right. \tag{5.4}$$

$$+\,2\sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b)) +$$

$$\left. +\,K(t_k,b,x_{m-1}(b),x_{m-1}(g(b)),x_{m-1}(a),x_{m-1}(b))\right] + f(t_k) + R^T_{m,k}\,,\quad k=\overline{0,n}\,,\quad m\in N$$

with the estimate of the rest:

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2}\cdot \max_{s\in[a,b]}\left|\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]''_s\right| .$$

According to condition (h_{11}) it results that there exists the derivative of the function K from the estimate of the rest $R^T{}_{m,k}$ and it has the following expression:

$$\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]''_s = \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s\cdot\partial x_{m-1}}\cdot x'_{m-1}(s) +$$

$$+\,2\frac{\partial^2 K}{\partial s\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial g}\cdot g'(s) + \frac{\partial^2 K}{\partial x^2_{m-1}}\cdot\left(x'_{m-1}(s)\right)^2 + 2\frac{\partial^2 K}{\partial x^2_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial g}\cdot x'_{m-1}(s)\cdot g'(s) +$$

$$+\,\frac{\partial K}{\partial x_{m-1}}\cdot x''_{m-1}(s) + \frac{\partial^2 K}{\partial x^2_{m-1}}\cdot\left(\frac{\partial x_{m-1}}{\partial g}\right)^2\cdot\left(g'(s)\right)^2 + \frac{\partial K}{\partial x_{m-1}}\cdot\frac{\partial^2 x_{m-1}}{\partial g\partial s}\cdot g'(s) + \frac{\partial K}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial g}\cdot g''(s)\,,$$

where

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s,x_{m-2}(s),x_{m-2}(g(s)),x_{m-2}(a),x_{m-2}(b))}{\partial t}\,ds + f'(t)$$

$$x''_{m-1}(t)=\int_a^b \frac{\partial^2 K(t,s,x_{m-2}(s),x_{m-2}(g(s)),x_{m-2}(a),x_{m-2}(b))}{\partial t^2}ds+f''(t).$$

Denote

$$M_1^T=\max_{\substack{|\alpha|\le 2\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right|,\quad |\alpha|=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$M_2^T=\max_{\substack{\alpha\le 2\\ t\in[a,b]}}\left|f^{(\alpha)}(t)\right| \quad\text{and}\quad M_3^T=\max_{\substack{\alpha\le 2\\ t\in[a,b]}}\left|g^{(\alpha)}(t)\right|.$$

Now, using the expressions of the derivatives of $x_{m-1}(t)$, it results that

$$\left|x_{m-1}^{(\alpha)}(t)\right|\le(b-a)M_1^T+M_2^T,\quad \alpha=\overline{1,2},$$

and for the derivative of the function K from the expression of the rest $R^T_{m,k}$ we obtain:

$$\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]''_s\le$$

$$\le M_1^T\left\{1+\left[M_1^T(b-a)+M_2^T\right]\cdot\left(3+4M_3^T\right)+\left(1+M_3^T\right)^2\cdot\left[M_1^T(b-a)+M_2^T\right]^2\right\}=M_0^T.$$

It is obvious that M_0^T doesn't depend on m and k, so we have the estimation of the rest:

$$\left|R_{m,k}^T\right|\le M_0^T\cdot\frac{(b-a)^3}{12n^2},\quad M_0^T=M_0^T(K,D^\alpha K,f,D^\alpha f,g,D^\alpha g),\quad |\alpha|\le 2 \tag{5.5}$$

and thus we obtain a formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence (5.3'). Using the successive approximations method and the formula (5.4) with the estimation of the rest resulted from (5.5), we suggest further on an algorithm in order to solve the integral equation (2.1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence. Thus we have:

$$x_0(t_k)=f(t_k)$$

$$x_1(t_k)=\int_a^b K(t_k,s,f(s),f(g(s)),f(a),f(b))ds+f(t_k)=$$

$$=\frac{b-a}{2n}\Big[K(t_k,a,f(a),f(g(a)),f(a),f(b))+2\sum_{i=1}^{n-1}K(t_k,t_i,f(t_i),f(g(t_i)),f(a),f(b))+$$

$$+K(t_k,b,f(b),f(g(b)),f(a),f(b))\Big]+f(t_k)+R_{1,k}^T=$$

$$=\widetilde{x}_1(t_k)+\widetilde{R}_{1,k}^T,\quad k=\overline{0,n}$$

$$x_2(t_k)=\int_a^b K(t_k,s,x_1(s),x_1(g(s)),x_1(a),x_1(b))ds+f(t_k)=$$

$$=\frac{b-a}{2n}\big[K(t_k,a,x_1(a),x_1(g(a)),x_1(a),x_1(b))+2\sum_{i=1}^{n-1}K(t_k,t_i,x_1(t_i),x_1(g(t_i)),x_1(a),x_1(b))+$$

$$+K(t_k,b,x_1(b),x_1(g(b)),x_1(a),x_1(b))\big]+f(t_k)+R_{2,k}^T=$$

$$=\frac{b-a}{2n}\big[K(t_k,a,\widetilde{x}_1(a)+R_{1,0}^T,\widetilde{x}_1(g(a))+R_{1,0}^T,\widetilde{x}_1(a)+R_{1,0}^T,\widetilde{x}_1(b)+R_{1,0}^T)+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i)+R_{1,i}^T,\widetilde{x}_1(g(t_i))+R_{1,i}^T,\widetilde{x}_1(a)+R_{1,i}^T,\widetilde{x}_1(b)+R_{1,i}^T)+$$

$$+K(t_k,b,\widetilde{x}_1(b)+R_{1,n}^T,\widetilde{x}_1(g(b))+R_{1,n}^T,\widetilde{x}_1(a)+R_{1,n}^T,\widetilde{x}_1(b)+R_{1,n}^T)\big]+$$

$$+f(t_k)+R_{2,k}^T=\frac{b-a}{2n}\big[K(t_k,a,\widetilde{x}_1(a),\widetilde{x}_1(g(a)),\widetilde{x}_1(a),\widetilde{x}_1(b))+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i),\widetilde{x}_1(g(t_i)),\widetilde{x}_1(a),\widetilde{x}_1(b))+K(t_k,b,\widetilde{x}_1(b),\widetilde{x}_1(g(b)),\widetilde{x}_1(a),\widetilde{x}_1(b))\big]+f(t_k)+\widetilde{R}_{2,k}^T=$$

$$=\widetilde{x}_2(t_k)+\widetilde{R}_{2,k}^T,\quad k=\overline{0,n}$$

and the following estimate of the rest:

$$\left|\widetilde{R}_{2,k}^T\right|=\left|x_2(t_k)-\widetilde{x}_2(t_k)\right|\le\frac{b-a}{2n}\cdot L_K\left(4\left|R_{1,0}^T\right|+8\sum_{i=1}^{n-1}\left|R_{1,i}^T\right|+4\left|R_{1,n}^T\right|\right)+\left|R_{2,k}^T\right|\le$$

$$\le 4(b-a)L_K M_0^T\cdot\frac{(b-a)^3}{12n^2}+M_0^T\cdot\frac{(b-a)^3}{12n^2}=$$

$$=\frac{(b-a)^3}{12n^2}\cdot M_0^T\left[4L_K(b-a)+1\right].$$

The reasoning continues for $m=3,\ldots$ and through induction we obtain:

$$x_m(t_k)=\frac{b-a}{2n}\big[K(t_k,a,\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(g(a)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_{m-1}(t_i),\widetilde{x}_{m-1}(g(t_i)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+K(t_k,b,\widetilde{x}_{m-1}(b),\widetilde{x}_{m-1}(g(b)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))\big]+f(t_k)+\widetilde{R}_{m,k}^T=$$

$$=\widetilde{x}_m(t_k)+\widetilde{R}_{m,k}^T,\quad k=\overline{0,n}\ ,$$

with the estimate of the rest:

$$\left|\widetilde{R}_{m,k}^{T}\right| = \left|x_m(t_k) - \widetilde{x}_m(t_k)\right| \le \frac{(b-a)^3}{12n^2} \cdot M_0^T \cdot \left[4^{m-1}(b-a)^{m-1} L_K^{m-1} + . \quad . \quad . + 1\right], \; k = \overline{0,n},$$

that according to the contraction condition (h_6), is

$$\left|\widetilde{R}_{m,k}^{T}\right| \le \frac{(b-a)^3}{12n^2\left[1 - 4L_K(b-a)\right]} \cdot M_0^T . \tag{5.6}$$

Thus, using an equidistant division of the interval $[a,b]$ through the points $a = t_0 < t_1 < . . . < t_n = b$, we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $\left(x_m(t_k)\right)_{m\in N}$, $k = \overline{0,n}$ with the following error in calculation:

$$\left|x_m(t_k) - \widetilde{x}_m(t_k)\right| \le \frac{(b-a)^3}{12n^2\left[1 - 4L_K(b-a)\right]} \cdot M_0^T \;. \tag{5.7}$$

Now, using the estimates (5.2) and (5.7) it is obtain the following result.

Theorem 5.2.1. *Suppose that, in the particular case* $\boldsymbol{B} = \boldsymbol{R}$, *the conditions of the theorem* 2.1.2 *are fulfilled. In addition, we assume that the exact solution* x^* *of the integral equation* (2.1) *is approximated by the sequence* $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k = \overline{0,n}$, *on the nodes* t_k, $k = \overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$, *using the successive approximations method* (5.3) *and the trapezoids method* (1.14)+(1.15).

Under these conditions, the error of approximation is given by the evaluation:

$$\left|x^*(t_k) - \widetilde{x}_m(t_k)\right| \le \frac{4^m L_K^m (b-a)^m}{1 - 4L_K(b-a)} \left| x_1 - x_0 \right| + \frac{(b-a)^3}{12n^2\left[1 - 4L_K(b-a)\right]} \cdot M_0^T . \tag{5.8}$$

5.3 The approximation of the solution using the rectangles formula

Suppose that the following conditions are fulfilled:

(h_{21}) $K \in C^1\left([a,b]\times[a,b]\times J^4\right)$, $J \subset \boldsymbol{R}$ is closed interval ;

(h_{22}) $f \in C^1[a,b]$;

(h_{23}) $g \in C^1\left([a,b],[a,b]\right)$

and we will approximate the terms of the successive approximations sequence (5.3') using the rectangles formula (1.21) with the rest given by (1.22), considering the intermediary points of the division of the interval $[a,b]$ on the left end of the partial intervals $\xi_I = t_i$.

In the general case for $x_m(t_k)$ we obtain:

$$x_m(t_k) = \frac{b-a}{n}\left[K(t_k, a, x_{m-1}(a), x_{m-1}(g(a)), x_{m-1}(a), x_{m-1}(b)) + \right. \tag{5.9}$$

$$+\sum_{i=1}^{n-1} K(t_k, t_i, x_{m-1}(t_i), x_{m-1}(g(t_i)), x_{m-1}(a), x_{m-1}(b))\big] + f(t_k) + R_{m,k}^D \ , \quad k = \overline{0,n} \ , \quad m \in N$$

with the estimate of the rest:

$$\left| R_{m,k}^D \right| \le \frac{(b-a)^2}{n} \cdot \max_{s\in[a,b]} \left| \big[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))\big]_s^{'} \right| \ .$$

According to condition (h_{21}), it results that there exists the derivative of the function K from the estimate of the rest $R_{m,k}^D$ and it has the following expression:

$$\big[K(t_k, s, x_{m-1}(s), x_{m-1}(g(t_k)), x_{m-1}(a), x_{m-1}(b))\big]_s^{'} = \frac{\partial K}{\partial s} + \frac{\partial K}{\partial x_{m-1}} \cdot x'_{m-1}(s) + \frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g'(s) \ .$$

In what follows, we use the expressions of the derivatives of $x_{m-1}(t)$ given in the previous paragraph and we note down:

$$M_1^D = \max_{\substack{|\alpha|\le 1 \\ t,s\in[a,b]}} \left| \frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1} \partial s^{\alpha_2} \partial u_1^{\alpha_3} \partial u_2^{\alpha_4} \partial u_3^{\alpha_5} \partial u_4^{\alpha_6}} \right| , \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \ ,$$

$$M_2^D = \max_{\substack{\alpha\le 1 \\ t\in[a,b]}} \left| f^{(\alpha)}(t) \right| \qquad \text{and} \qquad M_3^D = \max_{\substack{\alpha\le 1 \\ t\in[a,b]}} \left| g^{(\alpha)}(t) \right| \ ,$$

to obtain

$$\left| x_{m-1}^{'}(t) \right| \le (b-a) M_1^D + M_2^D \ ,$$

and for the derivative of the function K from the expression of the rest $R_{m,k}^D$ it results that:

$$\big[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))\big]_s^{'} \le M_1^D \left\{1 + \left[M_1^D (b-a) + M_2^D\right] \cdot \left(1 + M_3^D\right)\right\} = M_0^D \ .$$

It is clear that M_0^D doesn't depend on m and k, so we have the estimation of the rest:

$$\left| R_{m,k}^D \right| \le M_0^D \cdot \frac{(b-a)^2}{n}, \quad M_0^D = M_0^D(K, D^\alpha K, f, D^\alpha f, g, D^\alpha g), \ \alpha = 1 \qquad (5.10)$$

and thus we obtain a formula for the approximate calculation of the integrals that appear in the successive approximations sequence (5.3'). Using the successive approximations method and the formula (5.9) with the estimation of the rest resulted from (5.10), we suggest further on an algorithm in order to solve the integral equation (2.1) approximately.

To this end, we will calculate approximately the terms of the successive approximations sequence. So, we have:

$x_0(t_k) = f(t_k)$

$$x_1(t_k) = \int_a^b K(t_k, s, f(s), f(g(s)), f(a), f(b)) ds + f(t_k) =$$

$$=\frac{b-a}{n}\left[K(t_k,a,f(a),f(g(a)),f(a),f(b))+\sum_{i=1}^{n-1}K(t_k,t_i,f(t_i),f(g(t_i)),f(a),f(b))\right]+$$

$$+f(t_k)+R_{1,k}^D=\widetilde{x}_1(t_k)+\widetilde{R}_{1,k}^D\ ,\quad k=\overline{0,n}$$

$$x_2(t_k)=\int_a^b K(t_k,s,x_1(s),x_1(g(s)),x_1(a),x_1(b))ds+f(t_k)=$$

$$=\frac{b-a}{n}\left[K(t_k,a,x_1(a),x_1(g(a)),x_1(a),x_1(b))+\sum_{i=1}^{n-1}K(t_k,t_i,x_1(t_i),x_1(g(t_i)),x_1(a),x_1(b))\right]+$$

$$+f(t_k)+R_{2,k}^D=$$

$$=\frac{b-a}{n}[K(t_k,a,\widetilde{x}_1(a)+R_{1,0}^D,\widetilde{x}_1(g(a))+R_{1,0}^D,\widetilde{x}_1(a)+R_{1,0}^D,\widetilde{x}_1(b)+R_{1,0}^D)+$$

$$+\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i)+R_{1,i}^D,\widetilde{x}_1(g(t_i))+R_{1,i}^D,\widetilde{x}_1(a)+R_{1,i}^D,\widetilde{x}_1(b)+R_{1,i}^D)]+$$

$$+f(t_k)+R_{2,k}^D=$$

$$=\frac{b-a}{n}\left[K(t_k,a,\widetilde{x}_1(a),\widetilde{x}_1(g(a)),\widetilde{x}_1(a),\widetilde{x}_1(b))+\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i),\widetilde{x}_1(g(t_i)),\widetilde{x}_1(a),\widetilde{x}_1(b))\right]+$$

$$+f(t_k)+\widetilde{R}_{2,k}^D=\widetilde{x}_2(t_k)+\widetilde{R}_{2,k}^D,\quad k=\overline{0,n}$$

and

$$\left|\widetilde{R}_{2,k}^D\right|\le\frac{b-a}{n}\cdot L_K\left(4\left|R_{1,0}^D\right|+4\sum_{i=1}^{n-1}\left|R_{1,i}^D\right|\right)+\left|R_{2,k}^D\right|\le 4(b-a)L_K M_0^D\cdot\frac{(b-a)^2}{n}+M_0^D\cdot\frac{(b-a)^2}{n}=$$

$$=\frac{(b-a)^2}{n}\cdot M_0^D[4L_K(b-a)+1].$$

The reasoning continues for $m=3,\ldots$ and through induction we obtain:

$$x_m(t_k)=\frac{b-a}{n}[K(t_k,a,\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(g(a)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_{m-1}(t_i),\widetilde{x}_{m-1}(g(t_i)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))]+f(t_k)+\widetilde{R}_{m,k}^D=$$

$$=\widetilde{x}_m(t_k)+\widetilde{R}_{m,k}^D,\qquad k=\overline{0,n}\ ,$$

with the estimate of the rest:

$$\left|\widetilde{R}_{m,k}^D\right|=\left|x_m(t_k)-\widetilde{x}_m(t_k)\right|\le\frac{(b-a)^2}{n}\cdot M_0^D\cdot\left[4^{m-1}(b-a)^{m-1}L_K^{m-1}+.\ \ .\ \ .+1\right]\ ,\ k=\overline{0,n}\,,$$

that according to the contraction condition (h_6) is

$$\left|\widetilde{R}^{D}_{m,k}\right| \le \frac{(b-a)^2}{n\left[1-4L_K(b-a)\right]}\cdot M_0^D . \tag{5.11}$$

Thus, using an equidistant division of the interval $[a,b]$ through the points $a = t_0 < t_1 < \ldots < t_n = b$, we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $\left(x_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$ with the following error in calculation:

$$\left| x_m(t_k)-\widetilde{x}_m(t_k)\right| \le \frac{(b-a)^2}{n\left[1-4L_K(b-a)\right]}\cdot M_0^D . \tag{5.12}$$

Now, using the estimates (5.2) and (5.12) it is obtain the following result.

