EXACT SOLUTION OF ONE PHASE STEFAN PROBLEM
BY HEAT POLYNOMIALS AND INTEGRAL ERROR FUNCTIONS

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Abstract: -Solution of one phase Stefan Problem with degenerate domain at the initial time represented analytically. The developed method is based on the use of Integral Error Functions and heat polynomials. Elaborated method can be effectively used in the fields of engineering, which require consideration of phenomena with phase transformations, such as heat and mass transfer, low temperature plasma, filtration. The main idea of the method is to find coefficients of linear combination of Integral Error Functions and heat polynomials which a priori satisfy the heat equation.

Key-Words: - Stefan problem, free boundary, heat polynomials, degenerate domain

1 Introduction
The first analytical solution of one phase Stefan problem, which describes the dynamics of soil freezing has been published by Lame and Clayperon. Solution of two phase Stefan problem was represented by Stefan. Solutions of these problems were obtained for \( \alpha(t) = \alpha_0 \sqrt{t} \) case and some automodel cases.

Despite the quite extensive list of problems in literature which lead to the necessity to solve Stefan type problems see: e.g., [1]-[9] and a long bibliography [10] on methods for solving these problems lead to additional difficulties which occur due to the degeneracy of domains. In some specific cases particularly for free moving boundaries it is possible to construct Heat potentials and a problem can be reduced to the system of integral equations [2]-[3], however in the case of degeneracy, singularity in integral equations occur, and method of successive approximations is inapplicable in general. Moreover, the use of numerical methods is problematic when the number of parameters is great. Therefore, development of new analytical methods is very important especially for various applications because it enables one to analyze an interrelationship of different input parameters and their influence on the dynamics of investigating phenomena.

As for applications: a wide range of electric contact phenomena, in particular, the phenomena occurring at the interaction of electrical arc with electrode can be described in dynamic use of the presented method see e.g., [11]-[13] for very short arc duration (nanosecond diapason), when experimental investigation is very difficult.

In this study we will try to find solution of one Phase Stefan problem for degenerate domain with \( \alpha(t) = \sum_{n=1}^{\infty} \alpha_n t^n \) moving boundary.

Tracking answers of these questions will be organized as following. In the continuation of this section Integral Error Functions and its properties necessary for elaboration of new methods are presented. In subsection 1.4 a test problem is solved by proposed method. In sections two and three one phase Stefan problem, its analytical solution and convergence of series represented. For finding analytical solution we mainly follow the method proposed by S.N.Kharin in [14] applying Faa Di Bruno’s formula for Integral Error Functions. For Heat polynomials we utilize Newton’s polynomial (generalization of Newton’s binomial) and its multinomial coefficients. Section four devoted to discussion and conclusion.

1.2 Integral Error Functions
Integral Error Functions or Hartree functions were firstly introduced by Hartree in 1935 and reasonably sometimes called Hartree functions.

The Integral Error Functions determined by recurrent formulas
\[ i^n \text{erfc}(x) = \int_{x}^{\infty} i^n \text{erfc} v \, dv \quad n=1,2,\ldots \]
\[ i^n \text{erfcx} = \text{erfcx} = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \exp(-v^2) \, dv \]

where
\[ \text{erfc} x = 1 - \text{erf} x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \exp(-v^2) \, dv \]

One can obtain from (1)
\[ i^n \text{erfcx} = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} (v-x)^n \exp(-v^2) \, dv \quad (3) \]

They satisfy the differential equation
\[ \frac{d^2}{dx^2} i^n \text{erfcx} + 2x \frac{d}{dx} i^n \text{erfcx} - 2n^2 i^n \text{erfcx} = 0 \quad (4) \]
and recurrent formulas
\[ 2n^2 i^n \text{erfcx} = i^{n-2} \text{erfcx} - 2xi^{n-1} \text{erfcx} \quad (5) \]

On the base of Integral Error Functions and its properties so called IEF method [15] (Integral Error Functions method or Hartree functions method) was developed which is very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation
\[ \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (6) \]
in a region \( D(t > 0, 0 < x < \alpha(t)) \) with free boundary \( x = \alpha(t) \). Functions
\[ u_n(\pm x,t) = \sum_{m=0}^{\infty} \left[ A_n u_n(x,t) + B_n u_n(-x,t) \right] \quad (7) \]
suffice the equation (6) as well as their linear combination or even series
\[ u(x,t) = \sum_{n=0}^{\infty} \left[ A_n u_n(x,t) + B_n u_n(-x,t) \right] \]
for any constants \( A_n, B_n \). We can choose these constants to satisfy the boundary conditions at \( x = 0 \) and \( x = \alpha(t) \), if given boundary functions can be expanded into Taylor series with powers \( t \) or \( \sqrt{t} \).