Theorem 5.3.1. *Suppose that the conditions of the theorem* 2.1.2 *are fulfilled* (***B*** = ***R***). *In addition, we assume that the exact solution* x^* *of the integral equation* (2.1) *is approximated by the sequence* $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$ *on the nodes* t_k, $k=\overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$, *using the successive approximations method* (5.3) *and the rectangles method* (1.21)+(1.22). *Under these conditions, the error of approximation is given by the evaluation*:

$$\left|x^*(t_k)-\widetilde{x}_m(t_k)\right| \le \frac{4^m L_K^m (b-a)^m}{1-4L_K(b-a)}\left| x_1-x_0\right| + \frac{(b-a)^2}{n\left[1-4L_K(b-a)\right]}\cdot M_0^D . \tag{5.13}$$

5.4 The approximation of the solution using the Simpson's formula

Suppose that the following conditions are fulfilled:

(h_{31}) $K\in C^4\left([a,b]\times[a,b]\times J^4\right)$, $J\subset \boldsymbol{R}$ is closed interval ;

(h_{32}) $f\in C^4[a,b]$;

(h_{33}) $g\in C^4\left([a,b],[a,b]\right)$

and using the Simpson's formula (1.27) for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence (5.3'), with the estimate of the rest given by (1.28), we will approximate the terms of this sequence.

In general case for $x_m(t_k)$ we obtain:

$$\begin{aligned} x_m(t_k) = \frac{b-a}{6n}\Big[&K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b)) + \\ &+2\sum_{i=1}^{n-1}K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b)) + \\ &+4\sum_{i=0}^{n-1}K\left(t_k\,,\frac{t_i+t_{i+1}}{2}\,,x_{m-1}\left(\frac{t_i+t_{i+1}}{2}\right),x_{m-1}\left(g\left(\frac{t_i+t_{i+1}}{2}\right)\right),x_{m-1}(a)\,,x_{m-1}(b)\right) + \\ &+K(t_k,b,x_{m-1}(b),x_{m-1}(g(b)),x_{m-1}(a),x_{m-1}(b))\Big] + f(t_k)+R^S_{m,k}\,,\quad k=\overline{0,n}\,,\ m\in N \end{aligned} \tag{5.14}$$

with the estimate of the rest:

$$\left|R_{m,k}^{S}\right| \le \frac{(b-a)^5}{2880n^4}\cdot \max_{s\in[a,b]}\left|\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]_s^{iv}\right| ,$$

and according to condition (h_{31}) it results that there exists the derivative of the function K from this estimate and it has the expression:

$$\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]_s^{iv} = \frac{\partial^4 K}{\partial s^4} + 4\frac{\partial^4 K}{\partial s^3 \partial x_{m-1}}x'_{m-1}(s) +$$

$$+4\frac{\partial^4 K}{\partial s^3 \partial x_{m-1}}\frac{\partial x_{m-1}}{\partial g}(g'(s)) + 6\frac{\partial^4 K}{\partial s^2 \partial x_{m-1}^2}\left(x'_{m-1}(s)\right)^2 + 12\frac{\partial^4 K}{\partial s^2 \partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x'_{m-1}(s)g'(s) +$$

$$+6\frac{\partial^3 K}{\partial s^2 \partial x_{m-1}}x''_{m-1}(s) + 6\frac{\partial^4 K}{\partial s^2 \partial x_{m-1}^2}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 (g'(s))^2 + 6\frac{\partial^3 K}{\partial s^2 \partial x_{m-1}}\frac{\partial^2 x_{m-1}}{\partial g \partial s}g'(s) +$$

$$+6\frac{\partial^3 K}{\partial s^2 \partial x_{m-1}}\frac{\partial x_{m-1}}{\partial g}g''(s) + 4\frac{\partial^4 K}{\partial s \partial x_{m-1}^3}\left(x'_{m-1}(s)\right)^3 + 12\frac{\partial^4 K}{\partial s \partial x_{m-1}^3}\frac{\partial x_{m-1}}{\partial g}\left(x'_{m-1}(s)\right)^2 g'(s) +$$

$$+12\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}x'_{m-1}(s)x''_{m-1}(s) + 12\frac{\partial^4 K}{\partial s \partial x_{m-1}^3}\left(\frac{\partial x_{m-1}}{\partial g}\right)x'_{m-1}(s)(g'(s))^2 +$$

$$+9\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}\frac{\partial^2 x_{m-1}}{\partial g \partial s}x'_{m-1}(s)g'(s) + 12\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x''_{m-1}(s)g'(s) +$$

$$+12\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x'_{m-1}(s)g''(s) + 4\frac{\partial^2 K}{\partial s \partial x_{m-1}}x'''_{m-1}(s) + 4\frac{\partial^4 K}{\partial s \partial x_{m-1}^3}\left(\frac{\partial x_{m-1}}{\partial g}\right)^3 (g'(s))^3 +$$

$$+12\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}\frac{\partial^2 x_{m-1}}{\partial g \partial s}(g'(s))^2 + 12\frac{\partial^3 K}{\partial s \partial x_{m-1}^2}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 g'(s)g''(s) +$$

$$+4\frac{\partial^2 K}{\partial s \partial x_{m-1}}\frac{\partial^3 x_{m-1}}{\partial g \partial s^2}g'(s) + 5\frac{\partial^2 K}{\partial s \partial x_{m-1}}\frac{\partial^2 x_{m-1}}{\partial g \partial s}g''(s) + 4\frac{\partial^2 K}{\partial s \partial x_{m-1}}\frac{\partial x_{m-1}}{\partial g}g'''(s) +$$

$$+\frac{\partial^4 K}{\partial x_{m-1}^4}\left(x'_{m-1}(s)\right)^4 + 4\frac{\partial^4 K}{\partial x_{m-1}^4}\frac{\partial x_{m-1}}{\partial g}\left(x'_{m-1}(s)\right)^3 g'(s) + 6\frac{\partial^3 K}{\partial x_{m-1}^3}\left(x'_{m-1}(s)\right)^2 x''_{m-1}(s) +$$

$$+6\frac{\partial^4 K}{\partial x_{m-1}^4}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 \left(x'_{m-1}(s)\right)^2 (g'(s))^2 + 6\frac{\partial^3 K}{\partial x_{m-1}^3}\frac{\partial^2 x_{m-1}}{\partial g \partial s}\left(x'_{m-1}(s)\right)^2 g'(s) +$$

$$+12\frac{\partial^3 K}{\partial x_{m-1}^3}\frac{\partial x_{m-1}}{\partial g}x'_{m-1}(s)x''_{m-1}(s)g'(s) + 6\frac{\partial^3 K}{\partial x_{m-1}^3}\frac{\partial x_{m-1}}{\partial g}\left(x'_{m-1}(s)\right)^2 g''(s) +$$

$$+3\frac{\partial^2 K}{\partial x_{m-1}^2}\left(x_{m-1}''(s)\right)^2+3\frac{\partial^2 K}{\partial x_{m-1}^2}x_{m-1}'(s)x_{m-1}'''(s)+4\frac{\partial^4 K}{\partial x_{m-1}^4}\left(\frac{\partial x_{m-1}}{\partial g}\right)^3 x_{m-1}'(s)\left(g'(s)\right)^3+$$

$$+12\frac{\partial^3 K}{\partial x_{m-1}^3}\frac{\partial x_{m-1}}{\partial g}\frac{\partial^2 x_{m-1}}{\partial g\partial s}x_{m-1}'(s)\left(g'(s)\right)^2+6\frac{\partial^3 K}{\partial x_{m-1}^3}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 x_{m-1}''(s)\left(g'(s)\right)^2+$$

$$+12\frac{\partial^3 K}{\partial x_{m-1}^3}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 x_{m-1}'(s)g'(s)g''(s)+3\frac{\partial^3 K}{\partial x_{m-1}^2\partial s}\frac{\partial^2 x_{m-1}}{\partial g\partial s}x_{m-1}'(s)g'(s)+$$

$$+4\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial^3 x_{m-1}}{\partial g\partial s^2}x_{m-1}'(s)g'(s)+6\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial^2 x_{m-1}}{\partial g\partial s}x_{m-1}''(s)g'(s)+$$

$$+8\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial^2 x_{m-1}}{\partial g\partial s}x_{m-1}'(s)g''(s)+4\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x_{m-1}'''(s)g'(s)+$$

$$+3\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x_{m-1}''(s)g''(s)+4\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}x_{m-1}'(s)g'''(s)+$$

$$+\frac{\partial^4 K}{\partial x_{m-1}^4}\left(\frac{\partial x_{m-1}}{\partial g}\right)^4\left(g'(s)\right)^4+6\frac{\partial^3 K}{\partial x_{m-1}^3}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2\frac{\partial^2 x_{m-1}}{\partial g\partial s}\left(g'(s)\right)^3+$$

$$+3\frac{\partial^2 K}{\partial x_{m-1}^2}\left(\frac{\partial^2 x_{m-1}}{\partial g\partial s}\right)^2\left(g'(s)\right)^2+3\frac{\partial^2 K}{\partial x_{m-1}^2}\left(\frac{\partial^2 x_{m-1}}{\partial g\partial s}\right)^2\left(g'(s)\right)^2+\frac{\partial^2 K}{\partial x_{m-1}^2}x_{m-1}'(s)x_{m-1}'''(s)+$$

$$+\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}\frac{\partial^3 x_{m-1}}{\partial g\partial s^2}\left(g'(s)\right)^2+3\frac{\partial^2 K}{\partial x_{m-1}^2}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2\left(g''(s)\right)^2+\frac{\partial K}{\partial x_{m-1}}\frac{\partial^4 x_{m-1}}{\partial g\partial s^3}g'(s)+$$

$$+14\frac{\partial^2 K}{\partial x_{m-1}^2}\frac{\partial x_{m-1}}{\partial g}\frac{\partial^2 x_{m-1}}{\partial g\partial s}g'(s)g''(s)+4\frac{\partial^2 K}{\partial x_{m-1}^2}\left(\frac{\partial x_{m-1}}{\partial g}\right)^2 g'(s)g'''(s)+\frac{\partial K}{\partial x_{m-1}}x_{m-1}^{(4)}(s)+$$

$$+3\frac{\partial K}{\partial x_{m-1}}\frac{\partial^3 x_{m-1}}{\partial g\partial s^2}g''(s)+3\frac{\partial K}{\partial x_{m-1}}\frac{\partial^2 x_{m-1}}{\partial g\partial s}g'''(s)+\frac{\partial K}{\partial x_{m-1}}\frac{\partial x_{m-1}}{\partial g}g^{(4)}(s).$$

Now, we use the expressions of the derivatives of $x_{m-1}(t)$

$$x_{m-1}^{(\alpha)}(t)=\int_a^b\frac{\partial^\alpha K(t,s,x_{m-2}(s),x_{m-2}(g(s)),x_{m-2}(a),x_{m-2}(b))}{\partial t^\alpha}ds+f^{(\alpha)}(t)\,,\quad \alpha=\overline{1,4}$$

and if we denote

$$M_1^S=\max_{\substack{|\alpha|\le 4\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right|,\quad |\alpha|=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\,,$$

$$M_2^S = \max_{\substack{\alpha\le 4\\ t\in[a,b]}} \left| f^{(\alpha)}(t) \right| \quad \text{and} \quad M_3^S = \max_{\substack{\alpha\le 4\\ t\in[a,b]}} \left| g^{(\alpha)}(t) \right| ,$$

then we obtain

$$\left| x_{m-1}^{(\alpha)}(t) \right| \le (b-a)M_1^S + M_2^S , \quad \alpha = \overline{1,4} ,$$

and for the derivative of the function K from the expression of the rest $R_{m,k}^S$ it results that:

$$\left[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b)) \right]_s^{iv} \le M_0^S .$$

It is obvious that M_0^S doesn't depend on m and k, so we have the estimation of the rest:

$$\left| R_{m,k}^S \right| \le M_0^S \cdot \frac{(b-a)^5}{2880 n^4}, \quad M_0^S = M_0^S(K, D^\alpha K, f, D^\alpha f), \quad |\alpha| \le 4 \tag{5.15}$$

and thus we obtain a formula for the approximate calculation of the integrals that appear in the successive approximations sequence (5.3').

Using the method of successive approximations and the formula (5.14) with the estimation of the rest resulted from (5.15), we suggest further on an algorithm in order to solve the integral equation (2.1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence. Thus we have:

$$x_0(t_k) = f(t_k)$$

$$x_1(t_k) = \int_a^b K(t_k, s, f(s), f(g(s)), f(a), f(b))ds + f(t_k) =$$

$$= \frac{b-a}{6n}\Big[K(t_k, a, f(a), f(g(a)), f(a), f(b)) + 2\sum_{i=1}^{n-1} K(t_k, t_i, f(t_i), f(g(t_i)), f(a), f(b)) +$$

$$+ 4\sum_{i=0}^{n-1} K\left(t_k, \frac{t_i + t_{i+1}}{2}, f\left(\frac{t_i + t_{i+1}}{2}\right), f\left(g\left(\frac{t_i + t_{i+1}}{2}\right)\right), f(a), f(b) \right) +$$

$$+ K(t_k, b, f(b), f(g(b)), f(a), f(b)) \Big] + f(t_k) + R_{1,k}^S =$$

$$= \widetilde{x}_1(t_k) + \widetilde{R}_{1,k}^S , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_a^b K(t_k, s, x_1(s), x_1(g(s)), x_1(a), x_1(b))ds + f(t_k) =$$

$$= \frac{b-a}{6n}\left[K(t_k, a, x_1(a), x_1(g(a)), x_1(a), x_1(b)) + 2\sum_{i=1}^{n-1} K(t_k, t_i, x_1(t_i), x_1(g(t_i)), x_1(a), x_1(b)) \right] +$$

$$+ 4\sum_{i=0}^{n-1} K\left(t_k, \frac{t_i + t_{i+1}}{2}, x_1\left(\frac{t_i + t_{i+1}}{2}\right), x_1\left(g\left(\frac{t_i + t_{i+1}}{2}\right)\right), x_1(a), x_1(b) \right) +$$

$$+K(t_k,b,x_1(b),x_1(g(b)),x_1(a),x_1(b))\big]+f(t_k)+R_{2,k}^S=$$

$$=\frac{b-a}{6n}\big[K(t_k,a,\widetilde{x}_1(a)+R_{1,0}^S,\widetilde{x}_1(g(a))+R_{1,0}^S,\widetilde{x}_1(a)+R_{1,0}^S,\widetilde{x}_1(b)+R_{1,0}^S)+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i)+R_{1,i}^S,\widetilde{x}_1(g(t_i))+R_{1,i}^S,\widetilde{x}_1(a)+R_{1,i}^S,\widetilde{x}_1(b)+R_{1,i}^S)+$$

$$+4\sum_{i=0}^{n-1}K\left(t_k,\frac{t_i+t_{i+1}}{2},\widetilde{x}_1\left(\frac{t_i+t_{i+1}}{2}\right)+R_{1,i,i+1}^S,\widetilde{x}_1\left(g\left(\frac{t_i+t_{i+1}}{2}\right)\right)+R_{1,i,i+1}^S,\right.$$

$$\left.\widetilde{x}_1(a)+R_{1,i,i+1}^S,\widetilde{x}_1(b)+R_{1,i,i+1}^S\right)+$$

$$+K(t_k,b,\widetilde{x}_1(b)+R_{1,n}^S,\widetilde{x}_1(g(b))+R_{1,n}^S,\widetilde{x}_1(a)+R_{1,n}^S,\widetilde{x}_1(b)+R_{1,n}^S)\big]+f(t_k)+R_{2,k}^S=$$

$$=\frac{b-a}{6n}\big[K(t_k,a,\widetilde{x}_1(a),\widetilde{x}_1(g(a)),\widetilde{x}_1(a),\widetilde{x}_1(b))+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_1(t_i),\widetilde{x}_1(g(t_i)),\widetilde{x}_1(a),\widetilde{x}_1(b))+$$

$$+4\sum_{i=0}^{n-1}K\left(t_k,\frac{t_i+t_{i+1}}{2},\widetilde{x}_1\left(\frac{t_i+t_{i+1}}{2}\right),\widetilde{x}_1\left(g\left(\frac{t_i+t_{i+1}}{2}\right)\right),\widetilde{x}_1(a),\widetilde{x}_1(b)\right)+$$

$$+K(t_k,b,\widetilde{x}_1(b),\widetilde{x}_1(g(b)),\widetilde{x}_1(a),\widetilde{x}_1(b))\big]+f(t_k)+\widetilde{R}_{2,k}^S=$$

$$=\widetilde{x}_2(t_k)+\widetilde{R}_{2,k}^S\ ,\quad k=\overline{0,n}$$

where

$$\left|\widetilde{R}_{2,k}^S\right|\le\frac{b-a}{6n}\cdot L_K\left(4\left|R_{1,0}^S\right|+8\sum_{i=1}^{n-1}\left|R_{1,i}^S\right|+16\sum_{i=0}^{n-1}\left|R_{1,i,i+1}^S\right|+4\left|R_{1,n}^S\right|\right)+\left|R_{2,k}^S\right|\le$$

$$\le 4(b-a)L_KM_0^S\cdot\frac{(b-a)^5}{2880n^4}+M_0^S\cdot\frac{(b-a)^5}{2880n^4}=$$

$$=\frac{(b-a)^5}{2880n^4}\cdot M_0^S\left[4L_K(b-a)+1\right].$$

The reasoning continues for $m=3,\ldots$ and through induction we obtain:

$$x_m(t_k)=\frac{b-a}{6n}\big[K(t_k,a,\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(g(a)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+2\sum_{i=1}^{n-1}K(t_k,t_i,\widetilde{x}_{m-1}(t_i),\widetilde{x}_{m-1}(g(t_i)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))+$$

$$+4\sum_{i=0}^{n-1}K\left(t_k,\frac{t_i+t_{i+1}}{2},\widetilde{x}_{m-1}\left(\frac{t_i+t_{i+1}}{2}\right),\widetilde{x}_{m-1}\left(g\left(\frac{t_i+t_{i+1}}{2}\right)\right),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b)\right)+$$

$$+K(t_k,a,\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(g(a)),\widetilde{x}_{m-1}(a),\widetilde{x}_{m-1}(b))]+f(t_k)+\widetilde{R}^S_{m,k}=$$

$$=\widetilde{x}_m(t_k)+\widetilde{R}^S_{m,k}\ ,\qquad k=\overline{0,n}\ ,$$

where

$$\left|\widetilde{R}^S_{m,k}\right|\le\frac{(b-a)^5}{2880n^4}\cdot M_0^S\cdot\left[4^{m-1}(b-a)^{m-1}L_K^{m-1}+.\ \ .\ \ .+1\right],\qquad k=\overline{0,n}$$

and according to contraction condition (h_6) it results the following estimate of the rest:

$$\left|\widetilde{R}^S_{m,k}\right|\le\frac{(b-a)^5}{2880n^4[1-4L_K(b-a)]}\cdot M_0^S\ . \tag{5.16}$$

Thus, using an equidistant division of the interval $[a,b]$ through the points $a=t_0<t_1<...<t_n=b$, we obtain the sequence $(\widetilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m(t_k))_{m\in N}$, $k=\overline{0,n}$ with the following error in calculation:

$$|x_m(t_k)-\widetilde{x}_m(t_k)|\le\frac{(b-a)^5}{2880n^4[1-4L_K(b-a)]}\cdot M_0^S\ . \tag{5.17}$$

Now, using the estimates (5.2) and (5.17) it results the error of approximation and we obtain the following theorem.

Theorem 5.4.1. *Suppose that the conditions of the theorem* 2.1.2 (***B*** = ***R***) *are fulfilled. Moreover, assume that the exact solution* x^* *of the integral equation* (2.1) *is approximated on the nodes* t_k , $k=\overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$ *by the sequence* $(\widetilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$ *using the successive approximations method* (5.3) *and the Simpson's formula* (1.27)+(1.28). *Under these conditions, the error of approximation is given by the evaluation*:

$$\left|x^*(t_k)-\widetilde{x}_m(t_k)\right|\le\frac{4^mL_K^m(b-a)^m}{1-4L_K(b-a)}|\,x_1-x_0\,|+\frac{(b-a)^5}{2880n^4[1-4L_K(b-a)]}\cdot M_0^S\,. \tag{5.18}$$

5.5 Example

We consider the integral equation with modified argument:

$$x(t)=\int_0^1\left[\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right]ds+\cos t,\qquad t\in[0,1] \tag{5.19}$$

where $K\in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$, $K(t,s,u_1,u_2,u_3,u_4)=\dfrac{\sin(u_1)+\cos(u_2)}{7}+\dfrac{u_3+u_4}{5}$,

$f\in C[0,1]$, $f(t)=\cos t$,

$g\in C([0,1],[0,1])$, $g(s)=s/2$ and $x\in C[0,1]$.

The existence and uniqueness of the solution of this integral equation was studied in the paragraph 2.4 from the chapter 2. The conditions of theorems 2.1.1 and 2.1.2 are fulfilled and therefore we establish under what conditions the integral equation (5.19) has a unique solution in the space $C[0,1]$ and in the sphere $\overline{B}(\cos t;r) \subset C[0,1]$, respectively.

We consider the case when the conditions of theorem 2.1.2 are fulfilled. So the integral equation (5.19) has a unique solution x^* in the sphere $\overline{B}(\cos t;r) \subset C[0,1]$. From the contraction condition $4L_K(b–a) = 4L_K = \frac{24}{35} < 1$, it results that $L_K = \frac{6}{35}$.

To determine x^* we apply the successive approximations method, starting at any element $x_0 \in \overline{B}(\cos t;r) \subset C[0,1]$, and if x_n is the n-th successive approximation, then the following estimation is true:

$$\left| x_n - x^* \right| \leq \frac{24^n}{35^{n-1} \cdot 11} \left| x_1 - x_0 \right| .$$

To calculate the integrals that appear in the terms of the successive approximations sequence, there have been used the following quadrature formulas: the trapezoids formula, the rectangles formula and the Simpson's formula.

Now, we have the sequence of successive approximations:

$$x_0(t) = \cos t$$

$$x_1(t) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos t =$$

$$= \int_0^1 \left[\frac{\sin(\cos(s)) + \cos(\cos(s/2))}{7} + \frac{\cos(0) + \cos(1)}{5} \right] ds + \cos t =$$

$$= \int_0^1 \left[\frac{\sin(\cos(s)) + \cos(\cos(s/2))}{7} + \frac{1 + 0{,}5403}{5} \right] ds + \cos t =$$

$$= \int_0^1 \left[\frac{\sin(\cos(s)) + \cos(\cos(s/2))}{7} + 0{,}308 \right] ds + \cos t = \qquad (5.20)$$

$$= \int_0^1 \frac{\sin(\cos(s)) + \cos(\cos(s/2))}{7} \, ds + 0{,}308 + \cos t$$

$$x_2(t) = \int_0^1 \left[\frac{\sin(x_1(s)) + \cos(x_1(s/2))}{7} + \frac{x_1(0) + x_1(1)}{5} \right] ds + \cos t$$

. . . . .

$$x_m(t) = \int_a^b \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos t$$

. . . . .

Also, to get a better approximation of the solution, it was considered an equidistant division of the interval [0,1] through the following points $0 = t_0 < t_1 < \ldots < t_n = 1$ and now, the successive approximations sequence will be:

$$x_0(t_k) = \cos t_k$$

$$x_0(s/2) = \cos(s/2)$$

$$x_0(0) = \cos(0)$$

$$x_0(1) = \cos(1)$$

$$x_1(t_k) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos t_k$$

$$x_1(s/2) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos(s/2)$$

$$x_1(0) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos\,(0)$$

$$x_1(1) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos\,(1) \qquad (5.20')$$

. . . . .

$$x_{m-1}(t_k) = \int_0^1 \left[\frac{\sin(x_{m-2}(s)) + \cos(x_{m-2}(s/2))}{7} + \frac{x_{m-2}(0) + x_{m-2}(1)}{5} \right] ds + \cos t_k$$

$$x_{m-1}(s/2) = \int_0^1 \left[\frac{\sin(x_{m-2}(s)) + \cos(x_{m-2}(s/2))}{7} + \frac{x_{m-2}(0) + x_{m-2}(1)}{5} \right] ds + \cos(s/2)$$

$$x_{m-1}(0) = \int_0^1 \left[\frac{\sin(x_{m-2}(s)) + \cos(x_{m-2}(s/2))}{7} + \frac{x_{m-2}(0) + x_{m-2}(1)}{5} \right] ds + \cos\,(0)$$

$$x_{m-1}(1) = \int_0^1 \left[\frac{\sin(x_{m-2}(s)) + \cos(x_{m-2}(s/2))}{7} + \frac{x_{m-2}(0) + x_{m-2}(1)}{5} \right] ds + \cos\,(1)$$

$$x_m(t_k) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos t_k$$

$$x_m(s/2) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos(s/2)$$

$$x_m(0) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos\,(0)$$

$$x_m(1) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos\,(1)$$

. . . . .

Next we present the method for approximating the solution of integral equation (5.19), obtained by the successive approximations method (5.20), combined with the trapezoids formula, the rectangles formula and the Simpson's formula, respectively.

A. The approximation of the solution using the trapezoids formula

We observe that the conditions (h_{11}), (h_{12}) and (h_{13}) are fulfilled. Using the quadrature formula of the trapezoids (1.14) to calculate the integrals that appear in the terms of the successive approximations sequence (5.20'), with the estimate of the rest given by (1.15), we will approximate the terms of this sequence. In general case for $x_m(t_k)$ we have:

$$x_m(t_k) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos t_k = \tag{5.21}$$

$$= \frac{1}{2n} \left[\frac{\sin(x_{m-1}(0)) + \cos(x_{m-1}(0))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} + \right.$$

$$+ 2\sum_{i=1}^{n-1} \left(\frac{\sin(x_{m-1}(t_i)) + \cos(x_{m-1}(t_i/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right) +$$

$$\left. + \frac{\sin(x_{m-1}(1)) + \cos(x_{m-1}(1/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] + \cos(t_k) + R_{m,k}^T \;, \quad k = \overline{0,n} \;, \quad m \in \mathbf{N}$$

with the estimate of the rest:

$$\left| R_{m,k}^T \right| \le \frac{1}{12n^2} \cdot \max_{s \in [0,1]} \left| \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right]_s^{"} \right| .$$

Using the expression of the derivative of the function K from the estimate of the rest $R_{m,k}^T$:

$$\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right]_s^{"} =$$

$$= \frac{1}{28} \left[-4\sin(x_{m-1}(s)) \cdot (x_{m-1}^{'}(s))^2 + 4\cos(x_{m-1}(s)) \cdot x_{m-1}^{"}(s) - \right.$$

$$\left. - \cos(x_{m-1}(s/2)) \cdot (x_{m-1}^{'}(s/2))^2 - \sin(x_{m-1}(s/2)) \cdot x_{m-1}^{"}(s/2) \right]$$

and the expressions of the derivatives of $x_{m-1}(t)$ and $x_{m-1}(t/2)$:

$$x'_{m-1}(t) = \int_0^1 \frac{\partial}{\partial t} \left(\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right) ds + (\cos t)' = -\sin t$$

$$x''_{m-1}(t)=\int_0^1\frac{\partial^2}{\partial t^2}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos t)''=-\cos t$$

$$x'_{m-1}(t/2)=\int_0^1\frac{\partial}{\partial t}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))'=-\frac{1}{2}\sin(t/2)$$

$$x''_{m-1}(t/2)=\int_0^1\frac{\partial^2}{\partial t^2}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))''=-\frac{1}{4}\cos(t/2)$$

and denoting

$$M_1^T=\max_{\substack{|\alpha|\le 2\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right|=$$

$$=\max_{t,s\in[0,1]}\left\{\left|\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right|,\right.$$

$$\left|\frac{2\cos(x_{m-1}(s))\cdot x'_{m-1}(s)-\sin(x_{m-1}(s/2))\cdot x'_{m-1}(s/2)}{14}\right|,$$

$$\frac{1}{28}\left|-4\sin(x_{m-1}(s))\cdot(x'_{m-1}(s))^2+4\cos(x_{m-1}(s))\cdot x''_{m-1}(s)-\right.$$

$$\left.-\cos(x_{m-1}(s/2))\cdot(x'_{m-1}(s/2))^2-\sin(x_{m-1}(s/2))\cdot x''_{m-1}(s/2)\right|,\left|\frac{\cos(x_{m-1}(s))}{7}\right|,$$

$$\left.\left|-\frac{\sin(x_{m-1}(s))}{7}\right|,\left|-\frac{\sin(x_{m-1}(s/2))}{7}\right|,\left|-\frac{\cos(x_{m-1}(s/2))}{7}\right|\right\},$$

$$M_2^T=\max_{\substack{\alpha\le 2\\ t\in[0,1]}}\left|f^{(\alpha)}(t)\right|=\max_{t\in[0,1]}\{|\cos t|,|-\sin t|,|-\cos t|\}=1,$$

$$M_3^T=\max_{\substack{\alpha\le 2\\ t\in[0,1]}}\left|g^{(\alpha)}(t)\right|=\max_{t\in[0,1]}\left\{\left|\frac{t}{2}\right|,\frac{1}{2},0\right\}=\frac{1}{2},$$

we obtain the estimates:

$$\left|x'_{m-1}(t)\right|=|-\sin t|\le \sin 1\le 0{,}841470985\ ,\quad t\in[0,1]$$

$$\left|x''_{m-1}(t)\right|=|-\cos t|\le 1\ ,\quad t\in[0,1]\ ,$$

$$\left|x'_{m-1}(t/2)\right|=\left|-\frac{1}{2}\sin(t/2)\right|\le\frac{1}{2}\sin(1/2)\le 0{,}239712770,\quad t\in[0,1]$$

$$\left|x''_{m-1}(t/2)\right|=\left|-\frac{1}{4}\cos(t/2)\right|\le 0{,}25,\quad t\in[0,1]$$

and

$$\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]_s^{''}\le$$

$$\le\frac{1}{28}\Big|-4\sin(x_{m-1}(s))\cdot(x_{m-1}^{'}(s))^2+4\cos(x_{m-1}(s))\cdot x_{m-1}^{''}(s)-$$

$$-\cos(x_{m-1}(s/2))\cdot(x_{m-1}^{'}(s/2))^2-\sin(x_{m-1}(s/2))\cdot x_{m-1}^{''}(s/2)\Big|\le$$

$$\le\frac{1}{28}\left(4\cdot1\cdot\sin^2 1+4\cdot1\cdot1+1\cdot\left(\frac{1}{2}\right)^2\cdot\sin^2(1/2)+1\cdot\frac{1}{4}\cdot1\right)=$$

$$=\frac{16\sin^2 1+\sin^2(1/2)+17}{112}=M_0^T.$$

It is observed that M_0^T doesn't depend on *m* and *k*. Hence, we have the estimate of the rest:`

$$\left|R_{m,k}^T\right|\le\frac{16\sin^2 1+\sin^2(1/2)+17}{112}\cdot\frac{1}{12n^2}\le 0{,}0212492735\cdot\frac{1}{n^2} \tag{5.22}$$

and we obtain a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence.

Using the method of successive approximations and the formula (5.21) with the estimate of the rest resulted from (5.22), we suggest further on an algorithm in order to solve the integral equation (5.19) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we will obtain:

$x_0(t_k)=\cos t_k$

$$x_1(t_k)=\int_0^1\left[\frac{\sin(x_0(s))+\cos(x_0(s/2))}{7}+\frac{x_0(0)+x_0(1)}{5}\right]ds+\cos t_k=$$

$$=\frac{1}{2n}\left[\left(\frac{\sin(x_0(0))+\cos(x_0(0))}{7}+\frac{x_0(0)+x_0(1)}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(x_0(t_i))+\cos(x_0(t_i/2))}{7}+\frac{x_0(0)+x_0(1)}{5}\right)+$$

$$\left.+\left(\frac{\sin(x_0(1))+\cos(x_0(1/2))}{7}+\frac{x_0(0)+x_0(1)}{5}\right)\right]+\cos t_k+R_{1,k}^T=$$

$$=\widetilde{x}_1(t_k)+\widetilde{R}_{1,k}^T\ ,\quad k=\overline{0,n}$$

$$x_2(t_k)=\int_0^1\left[\frac{\sin(x_1(s))+\cos(x_1(s/2))}{7}+\frac{x_1(0)+x_1(1)}{5}\right]ds+\cos t_k=$$

$$=\frac{1}{2n}\left[\left(\frac{\sin(x_1(0))+\cos(x_1(0))}{7}+\frac{x_1(0)+x_1(1)}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(x_1(t_i))+\cos(x_1(t_i/2))}{7}+\frac{x_1(0)+x_1(1)}{5}\right)+$$

$$\left.+\left(\frac{\sin(x_1(1))+\cos(x_1(1/2))}{7}+\frac{x_1(0)+x_1(1)}{5}\right)\right]+\cos t_k+R_{2,k}^T=$$

$$=\frac{1}{2n}\left[\left(\frac{\sin(\widetilde{x}_1(0)+R_{1,0}^T)+\cos(\widetilde{x}_1(0)+R_{1,0}^T)}{7}+\frac{\widetilde{x}_1(0)+R_{1,0}^T+\widetilde{x}_1(1)+R_{1,0}^T}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_1(t_i)+R_{1,i}^T)+\cos(\widetilde{x}_1(t_i/2)+R_{1,i}^T)}{7}+\frac{\widetilde{x}_1(0)+R_{1,i}^T+\widetilde{x}_1(1)+R_{1,i}^T}{5}\right)+$$

$$\left.+\left(\frac{\sin(\widetilde{x}_1(1)+R_{1,n}^T)+\cos(\widetilde{x}_1(1/2)+R_{1,n}^T)}{7}+\frac{\widetilde{x}_1(0)+R_{1,n}^T+\widetilde{x}_1(1)+R_{1,n}^T}{5}\right)\right]+\cos t_k+R_{2,k}^T=$$

$$=\frac{1}{2n}\left[\left(\frac{\sin(\widetilde{x}_1(0))+\cos(\widetilde{x}_1(0))}{7}+\frac{\widetilde{x}_1(0)+\widetilde{x}_1(1)}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_1(t_i))+\cos(\widetilde{x}_1(t_i/2))}{7}+\frac{\widetilde{x}_1(0)+\widetilde{x}_1(1)}{5}\right)+$$

$$\left.+\left(\frac{\sin(\widetilde{x}_1(1))+\cos(\widetilde{x}_1(1/2))}{7}+\frac{\widetilde{x}_1(0)+\widetilde{x}_1(1)}{5}\right)\right]+\cos t_k+\widetilde{R}_{2,k}^T=$$

$$=\widetilde{x}_2(t_k)+\widetilde{R}_{2,k}^T\ ,\qquad k=\overline{0,n}\ ,$$

where

$$\left|\widetilde{R}_{2,k}^T\right|\le\frac{1}{2n}\cdot L_K\left(4\left|R_{1,0}^T\right|+8\sum_{i=1}^{n-1}\left|R_{1,i}^T\right|+4\left|R_{1,n}^T\right|\right)+\left|R_{2,k}^T\right|\le$$

$$\le\frac{24}{35}\cdot\frac{16\sin^2 1+\sin^2(1/2)+17}{1344n^2}+\frac{16\sin^2 1+\sin^2(1/2)+17}{1344n^2}\le$$

$$\le 0{,}035820204\cdot\frac{1}{n^2}.$$

The reasoning continues for $m=3,\ldots$ and through induction we obtain:

$$x_m(t_k)=\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds+\cos t_k=$$

$$=\frac{1}{2n}\left[\left(\frac{\sin(\widetilde{x}_{m-1}(0))+\cos(\widetilde{x}_{m-1}(0))}{7}+\frac{\widetilde{x}_{m-1}(0)+\widetilde{x}_{m-1}(1)}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_{m-1}(t_i))+\cos(\widetilde{x}_{m-1}(t_i/2))}{7}+\frac{\widetilde{x}_{m-1}(0)+\widetilde{x}_{m-1}(1)}{5}\right)+$$

$$\left.+\left(\frac{\sin(\widetilde{x}_{m-1}(1))+\cos(\widetilde{x}_{m-1}(1/2))}{7}+\frac{\widetilde{x}_{m-1}(0)+\widetilde{x}_{m-1}(1)}{5}\right)\right]+\cos t_k+\widetilde{R}^T_{m,k}=$$

$$=\widetilde{x}_m(t_k)+\widetilde{R}^T_{m,k}\ ,\quad k=\overline{0,n}\ ,$$

where

$$\left|\widetilde{R}^T_{m,k}\right|\le\frac{16\sin^2 1+\sin^2(1/2)+17}{1344n^2}\cdot\left[\left(\frac{24}{35}\right)^{m-1}+.\ \ .\ \ .+1\right],\qquad k=\overline{0,n}\ .$$