1.3 Properties of Integral Error Functions

It is possible to derive properties of Integral Error Functions.

1. If \( n \) is an integer, then
\[ \begin{align*}
  i^n \text{erfc}(-x) + (-1)^n i^n \text{erfcx} &= \frac{1}{2^{n-1} n!} H_n(ix) = \\
  &= \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} e^{x^2} \\
\end{align*} \]

with \( i = \sqrt{-1} \) and Hermite polynomials \( H_n(x) \) in the right side. Indeed, using formula (1) one can write
\[ \begin{align*}
  i^n \text{erfc}(-x) + (-1)^n i^n \text{erfcx} &= \frac{1}{2^{n-1} n!} \int_{x}^{\infty} (v-x)^n \exp(-v^2) \, dv + \\
  &= \frac{1}{2^{n-1} n!} \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} (v+x)^n \exp(-v^2) \, dv + \\
  &\quad + \frac{1}{2^{n-1} n!} \int_{x}^{\infty} (v-x)^n \exp(-v^2) \, dv = \frac{1}{2^{n-1} n!} H_n(ix) \\
\end{align*} \]

Using formula for Hermite polynomials one can derive
\[ \begin{align*}
  i^n \text{erfc}(-x) + (-1)^n i^n \text{erfcx} &= \sum_{m=0}^{\infty} \frac{x^{-2m}}{2^{2m-1} m!(n-2m)!} \\
\end{align*} \]

If \( n = 2k \), then
\[ \begin{align*}
  i^{2k} \text{erfc} x + i^{2k} \text{erfc}(-x) &= \sum_{m=0}^{k} \frac{x^{2(k-m)}}{2^{2m-1} m!(2k-2m)!} \\
\end{align*} \]

In particular
\[ \begin{align*}
  \text{erfc} x + \text{erfc}(-x) &= 2 \\
  i^2 \text{erfc} x + i^2 \text{erfc}(-x) &= \frac{1}{2} x^2 \\
  i^4 \text{erfc} x + i^4 \text{erfc}(-x) &= \frac{1}{4} x^4 + \frac{1}{12} x^4 \\
\end{align*} \]

If \( n = 2k + 1 \), then
\[ \begin{align*}
  i^{2k+1} \text{erfc}(-x) - i^{2k+1} \text{erfc} x &= \sum_{m=0}^{k} \frac{x^{2(k-m)+1}}{2^{2m-1} m!(2k-2m+1)!} \\
\end{align*} \]

In particular
\[ \begin{align*}
  i^n \text{erfc}(-x) - i^n \text{erfc} x &= 2x \\
  i^3 \text{erfc}(-x) - i^3 \text{erfc} x &= \frac{1}{2} x + \frac{1}{3} x^3 \\
  i^5 \text{erfc}(-x) - i^5 \text{erfc} x &= \frac{1}{2^2 \cdot 2!} x + \frac{1}{2 \cdot 2! \cdot 3!} x^3 + \frac{2}{5!} x^5 \\
\end{align*} \]

2. The proof of the formula
\[ \begin{align*}
  i^n \text{erfc}(-x) - (-1)^n i^n \text{erfcx} &= \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} (e^{x^2} \text{erfx}) \\
\end{align*} \]
can be obtained by mathematical induction method using recurrent formula (4).