Hence, we have the following estimate of the rest:

$$\left|\widetilde{R}^T_{m,k}\right|\le\frac{16\sin^2 1+\sin^2(1/2)+17}{1344n^2}\cdot\frac{1}{1-\dfrac{24}{35}}\le 0{,}0676113247\cdot\frac{1}{n^2}\tag{5.23}$$

and thus, we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, that approximate the successive approximations sequence $\left(x_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$ on the nodes t_k, $k=\overline{0,n}$, with the error:

$$\left|x_m(t_k)-\widetilde{x}_m(t_k)\right|\le 0{,}0676113247\cdot\frac{1}{n^2}\ .\tag{5.24}$$

Now, using the successive approximations method (5.20) combined with the trapezoids method (1.14)+(1.15) and the theorem 5.2.1, it results that the error of approximation of the exact solution x^* of the integral equation (5.19) through the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, on the nodes of an equidistant division of the interval [0,1], is given by the evaluation:

$$\left|x^*(t_k)-\widetilde{x}_m(t_k)\right|\le\frac{24^m}{35^{m-1}\cdot 11}\left|x_1-x_0\right|+0{,}0676113247\cdot\frac{1}{n^2}\ .\tag{5.25}$$

B. The approximation of the solution using the rectangles formula

We observe that the conditions (h_{21}), (h_{22}) and (h_{23}) are fulfilled. We will approximate the terms of the successive approximations sequence (5.20’) using the rectangles formula (1.21) with the estimate of the rest given by (1.22), considering the intermediary points of the division of the interval [0,1] on the left end of the partial intervals $\xi_I=t_i$.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k)=\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds+\cos t_k=\tag{5.26}$$

$$= \frac{1}{n}\left[\frac{\sin(x_{m-1}(0)) + \cos(x_{m-1}(0))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} + \right.$$

$$\left. + \sum_{i=1}^{n-1}\left(\frac{\sin(x_{m-1}(t_i)) + \cos(x_{m-1}(t_i/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right)\right] +$$

$$+ \cos(t_k) + R_{m,k}^D \ , \quad k = \overline{0,n} \ , \quad m \in N$$

with the estimate of the rest:

$$\left|R_{m,k}^D\right| \le \frac{1}{n} \cdot \max_{s\in[0,1]}\left|\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right]_s^{'}\right| .$$

Using the expression of the derivative of the function K from the estimate of the rest $R_{m,k}^D$:

$$\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right]_s^{'} =$$

$$= \frac{2\cos(x_{m-1}(s)) \cdot x_{m-1}^{'}(s) - \sin(x_{m-1}(s/2)) \cdot x_{m-1}^{'}(s/2)}{14}$$

and the expressions of the derivatives of $x_{m-1}(t)$ and $x_{m-1}(t/2)$:

$$x'_{m-1}(t) = \int_0^1 \frac{\partial}{\partial t}\left(\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right)ds + (\cos t)' = -\sin t$$

$$x'_{m-1}(t/2) = \int_0^1 \frac{\partial}{\partial t}\left(\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right)ds + (\cos(t/2))' = -\frac{1}{2}\sin(t/2)$$

and denoting

$$M_1^D = \max_{\substack{|\alpha|\le 1\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right| =$$

$$= \max_{t,s\in[0,1]}\left\{\left|\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right|,\right.$$

$$\left|\frac{2\cos(x_{m-1}(s)) \cdot x_{m-1}^{'}(s) - \sin(x_{m-1}(s/2)) \cdot x_{m-1}^{'}(s/2)}{14}\right| ,$$

$$\left.\left|\frac{\cos(x_{m-1}(s))}{7}\right| , \left|-\frac{\sin(x_{m-1}(s/2))}{7}\right|\right\} ,$$

$$M_2^D = \max_{\substack{\alpha\le 1\\ t\in[a,b]}}\left|f^{(\alpha)}(t)\right| = \max_{t\in[0,1]}\left\{\left|\cos t\right| , \left|-\sin t\right|\right\} = 1 \ ,$$

$$M_3^D = \max_{\substack{\alpha\le 1\\ t\in[a,b]}} \left| g^{(\alpha)}(t) \right| = \max_{t\in[0,1]} \left\{ \left| \frac{t}{2} \right|, \frac{1}{2} \right\} = \frac{1}{2} \ ,$$

we obtain the estimates:

$$\left| x'_{m-1}(t) \right| = \left| -\sin t \right| \le \sin 1 \le 0{,}841470985 \ , \quad t \in [0,1]$$

$$\left| x'_{m-1}(t/2) \right| \ = \ \left| -\frac{1}{2}\sin(t/2) \right| \ \le \ \frac{1}{2}\sin\,(1/2) \ \le \ 0{,}239712770 \ , \quad t \in [0,1]$$

and

$$\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right]'_s \le$$

$$\le \frac{1}{14} \left| 2\cos(x_{m-1}(s)) \cdot x'_{m-1}(s) - \sin(x_{m-1}(s/2)) \cdot x'_{m-1}(s/2) \right| \le$$

$$\le \frac{1}{14} \left(2 \cdot 1 \cdot \sin 1 + 1 \cdot \frac{1}{2} \cdot \sin(1/2) \right) = \frac{4\sin 1 + \sin(1/2)}{28} = M_0^D \ .$$

It is observed that M_0^D doesn't depend on *m* and *k*. Hence, we have the estimate of the rest:

$$\left| R_{m,k}^D \right| \le \frac{4\sin 1 + \sin(1/2)}{28} \cdot \frac{1}{n} \le 0{,}1373324814 \cdot \frac{1}{n} \ , \qquad (5.27)$$

and we obtain a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence.

Using the successive approximations method and the formula (5.26) with the estimate of the rest resulted from (5.27), we suggest further on an algorithm in order to solve the integral equation (5.19) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we will obtain:

$$x_0(t_k) = \cos t_k$$

$$x_1(t_k) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos t_k =$$

$$= \frac{1}{n} \left[\left(\frac{\sin(x_0(0)) + \cos(x_0(0))}{7} + \frac{x_0(0) + x_0(1)}{5} \right) + \right.$$

$$\left. + \sum_{i=1}^{n-1} \left(\frac{\sin(x_0(t_i)) + \cos(x_0(t_i/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right) \right] + \cos t_k \ + \ R_{1,k}^D \ =$$

$$= \widetilde{x}_1(t_k) + \widetilde{R}_{1,k}^D \ , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_0^1 \left[\frac{\sin(x_1(s)) + \cos(x_1(s/2))}{7} + \frac{x_1(0) + x_1(1)}{5} \right] ds + \cos t_k =$$

$$= \frac{1}{n}\left[\left(\frac{\sin(x_1(0)) + \cos(x_1(0))}{7} + \frac{x_1(0) + x_1(1)}{5}\right) + \right.$$

$$\left. + \sum_{i=1}^{n-1}\left(\frac{\sin(x_1(t_i)) + \cos(x_1(t_i/2))}{7} + \frac{x_1(0) + x_1(1)}{5}\right)\right] + \cos t_k + R_{2,k}^D =$$

$$= \frac{1}{n}\left[\left(\frac{\sin(\widetilde{x}_1(0) + R_{1,0}^D) + \cos(\widetilde{x}_1(0) + R_{1,0}^D)}{7} + \frac{\widetilde{x}_1(0) + R_{1,0}^D + \widetilde{x}_1(1) + R_{1,0}^D}{5}\right) + \right.$$

$$\left. + \sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_1(t_i) + R_{1,i}^D) + \cos(\widetilde{x}_1(t_i/2) + R_{1,i}^D)}{7} + \frac{\widetilde{x}_1(0) + R_{1,i}^D + \widetilde{x}_1(1) + R_{1,i}^D}{5}\right)\right] + \cos t_k + R_{2,k}^D =$$

$$= \frac{1}{n}\left[\left(\frac{\sin(\widetilde{x}_1(0)) + \cos(\widetilde{x}_1(0))}{7} + \frac{\widetilde{x}_1(0) + \widetilde{x}_1(1)}{5}\right) + \right.$$

$$\left. + \sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_1(t_i)) + \cos(\widetilde{x}_1(t_i/2))}{7} + \frac{\widetilde{x}_1(0) + \widetilde{x}_1(1)}{5}\right)\right] + \cos t_k + \widetilde{R}_{2,k}^D =$$

$$= \widetilde{x}_2(t_k) + \widetilde{R}_{2,k}^D, \qquad k = \overline{0,n}$$

with the estimate of the rest:

$$\left|\widetilde{R}_{2,k}^D\right| \le \frac{1}{n} \cdot L_K\left(4\left|R_{1,0}^D\right| + 4\sum_{i=1}^{n-1}\left|R_{1,i}^D\right|\right) + \left|R_{2,k}^D\right| \le$$

$$\le \frac{24}{35} \cdot \frac{4\sin 1 + \sin(1/2)}{28n} + \frac{4\sin 1 + \sin(1/2)}{28n} \le 0{,}23150332571\frac{1}{n}.$$

The reasoning continues for $m = 3, \ldots$ and through induction we obtain:

$$x_m(t_k) = \int_0^1\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right]ds + \cos t_k =$$

$$= \frac{1}{n}\left[\left(\frac{\sin(\widetilde{x}_{m-1}(0)) + \cos(\widetilde{x}_{m-1}(0))}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5}\right) + \right.$$

$$\left. + \sum_{i=1}^{n-1}\left(\frac{\sin(\widetilde{x}_{m-1}(t_i)) + \cos(\widetilde{x}_{m-1}(t_i/2))}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5}\right)\right] +$$

$$+ \cos t_k + \widetilde{R}_{m,k}^D = \widetilde{x}_m(t_k) + \widetilde{R}_{m,k}^D, \qquad k = \overline{0,n}$$

where

$$\left|\widetilde{R}_{m,k}^D\right| \le \frac{4\sin 1 + \sin(1/2)}{28n} \cdot \left[\left(\frac{24}{35}\right)^{m-1} + \ldots + 1\right], \qquad k = \overline{0,n}.$$

Hence, we have the following estimate of the rest:

$$\left|\widetilde{R}^{D}_{m,k}\right| \le \frac{4\sin 1+\sin(1/2)}{28n}\cdot\frac{1}{1-\frac{24}{35}} \le 0{,}43696698612\cdot\frac{1}{n} \tag{5.28}$$

and thus, we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, that approximate the successive approximations sequence $\left(x_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$ on the nodes t_k, $k=\overline{0,n}$, with the error:

$$\left|x_m(t_k)-\widetilde{x}_m(t_k)\right| \le 0{,}43696698612\cdot\frac{1}{n}\ . \tag{5.29}$$

Now, using the successive approximations method (5.20) combined with the rectangles method (1.21)+(1.22) and the theorem 5.3.1, it results that the error of approximation of the exact solution x^* of the integral equation (5.19) by the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k=\overline{0,n}$, on the nodes of an equidistant division of the interval [0,1], is given by the evaluation:

$$\left|x^*(t_k)-\widetilde{x}_m(t_k)\right| \le \frac{24^m}{35^{m-1}\cdot 11}\left|x_1-x_0\right| + 0{,}43696698612\cdot\frac{1}{n}. \tag{5.30}$$

C. The approximation of the solution using the Simpson's formula

We observe that the conditions (h_{31}), (h_{32}) şi (h_{33}) are fulfilled. Using the Simpson's quadrature formula (1.27) to calculate the integrals that appear in the terms of the successive approximations sequence (5.20'), with the estimate of the rest given by (1.28), we will approximate the terms of this sequence.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k)=\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds+\cos t_k = \tag{5.31}$$

$$=\frac{1}{6n}\left[\frac{\sin(x_{m-1}(0))+\cos(x_{m-1}(0))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(x_{m-1}(t_i))+\cos(x_{m-1}(t_i/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)+$$

$$+4\sum_{i=0}^{n-1}\left(\frac{\sin\left(x_{m-1}\left(\frac{t_i+t_{i+1}}{2}\right)\right)+\cos\left(x_{m-1}\left(\frac{t_i+t_{i+1}}{4}\right)\right)}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)+$$

$$\left.+\frac{\sin(x_{m-1}(1))+\cos(x_{m-1}(1/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]+\cos(t_k)+R^S_{m,k}\ ,\quad k=\overline{0,n}\ ,\quad m\in N$$

with the estimate of the rest:

$$\left|R^S_{m,k}\right| \le \frac{1}{2880\cdot n^4}\cdot\max_{s\in[a,b]}\left|\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]^{iv}_s\right|\ .$$

Using the expression of the derivative of the function K from the estimate of the rest $R^S_{m,k}$:

$$\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]_s^{iv}=$$

$$=\frac{1}{112}\Big[16\sin(x_{m-1}(s))\cdot(x'_{m-1}(s))^4-96\cos(x_{m-1}(s))\cdot(x'_{m-1}(s))^2\cdot x''_{m-1}(s)+$$

$$-48\sin(x_{m-1}(s))\cdot(x''_{m-1}(s))^2-64\sin(x_{m-1}(s/2))\cdot x'_{m-1}(s/2)\cdot x'''_{m-1}(s/2)-$$

$$+16\cos(x_{m-1}(s))\cdot x^{iv}_{m-1}(s)+\cos(x_{m-1}(s/2))\cdot(x'_{m-1}(s/2))^4-$$

$$+6\sin(x_{m-1}(s/2))\cdot(x'_{m-1}(s/2))^2\cdot x''_{m-1}(s/2)-3\cos(x_{m-1}(s/2))\cdot(x''_{m-1}(s/2))^2-$$

$$-4\cos(x_{m-1}(s/2))\cdot x'_{m-1}(s/2)\cdot x'''_{m-1}(s/2)-\sin(x_{m-1}(s/2))\cdot x^{iv}_{m-1}(s/2)\Big]$$

and the expressions of the derivatives of $x_{m-1}(t)$ and $x_{m-1}(t/2)$:

$$x'_{m-1}(t)=\int_0^1\frac{\partial}{\partial t}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos t)'=-\sin t$$

$$x''_{m-1}(t)=\int_0^1\frac{\partial^2}{\partial t^2}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos t)''=-\cos t$$

$$x'''_{m-1}(t)=\int_0^1\frac{\partial^3}{\partial t^3}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos t)'''=\sin t$$

$$x^{iv}_{m-1}(t)=\int_0^1\frac{\partial^4}{\partial t^4}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos t)^{iv}=\cos t$$

$$x'_{m-1}(t/2)=\int_0^1\frac{\partial}{\partial t}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))'=-\frac{1}{2}\sin(t/2)$$

$$x''_{m-1}(t/2)=\int_0^1\frac{\partial^2}{\partial t^2}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))''=-\frac{1}{4}\cos(t/2)$$

$$x'''_{m-1}(t/2)=\int_0^1\frac{\partial^3}{\partial t^3}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))'''=\frac{1}{8}\sin(t/2)$$

$$x^{iv}_{m-1}(t/2)=\int_0^1\frac{\partial^4}{\partial t^4}\left(\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)ds+(\cos(t/2))^{iv}=\frac{1}{16}\cos(t/2)$$

and denoting

$$M_1^S=\max_{\substack{|\alpha|\le 4\\ t,s\in[a,b]}}\left|\frac{\partial^\alpha K(t,s,u_1,u_2,u_3,u_4)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right|,$$

$$M_2^S = \max_{\substack{\alpha\le 4\\ t\in[a,b]}} \left| f^{(\alpha)}(t) \right| = \max_{t\in[0,1]} \{ |\cos t|, |-\sin t|, |-\cos t|, |-\sin t|, |-\cos t| \} = 1$$

$$M_3^S = \max_{\substack{\alpha\le 4\\ t\in[a,b]}} \left| g^{(\alpha)}(t) \right| = \max_{t\in[0,1]} \left\{ \left| \frac{t}{2} \right|, \frac{1}{2}, 0 \right\} = \frac{1}{2} ,$$

we obtain the estimates:

$$\left| x'_{m-1}(t) \right| = |-\sin t| \le \sin\ 1 \le 0{,}841470985 \ , \quad t\in[0,1]$$

$$\left| x''_{m-1}(t) \right| = |-\cos t| \le 1 \ , \quad t\in[0,1] \ ,$$

$$\left| x'''_{m-1}(t) \right| = |\sin t| \le \sin\ 1 \le 0{,}841470985 \ , \quad t\in[0,1]$$

$$\left| x^{iv}_{m-1}(t) \right| = |\cos t| \le 1 \ , \quad t\in[0,1] \ ,$$

$$\left| x'_{m-1}(t/2) \right| = \left| -\frac{1}{2}\sin(t/2) \right| \le \frac{1}{2}\sin\ (1/2) \le 0{,}239712770 \ , \quad t\in[0,1]$$

$$\left| x''_{m-1}(t/2) \right| = \left| -\frac{1}{4}\cos(t/2) \right| \le 0{,}25 \ , \quad t\in[0,1] \ ,$$

$$\left| x'''_{m-1}(t/2) \right| = \left| \frac{1}{8}\sin(t/2) \right| \le \frac{1}{8}\sin\ (1/2) \le 0{,}05992819233 \ , \quad t\in[0,1]$$

$$\left| x^{iv}_{m-1}(t/2) \right| = \left| \frac{1}{16}\cos(t/2) \right| \le 0{,}0625 \ , \quad t\in[0,1]$$

and

$$\left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right]_s^{iv} \le$$

$$\le \frac{1}{112}\Big[16\sin^4 1 + 96\sin^2 1 + 48 + 64\frac{1}{2}\sin(1/2)\cdot\frac{1}{8}\sin(1/2) + 16 + \frac{1}{2^4}\sin^4(1/2) +$$

$$+ 3\frac{1}{4^2} + 6\frac{1}{2^2}\sin^2(1/2)\cdot\frac{1}{4} + 4\frac{1}{2}\sin(1/2)\cdot\frac{1}{8}\sin(1/2) + \frac{1}{16} \Big] =$$

$$= \frac{256\sin^4 1 + 1536\sin^2 1 + \sin^4(1/2) + 80\sin^2(1/2) + 1028}{1792} = M_0^S$$

and we observe that M_0^T doesn't depend on *m* and *k*.

Now, we have the following estimation of the rest:

$$\left| R_{m,k}^S \right| \le \frac{256\sin^4 1 + 1536\sin^2 1 + \sin^4(1/2) + 80\sin^2(1/2) + 1028}{1792} \cdot \frac{1}{2880 n^4} \qquad (5.32)$$

and we obtain a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence.

Using the successive approximations method and the formula (5.31) with the estimate of the rest resulted from (5.32), we obtain an algorithm which solve the integral equation (5.19) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we will obtain:

$$x_0(t_k) = \cos t_k$$

$$x_1(t_k) = \int_0^1 \left[\frac{\sin(x_0(s)) + \cos(x_0(s/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right] ds + \cos t_k =$$

$$= \frac{1}{6n} \left[\left(\frac{\sin(x_0(0)) + \cos(x_0(0))}{7} + \frac{x_0(0) + x_0(1)}{5} \right) + \right.$$

$$+ 2\sum_{i=1}^{n-1} \left(\frac{\sin(x_0(t_i)) + \cos(x_0(t_i/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right) +$$

$$+ 4\sum_{i=0}^{n-1} \left(\frac{\sin\left(x_0\left(\frac{t_i + t_{i+1}}{2}\right)\right) + \cos\left(x_0\left(\frac{t_i + t_{i+1}}{4}\right)\right)}{7} + \frac{x_0(0) + x_0(1)}{5} \right) +$$

$$\left. + \left(\frac{\sin(x_0(1)) + \cos(x_0(1/2))}{7} + \frac{x_0(0) + x_0(1)}{5} \right) \right] + \cos t_k + R_{1,k}^S =$$

$$= \widetilde{x}_1(t_k) + \widetilde{R}_{1,k}^S, \qquad k = \overline{0,n}$$

$$x_2(t_k) = \int_0^1 \left[\frac{\sin(x_1(s)) + \cos(x_1(s/2))}{7} + \frac{x_1(0) + x_1(1)}{5} \right] ds + \cos t_k =$$

$$= \frac{1}{6n} \left[\left(\frac{\sin(x_1(0)) + \cos(x_1(0))}{7} + \frac{x_1(0) + x_1(1)}{5} \right) + \right.$$

$$+ 2\sum_{i=1}^{n-1} \left(\frac{\sin(x_1(t_i)) + \cos(x_1(t_i/2))}{7} + \frac{x_1(0) + x_1(1)}{5} \right) +$$

$$+ 4\sum_{i=0}^{n-1} \left(\frac{\sin\left(x_1\left(\frac{t_i + t_{i+1}}{2}\right)\right) + \cos\left(x_1\left(\frac{t_i + t_{i+1}}{4}\right)\right)}{7} + \frac{x_1(0) + x_1(1)}{5} \right) +$$

$$\left. + \left(\frac{\sin(x_1(1)) + \cos(x_1(1/2))}{7} + \frac{x_1(0) + x_1(1)}{5} \right) \right] + \cos t_k + R_{2,k}^S =$$

$$= \frac{1}{6n}\left[\left(\frac{\sin(\tilde{x}_1(0)+R^S_{1,0})+\cos(\tilde{x}_1(0)+R^S_{1,0})}{7}+\frac{\tilde{x}_1(0)+R^S_{1,0}+\tilde{x}_1(1)+R^S_{1,0}}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(\tilde{x}_1(t_i)+R^S_{1,i})+\cos(\tilde{x}_1(t_i/2)+R^S_{1,i})}{7}+\frac{\tilde{x}_1(0)+R^S_{1,i}+\tilde{x}_1(1)+R^S_{1,i}}{5}\right)+$$

$$+4\sum_{i=0}^{n-1}\left(\frac{\sin\left(\tilde{x}_1\left(\frac{t_i+t_{i+1}}{2}\right)+R^S_{1,i,i+1}\right)+\cos\left(\tilde{x}_1\left(\frac{t_i+t_{i+1}}{4}\right)+R^S_{1,i,i+1}\right)}{7}+\right.$$

$$\left.+\frac{\tilde{x}_1(0)+R^S_{1,i,i+1}+\tilde{x}_1(1)+R^S_{1,i,i+1}}{5}\right)+$$

$$\left.+\left(\frac{\sin(\tilde{x}_1(1)+R^S_{1,n})+\cos(\tilde{x}_1(1/2)+R^S_{1,n})}{7}+\frac{\tilde{x}_1(0)+R^S_{1,n}+\tilde{x}_1(1)+R^S_{1,n}}{5}\right)\right]+\cos t_k+R^S_{2,k}=$$

$$= \frac{1}{6n}\left[\left(\frac{\sin(\tilde{x}_1(0))+\cos(\tilde{x}_1(0))}{7}+\frac{\tilde{x}_1(0)+\tilde{x}_1(1)}{5}\right)+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(\tilde{x}_1(t_i))+\cos(\tilde{x}_1(t_i/2))}{7}+\frac{\tilde{x}_1(0)+\tilde{x}_1(1)}{5}\right)+$$

$$+4\sum_{i=0}^{n-1}\left(\frac{\sin\left(\tilde{x}_1\left(\frac{t_i+t_{i+1}}{2}\right)\right)+\cos\left(\tilde{x}_1\left(\frac{t_i+t_{i+1}}{4}\right)\right)}{7}+\frac{\tilde{x}_1(0)+\tilde{x}_1(1)}{5}\right)+$$

$$\left.+\left(\frac{\sin(\tilde{x}_1(1))+\cos(\tilde{x}_1(1/2))}{7}+\frac{\tilde{x}_1(0)+\tilde{x}_1(1)}{5}\right)\right]+\cos t_k+\tilde{R}^S_{2,k}=$$

$$= \tilde{x}_2(t_k)+\tilde{R}^S_{2,k}, \qquad k=\overline{0,n}$$

with the estimate of the rest:

$$\left|\tilde{R}^S_{2,k}\right| \le \frac{1}{6n}\cdot L_K\left(4\left|R^S_{1,0}\right|+8\sum_{i=1}^{n-1}\left|R^S_{1,i}\right|+16\sum_{i=0}^{n-1}\left|R^S_{1,i,i+1}\right|+4\left|R^S_{1,n}\right|\right)+\left|R^S_{2,k}\right| \le$$

$$\le \frac{24}{35}\cdot\frac{256\sin^4 1+1536\sin^2 1+\sin^4(1/2)+80\sin^2(1/2)+1028}{1792}\cdot\frac{1}{2880n^4}+$$

$$+\frac{256\sin^4 1+1536\sin^2 1+\sin^4(1/2)+80\sin^2(1/2)+1028}{1792}\cdot\frac{1}{2880n^4}\le$$

$$\leq \frac{59\left(256\sin^4 1 + 1536\sin^2 1 + \sin^4(1/2) + 80\sin^2(1/2) + 1028\right)}{180633600n^4} \leq$$

$$\leq 0{,}00073896059 \cdot \frac{1}{n^4} \ .$$

The reasoning continues for $m = 3, \ldots$ and through induction we obtain:

$$x_m(t_k) = \int_0^1 \left[\frac{\sin(x_{m-1}(s)) + \cos(x_{m-1}(s/2))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} \right] ds + \cos t_k =$$

$$= \frac{1}{6n}\left[\left(\frac{\sin(\widetilde{x}_{m-1}(0)) + \cos(\widetilde{x}_{m-1}(0))}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5} \right) + \right.$$

$$+ 2\sum_{i=1}^{n-1} \left(\frac{\sin(\widetilde{x}_{m-1}(t_i)) + \cos(\widetilde{x}_{m-1}(t_i/2))}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5} \right) +$$

$$+ 4\sum_{i=0}^{n-1} \left(\frac{\sin\left(\widetilde{x}_{m-1}\left(\frac{t_i + t_{i+1}}{2} \right) \right) + \cos\left(\widetilde{x}_{m-1}\left(\frac{t_i + t_{i+1}}{4} \right) \right)}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5} \right) +$$

$$\left. + \left(\frac{\sin(\widetilde{x}_{m-1}(1)) + \cos(\widetilde{x}_{m-1}(1/2))}{7} + \frac{\widetilde{x}_{m-1}(0) + \widetilde{x}_{m-1}(1)}{5} \right) \right] + \cos t_k + R_{1,k}^S =$$

$$= \widetilde{x}_m(t_k) + \widetilde{R}_{m,k}^S, \qquad k = \overline{0,n} \ ,$$

where

$$\left| \widetilde{R}_{m,k}^S \right| \leq \frac{256\sin^4 1 + 1536\sin^2 1 + \sin^4(1/2) + 80\sin^2(1/2) + 1028}{5160960n^4} \cdot \left[\left(\frac{24}{35} \right)^{m-1} + . \ \ . \ \ . + 1 \right], \quad k = \overline{0,n} \ .$$

Hence, we have the following estimate of the rest:

$$\left| \widetilde{R}_{m,k}^S \right| \leq \frac{35\left(256\sin^4 1 + 1536\sin^2 1 + \sin^4(1/2) + 80\sin^2(1/2) + 1028\right)}{56770560n^4} \leq \tag{5.33}$$

$$\leq 0{,}00139480234 \frac{1}{n^4}$$

and thus, we obtain the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in N}$, $k = \overline{0,n}$, that approximate the successive approximations sequence $\left(x_m(t_k)\right)_{m\in N}$, $k = \overline{0,n}$ on the nodes t_k, $k = \overline{0,n}$, with the error:

$$\left| x_m(t_k) - \widetilde{x}_m(t_k) \right| \leq 0{,}00139480234 \frac{1}{n^4} \ . \tag{5.34}$$

Now, using the successive approximations method (5.20) combined with the Simpson's formula (1.27)+(1.28) and the theorem 5.4.1, it results that the error of approximation of the exact solution x^* of the integral equation (5.19) by the sequence $(\widetilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, on the nodes of an equidistant division of the interval [0,1], is given by the evaluation:

$$\left|x^*(t_k)-\widetilde{x}_m(t_k)\right| \le \frac{24^m}{35^{m-1}\cdot 11}\left| x_1 - x_0 \right| + 0,00139480234\frac{1}{n^4}. \tag{5.35}$$

D. Conclusions

The integral equation with modified argument (5.19), considered in this example:

$$x(t)=\int_0^1\left[\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right]ds+\cos t, \quad t\in[0,1],$$

has a unique solution in the space $C[0,1]$, and in the sphere $\overline{B}(\cos t;r)\subset C[0,1]$ respectively (chapter 2, paragraph 2.4).

It was considered the case when the conditions of the theorem 2.1.2, for $\boldsymbol{B}=\boldsymbol{R}$, are fulfilled, i.e. the integral equation (5.19) has a unique solution x^* in the sphere $\overline{B}(\cos t;r)\subset C[0,1]$.

The solution x^* was determined using the method of successive approximations starting from the element $x_0(t)=\cos t$, $x_0\in\overline{B}(\cos t;r)\subset C[0,1]$, and for the approximate calculation of the integrals that appear in the terms of the sequence of successive approximations, the trapezoids formula, the rectangles formula and the Simpson's formula, respectively, were used.

It is observed that the functions K and f fulfill the conditions:

- (h_{11}), (h_{12}) and (h_{13}), necessary to apply the trapezoids formula;
- (h_{21}), (h_{22}) and (h_{23}), necessary to apply the rectangles formula,

and respectively

- (h_{31}), (h_{32}) and (h_{33}), necessary to apply the Simpson's formula.

Also, to get a better approximation of the solution, an equidistant division of the interval [0,1] through the points $0=t_0<t_1<\ldots<t_n=1$ was considered.

The approximate value of the integral that arise in the general term of the successive approximations sequence:

$$x_m(t_k)=\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds+\cos t_k,$$

was calculated as it follows:

a) when we use the trapezoids formula, we have the relation:

$$\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds=$$

$$=\frac{1}{2n}\left[\frac{\sin(x_{m-1}(0))+\cos(x_{m-1}(0))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}+\right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(x_{m-1}(t_i))+\cos(x_{m-1}(t_i/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)+$$

$$+\frac{\sin(x_{m-1}(1))+\cos(x_{m-1}(1/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\Bigg]+R^T_{m,k}\ ,\quad k=\overline{0,n}\ ,\quad m\in N\ ,$$

with the estimate of the rest:

$$\left|R^T_{m,k}\right|\le\frac{1}{12n^2}\cdot\max_{s\in[0,1]}\left|\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]''_s\right|\ .$$

b) when we use the rectangles formula, we have the relation:

$$\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds=$$

$$=\frac{1}{n}\Bigg[\frac{\sin(x_{m-1}(0))+\cos(x_{m-1}(0))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}+$$

$$+\sum_{i=1}^{n-1}\left(\frac{\sin(x_{m-1}(t_i))+\cos(x_{m-1}(t_i/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)\Bigg]+$$

$$+R^D_{m,k}\ ,\quad k=\overline{0,n}\ ,\quad m\in N\ ,$$

with the estimate of the rest:

$$\left|R^D_{m,k}\right|\le\frac{1}{n}\cdot\max_{s\in[0,1]}\left|\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]'_s\right|\ .$$

c) when we use the Simpson's formula, we have the relation:

$$\int_0^1\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]ds=$$

$$=\frac{1}{6n}\Bigg[\frac{\sin(x_{m-1}(0))+\cos(x_{m-1}(0))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}+$$

$$+2\sum_{i=1}^{n-1}\left(\frac{\sin(x_{m-1}(t_i))+\cos(x_{m-1}(t_i/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)+$$

$$+4\sum_{i=0}^{n-1}\left(\frac{\sin\left(x_{m-1}\left(\frac{t_i+t_{i+1}}{2}\right)\right)+\cos\left(x_{m-1}\left(\frac{t_i+t_{i+1}}{4}\right)\right)}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right)+$$

$$+\frac{\sin(x_{m-1}(1))+\cos(x_{m-1}(1/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\Bigg]+R^S_{m,k}\ ,\quad k=\overline{0,n}\ ,\quad m\in N\ ,$$

with the estimate of the rest:

$$\left|R^S_{m,k}\right|\le\frac{1}{2880\cdot n^4}\cdot\max_{s\in[0,1]}\left|\left[\frac{\sin(x_{m-1}(s))+\cos(x_{m-1}(s/2))}{7}+\frac{x_{m-1}(0)+x_{m-1}(1)}{5}\right]^{iv}_s\right|\ .$$

Thus, using an equidistant division of the interval [0,1] through the points $a = t_0 < t_1 < \ldots < t_n = b$, we obtain the sequence $(\widetilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m(t_k))_{m\in N}$, $k=\overline{0,n}$ with the following error in calculation:

a) when we use the trapezoids formula, the error is:

$$|x_m(t_k)-\widetilde{x}_m(t_k)| \le 0{,}0676113247\cdot\frac{1}{n^2}$$

b) when we use the rectangles formula, the error is:

$$|x_m(t_k)-\widetilde{x}_m(t_k)| \le 0{,}43696698612\cdot\frac{1}{n}$$

c) when we use the Simpson's formula, the error is:

$$|x_m(t_k)-\widetilde{x}_m(t_k)| \le 0{,}00139480234\frac{1}{n^4}\ .$$

The calculus of the approximate value of the integral from the expression of the general term of the successive aproximations sequence using the trapezoids formula, the rectangles formula and the Simpson's formula respectively, was performed with a software developed in MATLAB. The results that was obtained using this software product are given in appendices.

Knowing the number of sub-intervals with equal length, contained in the interval [0,1], the approximate solution of the integral equation (5.19) was determined in the following two situations:

- when we know the error and respectively
- when we know the number of iterations.

Thus, for a division of the interval [0,1] in 100 equal parts and an error er $\le 10^{-10}$, the following results have been obtained:

a) Using the trapezoids formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72289467701720$ (appendix 1a)

and it was obtained after 18 iterations, with a requested error.

b) Using the rectangles formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72281138567569$ (appendix 2a)

and it was obtained after 18 iterations, with a requested error.

c) Using the Simpson's formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72289470137956$ (appendix 3a)

and it was obtained after 18 iterations, with a requested error.

Finally, for a division of the interval [0,1] in 100 equal parts and after 20 iterations, the following approximate solutions of the integral equation were obtained:

a) Using the trapezoids formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72289467704154$ (appendix 1b)

and it was obtained after 20 iterations, with the error er = 5.008105041781619e-012 .

b) Using the rectangles formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72281138569953$ (appendix 2b)

and it was obtained after 20 iterations, with the error er = 4.898970118460966e-012 .

c) Using the Simpson's formula, the approximate solution of the integral equation is:

$x(t) \approx \cos t + 0{,}72289470140390$ (appendix 3b)

and it was obtained after 20 iterations, with the error er = 5.007882997176694e-012 .

5.6 References

[1] Ambro, M., Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat, *Studia Univ. Babeş-Bolyai, Mathematica*, 2(1978), 26-32

[2] Cerone, P., Dragomir, S. S., *Trapezoidal and Midpoint-Type Rules from Inequalities Point of View*, Handbook of Analytic Computational Methods in Applied Mathematics, Chapman&Hall/CRC, New York, 2000

[3] Coman, Gh., Rus, I., Pavel, G., Rus, I. A., *Introducere în teoria ecuaţiilor operatoriale*, Ed. Dacia, Cluj-Napoca, 1976

[4] Coman, Gh., *Analiză numerică*, Ed. Libris, Cluj-Napoca, 1995

[5] Dobriţoiu, M., *Aproximări ale soluţiei unei ecuaţii integrale Fredholm cu argument modificat*, Analele Univ. Aurel Vlaicu din Arad, seria Matematică, fasc. „Matematică-Informatică”, Arad, 28-30 nov. 2002, ISSN 1582-344X, 51-56

[6] Dobriţoiu, M., *Formule de cuadratură utilizate în aproximarea soluţiei unei ecuaţii integrale cu argument modificat*, Lucr. Şt. ale Simpozionului Internaţional “Universitaria ROPET 2003”, Petroşani, 16-18 oct. 2003, Ed. Universitas Petroşani, fasc. „Matematică-Informatică-Fizică”, ISBN 973-8260-37-X, 53-56

[7] Dobriţoiu, M., *The rectangle method for approximating the solution to a Fredholm integral equation with a modified argument*, Lucr. Şt. a celei de *a XXX-a Sesiuni de comunicări ştiinţifice cu participare internaţională* “TEHNOLOGII MODERNE ÎN SECOLUL XXI”, Academia Tehnică Militară Bucureşti, secţ. 16, “Matematică”, Bucureşti, 6-7 nov. 2003, ISBN 973-640-012-3, 36-39

[8] Dobriţoiu, M., A Fredholm integral equation - numerical methods, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM - CVI / 2004, Nr. 2188, ISSN 0133-3526, 285-292

[9] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. 51, nr. 1/2006, 81-94

[10] Hackbusch, W., *Integral equations*, Birkhäuser, Berlin, 1995

[11] Iancu, C., A numerical method for approximating the solution of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIII, 4(1998), 37-45

[12] Iancu, C., *Modelare matematică. Teme speciale*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2002

[13] Ionescu, D. V., *Cuadraturi numerice*, Ed. Tehnică, Bucureşti, 1957

[14] Marinescu, Gh., *Analiză matematică*, vol. I, Ed. Didactică şi Pedagogică, Bucureşti, 1980

[15] Pavel, G., Rezolvarea aproximativă a problemei lui Dirichlet relativă la un sistem de ecuaţii diferenţiale neliniare de ordinul al doilea, *Studia Univ. Babeş-Bolyai, Mathematica*, 1(1977), 47-52

[16] Polyanin, A. D., Manzhirov, A. V., *Handbook of integral equations*, CRC Press, London, 1998

[17] Precup, R., Existence and approximation of positive fixed points of nonexpansive maps, *Rev. Anal. Numer. Theor. Approx.*, 26(1997), 203-208

[18] Rus, I. A., *Approximation of fixed points of generalized contraction mappings*, Academic Press, London, 1975, 157-161

[19] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[20] Rus, I. A., Şerban, M. A., D. Trif, Step method for some integral equations from biomathematics, *Bull. Math. Soc. Sci. Math. Roumanie*, Tome 54(102)(2011) No. 2, 167-183

[21] Stancu, D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză numerică şi teoria aproximării*, vol. 1, Presa Univ. Clujeană, Cluj-Napoca, 2001

[22] Stancu, D. D., Coman, Gh., Blaga, P., *Analiză numerică şi teoria aproximării*, vol. 2, Presa Univ. Clujeană, Cluj-Napoca, 2002

6 An integral equation from the theory of epidemics

In the study of certain problems of the dynamics of population, with phenomena that occur periodically, often occurs the following nonlinear integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \quad t \in \boldsymbol{R}, \tag{6.1}$$

where the function $f \in C(\boldsymbol{R} \times \boldsymbol{R}_+)$ satisfies the condition of periodicity with respect to t ($\omega > 0$), $f(t+\omega, x) = f(t,x)$, for each $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}_+$ and $\tau > 0$ is a parameter.

According to presentation in paper [12], this equation can be described in the terms of epidemics, when the number of members of the population is constant, and also in the terms of one population increase, when the birth rate varies periodically (chapter 1, paragraph 1.8.2).

Therefore, the integral equation (6.1) can be used as a mathematical model, important for the study of the spreading of an infectious disease, which has a periodic contact rate and which varies seasonally. In this situation, $x(t)$ is a continuous quantity, which represents the number of members of the population that were infected at a certain moment, t, function $f(t,x(t))$ represents the number of the individuals new infected per unit of time ($f(t, 0) = 0$), and τ is the length of time in which an individual remains infectious.

K. L. Cooke and J. L. Kaplan have proposed this integral equation as a mathematical model of epidemics and of population increase, respectively. This model has been intensely studied, and the conditions of existence and uniqueness of the non-trivial positive and periodic solutions with period $\omega > 0$ were obtained, emphasizing some of the interesting properties of the solutions.

Among those who studied this equation one can mention K. L.Cooke and J. L.Kaplan [3], D. Guo and V. Lakshmikantham [5], R. Torrejón [20], R. Precup [9], [10], [11], E. Kirr [7], [8], A. Cañada and A. Zertiti [1], [2], R. Precup and E. Kirr [12], I. A. Rus [13], [14], [15], [16], I. A. Rus and C. Iancu [17], C. Iancu [6], M. Dobriţoiu, I. A. Rus and M. A. Şerban [4], I. A. Rus, M. A. Şerban and D. Trif [19].

In what follows, we present the results obtained by I. A Rus, M. A. Şerban and M. Dobriţoiu in a study of the integral equation (6.1), using the Picard operators technique, and published in paper [4]. This study contains the results regarding to existence and uniqueness of the solution, lower-solutions and upper-solutions of the integral equation (6.1), the results regarding to the data dependence of the solution and the differentiability of the solution with respect to a parameter.

6.1 The existence and uniqueness of the solution in a subset of the space $C(\boldsymbol{R},\boldsymbol{I})$

In order to study the existence and uniqueness of the solution in a subset of the space $C(R, I)$ we consider the nonlinear integral equation (6.1) under the following conditions:

(e_1) $I, J \subset \boldsymbol{R}$ are compact intervals and $f \in C(\boldsymbol{R} \times I, J)$;

(e_2) $f(t, \cdot) : I \to J$ is L_f – Lipschitz for each $t \in \boldsymbol{R}$;

(e_3) $L_f \cdot \tau < 1$;

(e_4) there exists $U \subset C(\boldsymbol{R},I)$ such that $U \in I_{cl}(A)$, where the operator A is defined by the relation:

$$A(x)(t) := \int_{t-\tau}^{t} f(s,x(s))ds\,, \quad t \in \boldsymbol{R}\,. \tag{6.2}$$

The following theorem of existence and uniqueness is true.

Theorem 6.1.1. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Under the conditions* (e_1)–(e_4), *the integral equation* (6.1) *has a unique solution in U.*

Proof. We consider the Banach space ($C(\boldsymbol{R},I)$, $\|\cdot\|_C$) endowed with the supremum norm

$$\| x \|_C = \sup_{t \in R} | x(t) |,$$

and the operator A defined by the relation (6.2).

From the condition (e_4) it results that $A(U) \subset U$, so one can consider the operator $A : U \to U$, defined by the relation (6.2). The solutions set of the integral equation (6.1) coincides with the fixed point set of the operator A.

From the condition (e_2) we have:

$$| A(x_1)(t) - A(x_2)(t) | = \left| \int_{t-\tau}^{t} \big(f(s,x_1(s)) - f(s,x_2(s))\big)ds \right| \le$$

$$\le \left| \int_{t-\tau}^{t} | f(s,x_1(s)) - f(s,x_2(s)) | ds \right| \le \left| \int_{t-\tau}^{t} L_f | x_1(s) - x_2(s) | ds \right|$$

and using the supremum norm, we obtain:

$$\| A(x_1) - A(x_2) \|_C \le L_f \cdot \tau \| x_1 - x_2 \|_C \;,$$

and according to condition (e_3) it results that the operator A is an α-contraction with the coefficient $\alpha = L_f \cdot \tau$.

Now, we obtain the conclusion of the theorem by applying *the Contraction Principle* 1.3.1.

Remark 6.1.1. If the conditions (e_1)–(e_4) are fulfilled, then the operator

$$A : (U,\, d_{\|\cdot\|_C}) \to (U,\, d_{\|\cdot\|_C})$$

is a Picard operator.

Let be $0 < m < M$, $0 < \alpha < \beta$, $I = [\alpha,\beta]$, $J = [m,M]$.

Corollary 6.1.1. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that:*

(*i*) *the conditions* (e_1)–(e_3) *are fulfilled*;

(*ii*) $\alpha \le m \cdot \tau$, $\beta \ge M \cdot \tau$.

Then the integral equation (6.1) *has a unique solution in* $C(\boldsymbol{R},I)$.

Proof. We consider $U := C(\boldsymbol{R},I)$, where $I = [\alpha,\beta]$ and the operator A is defined by the relation (6.2).
From the definition of the function f it results that

$f(t, x(t)) \in [m,M]$,   for all $t \in \boldsymbol{R}$, $x \in U$

and we obtain

$$\int_{t-\tau}^{t} f(s,x(s))ds \in [m\tau, M\tau] ,\quad \text{for all } t \in \boldsymbol{R}, x \in U ,$$

i. e.

$A(x)(t) \in [m\tau, M\tau]$,   for all $t \in \boldsymbol{R}, x \in U$.

From the condition (*ii*) it results that

$A(x)(t) \in [\alpha,\beta]$,   for all $t \in \boldsymbol{R}, x \in U$.

Therefore, U is an invariant subset for the operator A and now, applying the theorem 6.1.1 the proof is complete.

Corollary 6.1.2. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that the conditions of the corollary* 6.1.1 *are fulfilled. In addition, we suppose that there exists* $\omega > 0$ *such that*:

$f(t + \omega, u) = f(t, u)$,   *for all* $t \in \boldsymbol{R}, u \in I$.

Then, the integral equation (6.1) *has a unique periodic solution, that has the period* $\omega > 0$.

Proof. Consider

$U := X_\omega := \{ x \in C(\boldsymbol{R},I) \mid x(t + \omega) = x(t) \text{ , for all } t \in \boldsymbol{R} \}$

and the operator A defined by the relation (6.2).

Using the condition (e_1) and the condition (*ii*) of the corollary 6.1.1 and since the function f is ω-periodic with respect to t, we deduce that $A(U) \subset U$, i. e. $U \in I(A)$. Thus, the conditions of the theorem 6.1.1 are fulfilled and therefore it results the conclusion of the corollary. The proof is complete.

6.2 Lower-solutions and upper-solutions

Consider the integral equation (6.1) under the conditions (e_1)–(e_4) and we denote by $x_A^* \in U$ the unique fixed point of the operator A. In addition, we suppose that:

(e_5) $f(t, \cdot) : I \to J$ is increasing for all $t \in \boldsymbol{R}$.

We have:

Theorem 6.2.1. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that the conditions* (e_1)–(e_5) *are fulfilled. If*

$$x \in U ,\quad x(t) \le \int_{t-\tau}^{t} f(s,x(s))ds ,$$

then

$x \le x_A^*$.

Proof. We consider the operator $A : U \to U$, defined by the relation (6.2). From the conditions (e_1)–(e_4) it results that A is a Picard operator, and from the condition (e_5) it results that A is an increasing operator. Since the conditions of *the abstract Gronwall's lemma*, 1.4.1, are fulfilled, we obtain:

$$x \le x_A^*$$

and the proof is complete.

Let be $0 < m < M$, $0 < \alpha < \beta$, $I = [\alpha,\beta]$, $J = [m,M]$. The following theorem is true.

Theorem 6.2.2. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Let* f_i , $i = 1, 2, 3$ *be three functions and suppose that the following conditions are fulfilled*:

(i) $f_i \in C(\mathbf{R}\times I, J)$, $i = 1, 2, 3$, *where* $I, J \subset \mathbf{R}$ *compact intervals*;

(ii) $f_2(t, \cdot)$ *is increasing for all* $t \in \mathbf{R}$;

(iii) $f_1 \le f_2 \le f_3$;

(iv) $f_i(t, \cdot) : I \to J$ *is* L_{f_i} *– Lipschitz for all* $t \in \mathbf{R}$, $i = 1, 2, 3$;

(v) $L_{f_i} \cdot \tau < 1$, $i = 1, 2, 3$;

(vi) $\alpha \le m\cdot\tau$, $\beta \ge M\cdot\tau$.

Let x_i^* , $i = 1, 2, 3$ *be the unique solutionof the integral equation* (6.1) *for each of the three functions* f_i , $i = 1, 2, 3$. *Then*

$$x_1^* \le x_2^* \le x_3^* .$$

Proof. We consider the operators $A_i : C(\mathbf{R},I) \to C(\mathbf{R},I)$, defined by the relations:

$$A_i(x)(t) := \int_{t-\tau}^{t} f_i(s,x(s))ds\,, \quad t \in \mathbf{R}\,, \; i = 1, 2, 3. \tag{6.3}$$

From the condition (ii) we deduce that the operator A_2 is increasing, and from the condition (iii) it results that

$$A_1 \le A_2 \le A_3 .$$

Using the conditions (i), (iv) and (v) we obtain that the operators A_i are α_i–contractions with the constants $\alpha_i = L_{f_i} \cdot \tau$, $i = 1, 2, 3$ and therefore A_i , $i = 1, 2, 3$ are Picard operators.

According to *the abstract comparison lemma*, 1.4.5, it results that the following implication is fulfilled:

$$x_1 \le x_2 \le x_3 \;\Rightarrow\; A_1^\infty(x_1) \le A_2^\infty(x_2) \le A_3^\infty(x_3),$$

and since A_i , $i = 1, 2, 3$, are Picard operators, we obtain that

$$x_1^* \le x_2^* \le x_3^* ,$$

and, finally, the proof is complete.

Theorem 6.2.3. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that the conditions* (e_1)–(e_3) *and* (e_5). *Then*

$$x \le x_A^* \le y \ ,$$

for all $x \in (LF)_A$ *and* $y \in (UF)_A$.

Proof. We consider $U := (LF)_A \cup (UF)_A$ and the operator A defined by the relation (6.2):

$$A(x)(t) := \int_{t-\tau}^{t} f(s, x(s))ds\,, \quad t \in \boldsymbol{R}\ .$$

From the condition (e_5) it results that the operator A is increasing and therefore we have that $(LF)_A \in I(A)$ and $(UF)_A \in I(A)$. Hence it results that $(LF)_A \cup (UF)_A \in I(A)$ and, so, $(LF)_A \cup (UF)_A$ is an invariant subset for the operator A.

Now, we consider the operator $A : U \to U$, defined by the same relation (6.2).

From the conditions (e_1)–(e_3) and the condition above, it results that A is a Picard operator.

Applying the theorem 6.1.1 it results that the operator A has in U a unique fixed point, which we denote by x_A^* .

Since the conditions of lemma 1.4.2 are fulfilled, we obtain the conclusion of this theorem and the proof is complete.

6.3 The data dependence

In what follows, we study the dependence of the solution of the integral equation (6.1) with respect to the function f , and for this, we consider the following perturbed integral equation:

$$y(t) = \int_{t-\tau}^{t} g(s, y(s))ds\,, \quad t \in \boldsymbol{R}, \qquad (6.4)$$

where $g \in C(\boldsymbol{R} \times I, J)$, and $I, J \subset \boldsymbol{R}$ are compact intervals.

Theorem 6.3.1. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that:*

(i) *the conditions of the theorem* 6.1.1 *are fulfilled and we denote by* x^* *the unique solution of the integral equation* (6.1);

(ii) *there exists* $\eta > 0$, *such that*

$$\left| f(t,u) - g(t,u) \right| \le \eta\,, \quad \text{for all } t \in \boldsymbol{R}\,, u \in I\,.$$

Under these conditions, if y^* *is a solution of the integral equation* (6.4), *then we have*:

$$\left\| x^* - y^* \right\|_C \le \frac{\eta \cdot \tau}{1 - L_f \cdot \tau}\ .$$

Proof. We consider the operator $A : U \to U$, defined by the relation (6.2):

$$A(x)(t) := \int_{t-\tau}^{t} f(s,x(s))ds\,, \quad t \in \boldsymbol{R}\,.$$

Also, let $B : U \to U$ be an operator attached to the perturbed integral equation (6.4), defined by the relation:

$$B(y)(t) := \int_{t-\tau}^{t} g(s,y(s))ds\,, \quad t\in \boldsymbol{R}\,. \qquad (6.5)$$

From the condition (*ii*) we have:

$$|\,A(x)(t) - B(x)(t)\,| = \left|\int_{t-\tau}^{t} [\, f(s,x(s)) - g(s,x(s))\,]ds\right| \le$$

$$\le \left|\int_{t-\tau}^{t} |\, f(s,x(s)) - g(s,x(s))\,|ds\right| \le \left|\int_{t-\tau}^{t} \eta\, ds\right| = \eta\cdot\tau$$

and using the supremum norm, we obtain:

$$\|\,A(x) - B(x)\,\|_C \le \eta\cdot\tau \;.$$

Now, the proof of the theorem it results by applying *the abstract data dependence theorem*, 1.3.5.

We have, also, the following theorem of data dependence of the periodic solution of the integral equation (6.1).

Theorem 6.3.2. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that:*

(*i*) *the conditions of the corollary* 6.1.2 *are fulfilled and we denote by* x^* *the unique* ω*–periodic solution of the integral equation* (6.1);

(*ii*) $g(t + \omega, u) = g(t, u)$, *for all* $t\in \boldsymbol{R}, u\in I$;

(*iii*) *there exists* $\eta > 0$, *such that*

$$|\, f(t,u) - g(t,u)\,| \le \eta\,, \quad \textit{for all } t\in \boldsymbol{R}, u\in I\,.$$

Under these conditions, if y^* *is an* ω*–periodic solution of the perturbed integral equation* (6.4), *then we have*:

$$\|x^* - y^*\|_C \le \frac{\eta\cdot\tau}{1 - L_f\cdot\tau}\;.$$

Proof. We consider

$$U := X_\omega := \{\, x\in C(\boldsymbol{R},I) \mid x(t + \omega) = x(t)\,, \text{ for all } t\in \boldsymbol{R}\,\}$$

and the operator $A : U \to U$ defined by the relation (6.2):

$$A(x)(t) := \int_{t-\tau}^{t} f(s,x(s))ds\,, \quad t \in \boldsymbol{R}\,.$$

Let $B : U \to U$ be an operator defined by the relation (6.5):

$$B(y)(t) := \int_{t-\tau}^{t} g(s, y(s))ds\,, \quad t\in \boldsymbol{R}\,.$$

From the condition (*iii*) we have:

$$|\,A(x)(t) - B(x)(t)\,| = \left| \int_{t-\tau}^{t} [\, f(s,x(s)) - g(s,x(s))\,]ds \right| \le$$

$$\le \left| \int_{t-\tau}^{t} |\ f(s,x(s)) - g(s,x(s)) |ds \right| \le \left| \int_{t-\tau}^{t} \eta\, ds \right| = \eta \cdot \tau$$

and using the supremum norm, we obtain:

$$\|\, A(x) - B(x) \,\|_C \le \eta \cdot \tau \ .$$

Now, we obtain the conclusion of the theorem by applying *the abstract data dependence theorem* 1.3.5.

6.4 The differentiability of the solution with respect to a parameter

In what follows, we study the differentiability of the solution of the integral equation (6.1) (see [7], [12], [18]) with respect to the parameter λ:

$$x(t,\lambda) = \int_{t-\tau}^{t} f(s,x(s);\lambda)ds\,, \quad t\in \boldsymbol{R},\ \lambda\in K\,, \tag{6.6}$$

where $f\in C(\boldsymbol{R}\times I\times K, J)$, with $I = [\alpha,\beta]$, $0 < \alpha < \beta$, $J = [m,M]$, $0 < m < M$ and $K \subset \boldsymbol{R}$ is an compact interval.

Let be

$X_\omega := \{\, x\in C(\boldsymbol{R} \times K, I) \mid x(t + \omega, \lambda) = x(t, \lambda)$, for all $t\in \boldsymbol{R},\ \lambda\in K \,\}$,

where $\omega > 0$.

Theorem 6.4.1. (M. Dobriţoiu, I. A. Rus and M. A. Şerban [4]) *Suppose that the following conditions are fulfilled:*

(*i*) $\alpha \le m\cdot \tau$, $\beta \ge M\cdot \tau$;

(*ii*) $f(t, u; \lambda) \in [m,M]$, *for all* $t\in \boldsymbol{R},\ u\in I,\ \lambda\in K$;

(*iii*) $f(t + \omega, u; \lambda) = f(t, u; \lambda)$, *for all* $t\in \boldsymbol{R},\ u\in I,\ \lambda\in K$;

(*iv*) $f(t, \cdot\, ; \lambda) : I \to J$ *is* L_f*– Lipschitz for all* $t\in \boldsymbol{R},\ \lambda\in K$;

(*v*) $L_f \cdot \tau < 1$.

Then

(*a*) *the integral equation* (6.1) *has a unique solution* x^* *in* X_ω .

(*b*) *for all* $x_0\in X_\omega$, *the sequence* $(x_n)_{n\in \boldsymbol{N}}$ *defined by the relation:*

$$x_{n+1}(t,\lambda) = \int_{t-\tau}^{t} f(s, x_n(s,\lambda))ds$$

converges uniformly to x^* ;

(c) *if* $f(t, \cdot, \cdot) \in C^1(I \times K)$, *then* $x^*(t, \cdot) \in C^1(K)$.

Proof. (*a*) + (*b*). We consider the operator $B : X_\omega \to C(\boldsymbol{R}\times K)$ defined by the relation:

$$B(x)(t,\lambda) := \int_{t-\tau}^{t} f(s, x(s;\lambda))ds \quad .$$

From the conditions (*i*) and (*iii*) it results that X_ω is an invariant subset for the operator B, i. e. $X_\omega \in I(B)$.

From the conditions (*iv*) and (*v*) it results that the operator B is an α–contraction with the constant $\alpha = L_f \cdot \tau$.

Applying now, *the Contraction Principle* 1.3.1, it results that B is a Picard operator.

(*c*). We prove that there exists $\dfrac{\partial x^*}{\partial \lambda}$ and that $\dfrac{\partial x^*}{\partial \lambda} \in C(\boldsymbol{R}\times K)$.

If we suppose that there exists $\dfrac{\partial x^*}{\partial \lambda}$, then from

$$x(t,\lambda) = \int_{t-\tau}^{t} f(s, x(s,\lambda);\lambda)ds$$

we have:

$$\frac{\partial x(t,\lambda)}{\partial \lambda} = \int_{t-\tau}^{t} \frac{\partial f(s, x(s,\lambda);\lambda)}{\partial x} \cdot \frac{\partial x(s;\lambda)}{\partial \lambda} ds + \int_{t-\tau}^{t} \frac{\partial f(s, x(s,\lambda);\lambda)}{\partial \lambda} ds \; .$$

This relation suggests us to consider the operator $T : X_\omega \times X_\omega \to X_\omega \times X_\omega$, defined by the relation:

$T = (B, C)$, $\qquad T(x, y) = (B(x), C(x, y))$,

where

$$C(x,y)(t,\lambda) := \int_{t-\tau}^{t} \frac{\partial f(s, x(s,\lambda);\lambda)}{\partial x} \cdot y(s,\lambda)ds + \int_{t-\tau}^{t} \frac{\partial f(s, x(s,\lambda);\lambda)}{\partial \lambda} ds \, .$$

We have:

$$\left| C(x,y)(t,\lambda) - C(x,z)(t,\lambda) \right| \le \int_{t-\tau}^{t} \left| \frac{\partial f(s, x(s,\lambda);\lambda)}{\partial x} \right| \cdot \left| y(s,\lambda) - z(s,\lambda) \right| ds \le$$

$$\le L_f \cdot \left\| y - z \right\|_C \int_{t-\tau}^{t} ds = L_f \cdot \tau \cdot \left\| y - z \right\|_C \; ,$$

for all $x, y, z \in X_\omega$.

Since the conditions of *the fiber Picard operators theorem* are fulfilled, it results that T is a Picard operator and the sequences:

$x_{n+1} = B(x_n)$ $\qquad$ and $\qquad$ $y_{n+1} = C(x_n, y_n)$

converge uniformly to $(x^*, y^*) \in F_T$, for all $x_0, y_0 \in X_\omega$.

If we consider $x_0, y_0 \in X_\omega$ such that $y_0 = \frac{\partial x_0}{\partial \lambda}$, then it results that

$$y_n = \frac{\partial x_n}{\partial \lambda}\ , \quad \text{for all } n \in \boldsymbol{N}.$$

So

$$x_n \xrightarrow{unif.} x^*, \quad \text{as } n \to \infty.$$

$$\frac{\partial x_n}{\partial \lambda} \xrightarrow{unif.} y^* \quad \text{as } n \to \infty.$$

Now, using a Weierstrass argument, we deduce that x^* is differentiable, i. e. there exists $\frac{\partial x^*}{\partial \lambda}$ and $y^* = \frac{\partial x^*}{\partial \lambda}$.

6.5 References

[1] Cañada, A., Zertiti, A., Systems of nonlinear delay integral equations modelling population growth in a periodic environment, *Comment. Math. Univ. Carolinae* 35, 4(1994), 633-644

[2] Cañada, A., Zertiti, A., Topological methods in the study of positive solutions for some nonlinear delay integral equations, *Nonlinear Analysis, Theory, Methods and Applications*, vol.23, 9(1994), 1153-1165

[3] Cooke, K. L., Kaplan, J. L., A periodicity thereshald theorem for epidemics and population growth, *Math. Biosciences*, 31(1976), no.1-2, 87-104

[4] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[5] Guo, D., Lakshmikantham, V., Positive solutions of nonlinear integral equations arising in infectious diseases, *J. Math. Anal. Appl.* 134 (1988), 1-8

[6] Iancu, C., A numerical method for approximating the solution of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIII, 4(1998), 37-45

[7] Kirr, E., Existence of periodic solutions for some integral equations arising in infectious diseases, *Studia Univ. Babeş-Bolyai, Mathematica,* 39, No.2 (1994), 107-119

[8] Kirr, E., Existence and continuous dependence on data of the positive solutions of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica,* 41, No.2 (1996)

[9] Precup, R., Positive solutions of the initial value problem for an integral equation modelling infectious disease, în *"Seminarul de Ecuaţii Diferenţiale: Preprint* Nr.3, 1991", I. A. Rus ed., Universitatea Babeş-Bolyai, Cluj, 1991, 25-30

[10] Precup, R., Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle, *Studia Univ. Babeş-Bolyai, Mathematica,* 39, No.1 (1994), 47-58

[11] Precup, R., Monotone technique to the initial values problem for a delay integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica,* 40, No.2 (1995), 63-73

[12] Precup, R., Kirr, E., *Analysis of a nonlinear integral equation modelling infection diseases*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 178-195

[13] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[14] Rus, I. A., A delay integral equation from biomathematics, Babeş-Bolyai University of Cluj-Napoca, *Preprint* Nr.3, 1989, 87-90

[15] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334

[16] Rus, I. A., Fiber Picard operators theorem and applications, *Studia Univ. Babeş-Bolyai, Mathematica*, **44**(1999), No. 3, 89-97

[17] Rus, I. A., Iancu, C., *Modelare matematică*, Transilvania Press, Cluj-Napoca, 2000

[18] Rus, I. A., Weakly Picard operators and applications, *Seminar on Fixed Point Theory*, Babeş-Bolyai University of Cluj-Napoca, Vol. 2, 2001, 41-58

[19] Rus, I. A., Şerban, M. A., D. Trif, Step method for some integral equations from biomathematics, *Bull. Math. Soc. Sci. Math. Roumanie*, Tome 54(102)(2011) No. 2, 167-183

[20] Torrejón, R., A note on a nonlinear integral equation from the theory of epidemics, *Nonlinear Analysis*, 14(1990), No.6, 483-488

Appendices

Appendix 1a

The results obtained using the program MAS_TrapezE.m

>> **The successive approximations method and the trapezoids formula**

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The error, **er = 0.0000000001**

<u>The results:</u>

The approximate solution of the integral equation is:

x(t) = cost +

ans =

0.72289467701720

and it was obtained for the required value of the error, after

ni =

18

iterations.

>>

Appendix 1b

The results obtained using the program MAS_TrapezI.m

>> **The successive approximations method and the trapezoids formula**

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The numbers of iterations, **ni = 20**

<u>The results:</u>

The approximate solution of the integral equation, was calculated after:

ni =

20

iterations and it is:

x(t) = cost +

ans =

0.72289467704154

with the error:

er =

5.008105041781619e-012

>>

Appendix 2a

The results obtained using the program MAS_DreptunghiE.m

>> The successive approximations method and the rectangles formula

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The error, **er = 0.0000000001**

<u>The results:</u>

The approximate solution of the integral equation is:

x(t) = cost +

ans =

0.72281138567569

and it was obtained for the required value of the error, after

ni =

18

iterations.

>>

Appendix 2b

The results obtained using the program MAS_DreptunghiI.m

>> The successive approximations method and the rectangles formula

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The number of iterations, **ni = 20**

<u>The results:</u>

The approximate solution of the integral equation, was calculated after:

ni =

20

iterations and it is:

x(t) = cost +

ans =

0.72281138569953

with the error:

er =

4.898970118460966e-012

>>

Appendix 3a

The results obtained using the program MAS_SimpsonE.m

>> **The successive approximations method and the Simpson's formula**

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The error, **er = 0.0000000001**

<u>The results:</u>

The approximate solution of the integral equation is:

x(t) = cost +

ans =

0.72289470137956

and it was obtained for the required value of the error, after

ni =

18

iterations.
>>

Appendix 3b

The results obtained using the program MAS_SimpsonI.m

>> **The successive approximations method and the Simpson's formula**

<u>The input data:</u>

We divide the interval [0,1] into n equal parts, **n = 100**

The number of iterations, **ni = 20**

<u>The results:</u>

The approximate solution of the integral equation, was calculated after:

ni =

20

iterations and it is:

x(t) = cost +

ans =

0.72289470140390

with the error:

er =

5.007882997176694e-012

References

[1] Albu, M., A fixed point theorem of Maia-Perov type, *Studia Univ. Babeş-Bolyai, Mathematica*, 3(1978), 76-79

[2] Ambro, M., Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat, *Studia Univ. Babeş-Bolyai, Mathematica*, 2(1978), 26-32

[3] András, Sz., Fredholm-Volterra integral equations, *Pure Math. Appl.*, 13(2002)1:2, 21-30

[4] András, Sz., Gronwall type inequalities via subconvex sequences, *Seminar on Fixed Point Theory*, 3(2002), 183-188

[5] András, Sz., Fiber φ-contractions on generalized metric spaces and applications, *Mathematica*, 45(68)(2003):1, 3-8, Cluj-Napoca

[6] András, Sz., *Ecuaţii integrale Fredholm-Volterra*, Ed. Didactică şi Pedagogică, Bucureşti, 2005

[7] Bainov, D., Simeonov, P., *Integral inequalities and applications*, Kluwer, Dordrecht, 1992

[8] Beesak, P. R., Gronwall inequalities, *Carleton Math. Lecture Notes*, 1975, Nr. 11

[9] Bica, A., *Modele matematice în biologie guvernate de ecuaţii diferenţiale*, Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică, Teză de doctorat, 2004

[10] Berinde, V., *Generalized contractions and applications*, Ed. Cub Press 22, Baia Mare, 1997

[11] Buică, A., Gronwall-type nonlinear integral inequalities, *Mathematica*, 44(2002), Nr. 1, 19-23

[12] Cañada, A., Zertiti, A., Systems of nonlinear delay integral equations modelling population growth in a periodic environment, *Comment. Math. Univ. Carolinae* 35, 4(1994), 633-644

[13] Cañada, A., Zertiti, A., Topological methods in the study of positive solutions for some nonlinear delay integral equations, *Nonlinear Analysis, Theory, Methods and Applications*, vol.23, 9(1994), 1153-1165

[14] Cerone, P., Dragomir, S. S., *Trapezoidal and Midpoint-Type Rules from Inequalities Point of View*, Handbook of Analytic Computational Methods in Applied Mathematics, Chapman&Hall/CRC, New York, 2000

[15] Coman, Gh., Rus, I., Pavel, G., Rus, I. A., *Introducere în teoria ecuaţiilor operatoriale*, Ed. Dacia, Cluj-Napoca, 1976

[16] Coman, Gh., *Analiză numerică*, Ed. Libris, Cluj-Napoca, 1995

[17] Constantin, A., A Gronwall-like inequality and its applications, *Rend. Mat. Acc. Lincei* 1(1990), 111-115

[18] Cooke, K. L., Kaplan, J. L., A periodicity thereshald theorem for epidemics and population growth, *Math. Biosciences*, 31(1976), no.1-2, 87-104

[19] Corduneanu, C., *Ecuaţii diferenţiale şi integrale*, Univ. Iaşi, 1971

[20] Corduneanu, C., *Integral Equations and Applications*, Cambridge University Press, 1991

[21] Crăciun, C., On some Gronwall inequalities, *Seminar on Fixed Point Theory*, 1 (2000), 31-34

[22] Dobriţoiu, M., Aproximări ale soluţiei unei ecuaţii integrale Fredholm cu argument modificat, *Analele Univ. Aurel Vlaicu din Arad, seria Matematică, fasc. „Matematică-Informatică”*, Arad, 28-30 nov. 2002, ISSN 1582-344X, 51-56

[23] Dobriţoiu, M., *Formule de cuadratură utilizate în aproximarea soluţiei unei ecuaţii integrale cu argument modificat*, Lucr. Şt. ale Simp. Int. “Universitaria ROPET 2003”, Petroşani, 16-18 oct. 2003, Ed. Universitas Petroşani, fasc. „Matematică-Informatică-Fizică”, ISBN 973-8260-37-X, 53-56

[24] Dobriţoiu, M., *The rectangle method for approximating the solution to a Fredholm integral equation with a modified argument*, Lucr. Şt. a celei de *a XXX-a Sesiuni de comunicări ştiinţifice cu participare internaţională* "TEHNOLOGII MODERNE ÎN SECOLUL XXI", Academia Tehnică Militară Bucureşti, secţ. 16, "Matematică", Bucureşti, 6-7 nov. 2003, ISBN 973-640-012-3, 36-39

[25] Dobriţoiu, M., The solution to a Fredholm implicit integral equation in the $\overline{B}(0;R)$ sphere, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM CV/2003, Nr.2162, , ISSN 0133-3526, 27-32

[26] Dobriţoiu, M., A Fredholm integral equation - numerical methods, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM - CVI / 2004, Nr. 2188, ISSN 0133-3526, 285-292

[27] Dobriţoiu, M., An integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIX, nr. 3/2004, 27-33

[28] Dobriţoiu, M., *The generalization of an integral equation*, Proc. of the 9th National Conference of the Romanian Math. Soc., Ed. Universităţii de Vest Timisoara, 2005, Seria Alef, ISBN 973-7608-37-2, 392-396

[29] Dobriţoiu, M., Existence and continuous dependence on data of the solution of an integral equation, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM - CVI / 2005 , Nr. ISSN 0133-3526

[30] Dobriţoiu, M., On an integral equation with modified argument, *Acta Univ. Apulensis*, *Mathematics-Informatics*, No.11/2006, ISSN 1582-5329, 387-391

[31] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. 51, nr. 1/2006, 81-94

[32] Dobriţoiu, M., A Fredholm-Volterra integral equation with modified argument, *Analele Univ. din Oradea, Fasc. Matematica*, tom XIII, 2006, ISSN 1221-1265, 133-138

[33] Dobriţoiu, M., W. W. Kecs, A. Toma, *The Differentiability of the Solution of a Nonlinear Integral Equation*, Proc. of the 8th WSEAS Int. Conf. on Math. Methods and Comp. Techniques in Electrical Eng., Bucharest, Romania, Oct. 16-17, 2006, ISSN 1790-5117, ISBN 960-8457-54-8, 155-158

[34] Dobriţoiu, M., W. W. Kecs, A. Toma, An Application of the Fiber Generalized Contractions Theorem, *WSEAS Transactions on Mathematics*, Issue 12, Vol. 5, Dec. 2006, ISSN 1109-2769, 1330-1335

[35] Dobriţoiu, M., Properties of the solution of an integral equation with modified argument, *Carpathian Journal of Mathematics*, 23(2007), No. 1-2, 77-80

[36] Dobriţoiu, M., Rus, I. A., Şerban, M. A., An integral equation arising from infectious diseases, via Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. LII, nr. 3/2007, 81-94

[37] Dobriţoiu, M., System of integral equations with modified argument, *Carpathian Journal of Mathematics*, vol. 24 (2008), No. 2, 26-36

[38] Dobriţoiu, M., *Gronwall-type lemmas for an integral equation with modified argument*, Proc. of the Int. Conf. on Theory and Applications of Mathematics and Informatics, Alba Iulia, 2007

[39] Dobriţoiu, M., A class of nonlinear integral equations, *Transylvanian Journal of Mathematics and Mechanics*, 4 (2012), No. 2, 117-123

[40] Dragomir, S. S., The Gronwall type lemmas and applications, *Monografii Matematice*, No. 29, Univ. Timişoara, 1987

[41] Goursat, E., *Cours d'Analyse Mathématique*, Gauthier-Villars, Paris, 1923

[42] Guo, D., Lakshmikantham, V., Positive solutions of nonlinear integral equations arising in infectious diseases, *J.Math.Anal.Appl.* 134 (1988), 1-8

[43] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear Integral Equation in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996
[44] Hackbusch, W., *Integral equations*, Birkhäuser, Berlin, 1995
[45] Haimovici, A., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1965
[46] Hirsch, M. W., Pugh, C. C., *Stable manifolds and hyperbolic sets*, Proc. Symp. In Pure Math., 14(1970), AMS, 133-143
[47] Iancu, C., A numerical method for approximating the solution of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, vol. XLIII, 4(1998), 37-45
[48] Iancu, C., *Modelare matematică. Teme speciale*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2002
[49] Ionescu, D. V., *Cuadraturi numerice*, Ed. Tehnică, Bucureşti, 1957
[50] Ionescu, D. V., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti,1972
[51] Istrăţescu, V. I., *Introducere în teoria punctelor fixe*, Ed. Academiei R.S.R., Bucureşti, 1973
[52] Kirk, W. A., Sims, B., *Handbook of metric fixed point theory*, Kluwer, Dordrecht, 2001
[53] Kirr, E., Existence of periodic solutions for some integral equations arising in infectious diseases, *Studia Univ. Babeş-Bolyai, Mathematica*, 39, No.2 (1994), 107-119
[54] Kirr, E., Existence and continuous dependence on data of the positive solutions of an integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica,* 41, No.2 (1996)
[55] Lakshmikantham, V., Leela, S., *Differential and Integral Inequalities*, Acad. Press, New York, 1969
[56] Lalescu, T., *Introducere în teoria ecuaţiilor integrale*, Ed. Academiei, Bucureşti, 1956
[57] Lungu, N., Rus, I. A., Gronwall inequalities via Picard operators, *Analele Ştiinţifice ale Univ. "Al.I. Cuza" din Iaşi, Matematica*, Tomul LVIII, 2012, f. 2, 269-278
[58] Maia, M. G., Un'osservazione sulle contrazioni metriche, *Rend. Sem. Mat. Univ. Padova*, 40(1968), 139-143
[59] Marinescu, Gh., *Teoria ecuaţiilor diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1963
[60] Marinescu, Gh., *Analiză matematică*, vol. I, Ed. Didactică şi Pedagogică, Bucureşti, 1980
[61] Mirică, Şt., *Ecuaţii diferenţiale şi integrale*, vol. I, II, III, Ed. Universităţii Bucureşti, 1999
[62] Mitrinović, D. S., Pečarić, J. E., Fink, A. M., *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991
[63] Mureşan, V., *Ecuaţii diferenţiale cu modificarea afină a argumentului*, Transilvania Press, Cluj-Napoca, 1997
[64] Mureşan, V., A Gronwall type inequality for Fredholm operators, *Mathematica*, 41(1999), Nr. 2, 227-231
[65] Mureşan, V., *Functional-Integral Equations*, Ed. Mediamira, Cluj-Napoca, 2003
[66] Pachpatte, B. G., *Inequalities for differential and integral equations*, Academic Press, New York, 1998
[67] Pavel, P., Rus, I. A., *Ecuaţii diferenţiale şi integrale*, Ed. Didactică şi Pedagogică, Bucureşti, 1975
[68] Pavel, G., *Rezolvarea aproximativă a problemei lui Dirichlet relativă la un sistem de ecuaţii diferenţiale neliniare de ordinul al doilea*, Studia Univ. Babeş-Bolyai, Mathematica, 1(1977), 47-52
[69] Petrovskii, I. G., *Lectures on the theory of integral equations*, Graylock Press, Rochester, 1957.
[70] Petruşel, A., Fredholm-Volterra integral equations and Maia's theorem, *Preprint* Nr.3, 1988, Univ. Babeş-Bolyai, 79-82
[71] Petruşel, A., Rus, I. A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. 134 (2005), 411-418
[72] Polyanin, A. D., Manzhirov, A. V., *Handbook of integral equations*, CRC Press, London, 1998

[73] Precup, R., Positive solutions of the initial value problem for an integral equation modelling infectious disease, în *"Seminarul de Ecuaţii Diferenţiale: Preprint* Nr.3, 1991", I. A. Rus ed., Univ. Babeş-Bolyai, Cluj, 1991, 25-30

[74] Precup, R., *Ecuaţii integrale neliniare*, Univ. Babeş-Bolyai Cluj-Napoca, 1993

[75] Precup, R., Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle, *Studia Univ. Babeş-Bolyai, Mathematica,* 39, No.1 (1994), 47-58

[76] Precup, R., Monotone technique to the initial values problem for a delay integral equation from biomathematics, *Studia Univ. Babeş-Bolyai, Mathematica*, 40, No.2 (1995), 63-73

[77] Precup, R., Existence and approximation of positive fixed points of nonexpansive maps, *Rev. Anal. Numer. Theor. Approx.*, 26(1997), 203-208

[78] Precup, R., Kirr, E., *Analysis of a nonlinear integral equation modelling infection diseases*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 178-195

[79] Precup, R., Continuation results for mappings of contractive type, *Seminar on Fixed Point Theory* Cluj-Napoca, 2(2001), 23-40

[80] Precup, R., Some existence results for differential equations with both retarded and advanced arguments, *Mathematica*, Tome 44(67), Nr. 1, 2002, 31-38

[81] Precup, R., *Methods in nonlinear integral equations*, Kluwer Academic Publishers, 2002

[82] Ramalho, R., Existence and uniqueness theorems for nonlinear integral equation, *Univ. Federal de Pernambuco, Notas e Comun. de Matematica*, 40(1972), 1-42

[83] O'Regan, D., Petruşel, A., Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal*. Appl., 2008

[84] O'Regan, D., Precup, R., *Theorems of Leray-Schauder Type and Applications*, Series in Mathematical Analysis and Applications, vol. 3, Gordon and Breach Science Publishers, Amsterdam, 2001

[85] Rus, I. A., *Teoria punctului fix II, Teoria punctului fix în analiza funcţională*, Univ. Babeş-Bolyai Cluj, 1973

[86] Rus, I. A., *Approximation of fixed points of generalized contraction mappings*, Academic Press, London, 1975, 157-161

[87] Rus, I. A., On a fixed point theorem of Maia, *Studia Univ. Babeş-Bolyai*, *Mathematica*, 1(1977), 40-42

[88] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979

[89] Rus, I. A., *Metrical fixed point theorems*, University of Cluj-Napoca, 1979

[90] Rus, I. A., Teoreme de punct fix în spaţii Banach, *Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate*, Univ. Babeş-Bolyai Cluj-Napoca, 1982, 327-332

[91] Rus, I. A., Generalized contractions, *Seminar on Fixed Point Theory*, Babeş-Bolyai University, *Preprint* Nr.3, 1983, 1-130

[92] Rus, I. A., Picard mappings, *Studia Univ. Babeş-Bolyai, Mathematica*, 33(1988), 70-73

[93] Rus, I. A., A delay integral equation from biomathematics, *Babeş-Bolyai University of Cluj-Napoca, Preprint* Nr.3, 1989, 87-90

[94] Rus, I. A., Weakly Picard mappings, *Comment. Math. Univ. Caroline*, **34**, 3(1993), 769-773

[95] Rus, I. A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Casa de editură Transilvania Press, Cluj-Napoca, 1996

[96] Rus, I. A., Picard operators and applications, *Babeş-Bolyai Univ. of Cluj-Napoca, Preprint* Nr. 3, 1996

[97] Rus, I. A., *An abstract point of view for some integral equations from applied mathematics*, Proc. of the Int. Conf., University of the West, Timişoara, 1997, 256-270

[98] Rus, I. A., A general functional inequality and its applications, *Revue d'An. Num. Th. l'Approx.* 26(1997), 209-213

[99] Rus, I. A., Mureşan, S., Data dependence of the fixed point set of weakly Picard operators, *Studia Univ. Babeş-Bolyai, Mathematica*, 43(1998):1, 79-83

[100] Rus, I. A., A Fiber generalized contraction theorem and applications, *Académie Roumaine, Filiale de Cluj-Napoca, Mathematica*, Tome 41(64), Nr.1, 1999, 85-90

[101] Rus, I. A., Fiber Picard operators on generalized metric space and applications, *Scripta Scientiarum Mathematicarum*, Tomus I, Fasciculus II, anno MCMXCIX, 1999:2, 326-334

[102] Rus, I. A., Fiber Picard operators theorem and applications, *Studia Univ. Babeş-Bolyai, Mathematica*, **44**(1999), No. 3, 89-97

[103] Rus, I. A., Iancu, C., *Modelare matematică*, Transilvania Press, Cluj-Napoca, 2000

[104] Rus, I. A., On a class of functional-integral equations, *Sem. On Best Approximation Theory*, Cluj-Napoca, 2000

[105] Rus, I. A., Who authored the first integral equations book in the world, *Seminar on Fixed Point Theory*, 1(2000):1-4, 81-86

[106] Rus, I. A., *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001

[107] Rus, I. A., Weakly Picard operators and applications, *Seminar on Fixed Point Theory*, Babeş-Bolyai Univ. of Cluj-Napoca, Vol. 2, 2001, 41-58

[108] Rus, I. A., *Mathematical contributions of D. V. Ionescu*, Babeş-Bolyai University, Cluj-Napoca, 2001

[109] Rus, I. A., Petruşel, A., Petruşel, G., *Fixed point theory 1950-2000 romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002

[110] Rus, I. A., Picard operators and applications, *Scientiae Mathematicae Japonicae*, 58(2003):1, 191-219

[111] Rus, I. A., Fixed points, upper and lower fixed points: abstract Gronwall lemmas, *Carpathian Journal of Mathematics*, 20(2004), No. 1, 125-134

[112] Rus, I. A., Mureşan, S., Mureşan, V., Weakly Picard operators on a set with two metrics, *Fixed Point Theory*, Vol. 6, No. 2, 2005, 323-331

[113] Rus, I. A., Data dependence of the fixed points in a set with two metrics, *Fixed Point Theory*, Vol. 8, No. 1, 2007, 115-123

[114] Rus, I. A., Şerban, M. A., D. Trif, Step method for some integral equations from biomathematics, *Bull. Math. Soc. Sci. Math. Roumanie*, Tome 54(102)(2011) No. 2, 167-183

[115] Rzepecki, B., On the existence of exactly one solution of integral equations in the space L^p weith a mixed norm, *Ann. Polon. Math*., XXX, 3, 1975

[116] Schröder, J., *Operator Inequalities*, Acad. Press, New York, 1980

[117] Smirnov, V. I., *A course in higher mathematics*, vol. 4, Part 1, Ed. Nauka, Moscow, 1974

[118] Sotomayor, J., Smooth dependence of solution of differential equation on initial data: a simple proof, *Bol. Soc. Brasil*, 4, 1(1973), 55-59

[119] Stancu, D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză numerică şi teoria aproximării*, vol. 1, Presa Univ. Clujeană, Cluj-Napoca, 2001

[120] Stancu, D. D., Coman, Gh., Blaga, P., *Analiză numerică şi teoria aproximării*, vol. 2, Presa Univ. Clujeană, Cluj-Napoca, 2002

[121] Stuart, C. A., Existence theorems for a class of nonlinear integral equations, *Math. Z.* 137(1974), 49-64

[122] Şerban, M. A., Existence and uniqueness theorems for the Chandrasekhar's equation, *Académie Roumaine Filiale de Cluj-Napoca, Mathematica*, Tome **41**(64), Nr. 1, 1999, 91-103

[123] Şerban, M. A., Fiber φ-contractions, *Studia Univ. Babeş-Bolyai, Mathematica*, 34(1999), Nr.3, 99-108

[124] Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj-Napoca, 2002

[125] Şerban, M. A., Application of fiber Picard operators to integral equations, *Bul. Ştiinţific Univ. Baia Mare, Seria B, Matematică-Informatică*, Vol. XVIII(2002), Nr.1, 119-128

[126] Şerban, M. A., Rus, I. A., Petruşel, A., A class of abstract Volterra equations, via weakly Picard operators technique, *Math. Inequal. Appl*., 13(2010), no. 2, 255-269

[127] Torrejón, R., A note on a nonlinear integral equation from the theory of epidemics, *Nonlinear Analysis*, 14(1990), No.6, 483-488

[128] Walter, W., *Differential and integral inequalities*, Springer, Berlin, 1979

[129] Yosida, K., *Équations différentielles et integrales*, DUNOD, Paris, 1971

[130] Yosida, K., *Lectures on differential and integral equations*, Interscience, 1960

[131] Zima, M., The abstract Gronwall lemma for some nonlinear operators, *Demonstratio Math*., 31(1998), 325-332

Subject index

Index of authors

Dr. Maria Dobrițoiu is associate professor at the Department of Mathematics, University of Petrosani, Romania.

She graduated from the Faculty of Mathematics at the "Babes-Bolyai"University of Cluj-Napoca in 1978 and then, in 1979, she completed one year of specialization in Numerical Analysis. In 2000 she majored in Management at the Faculty of Science from the University of Petrosani.

In 2008 she received a Ph.D. in Mathematics at the "Babes-Bolyai" University of Cluj-Napoca. Fields of interest: Differential equations and integral equations, Numerical Analysis, Programming.

This book is a monograph of integral equations with modified argument and contains the results obtained by the author within a period that begins from her college years until to her doctoral studies, both carried out under the scientific coordination of Prof. Dr. Ioan A. Rus from the"Babes-Bolyai" University of Cluj-Napoca. This monograph is addressed to all who are interested of the study of integral equations with modified argument, in order to improve their knowledge, or to obtain new results in the field. The book is also useful to those concerned with the study of mathematical models governed by integral equations in general, and by integral equations with modified argument in particular.

978-1-61804-372-6