3. Differentiating the right side of (10), we obtain
\[ i^n \text{erfc}(x) - (-1)^n i^n \text{erfc}x = P_n(x) \text{erfc}x - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2) \] (11)

where polynomials \( P_n(x) \) and \( Q_n(x) \) are defined by formulas
\[ P_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2m}}{2^{m-1} \cdot m!(n-2m)!}, \]
\[ Q_n(x) = \sum_{k=0}^{n} (-1)^k \frac{1}{2^{n-k}} H_{n-k}(x) \]

4. From (10), (11) we can obtain the explicit expressions for Integral Error Functions of an integer index
\[ i^n \text{erfc}x = \frac{(-1)^n}{2} [P_n(x) \text{erfc}x + Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \] (12)
\[ i^n \text{erfc}(-x) = \frac{1}{2} [P_n(x) \text{erfc}(-x) - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \] (13)

5. Using L’Hopital rule and representation (1), it is not difficult to show that
\[ \lim_{x \to \infty} i^n \text{erfc}(-x) = \frac{2}{n!} \] (14)

6. Using property 2 one can derive following formula
\[ u(x,t) = \sum_{n=0}^{\infty} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} \] (15)

where \( u(x,t) \) is a solution of Heat Equation and called Heat Polynomials
\[ \beta(n,m) = \frac{1}{2^{n-1} \cdot m!(n-2m)!} \]

1.4 A domain with moving boundary. Approximate solution of test problem.

Let the heat equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \] (16)

be given on the interval with the moving boundary
\( D: \{0 < x < t, \quad 0 < t < 1\} \) subjected to boundary conditions
\[ u|_{x=0} = e^t \] (17)
\[ u|_{x=t} = 1 \] (18)

and fitting condition
\[ u|_{x=0} = 1 \] (19)

Using property (15) we consider approximate solution in the following form
\[ u(x,t) = \sum_{n=0}^{\infty} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} \] (20)

This function satisfies Heat equation and it is required to determine even and odd coefficients \( A_{2n}, A_{2n+1}. \)

Satisfying the boundary conditions (17) and (18) for \( x=0 \) we get,
\[ e^t = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} \] (21)

For \( x=t \)
\[ 1 = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} \] (22)

If we take \( k=5 \) and satisfy the equations (21) and (22) at the \( t = t_i = \frac{i}{5}, \quad i = 0,1,2,3,4,5 \) we get the values for \( A_{2n}, A_{2n+1} \).

Fig. 1 depicts the graph of approximate function
\[ v(t) = \left\{ \sum_{n=0}^{5} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} \right\} \] (23)
and original function \( e^t = \exp(t) \) at the boundary \( x=0 \), which are almost identical.
The greatest error of approximation is in the neighbourhood of zero. The graphs for this neighbourhood are presented in Fig. 2.

One can see that the error of approximation is less than 1%.

Similar situation can be observed at the second boundary $x = t$. The graphs of the functions

$$g(t) = 1$$

and

$$W(t) = \left\{ \sum_{n=0}^{5} \left( A_{2n} \sum_{m=0}^{n} x^{2n-2m+1} \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} \beta_{2n+1,m} \right) \right\}_{t=0}$$

are presented in Fig. 3 and Fig. 4.

The greatest error of approximation is less than 0.15%.

Thus if we replace the original functions $e^t$ and $1$ by approximate functions, then according to the Maximum Principle for the heat equation the error of approximation of the solution in the whole domain is not greater than the error on the boundaries.

2 Problem Formulation

In the following two sections we will deal with the analytical solution of heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \alpha(t), \quad 0 < t < \infty, \quad (23)$$

subjected to the following conditions:

$$u(0,t) = 0 \quad (24)$$

$$-\lambda \frac{\partial u}{\partial x}_{x=\alpha(t)} = P(t), \quad 0 < t < \infty, \quad (25)$$

$$u_{x=\alpha(t)} = U_m, \quad 0 < t < \infty, \quad (26)$$

The Stefan’s condition:

$$-\lambda \frac{\partial u}{\partial x}_{x=\alpha(t)} = L \frac{d\alpha(t)}{dt}, \quad 0 < x < \alpha(t), \quad (27)$$

where $P(t)$ is a heat flux coming from electric arc and $U_m$ is a melting temperature of electric contact material. Power balance is described by Stefan’s condition (27) and heat is consumed only for melting the solid region. The heat spread in the solid is negligible because of the physical properties of contact material. This condition is valid for refractory metals like wolfram.

3 Problem Solution

Along with the desired unknown function $u(x,t)$ an unknown moving boundary $\alpha(t) = \sum_{n=1}^{\infty} \frac{\xi_n^2}{\beta_n}$ has to be found.
3.1 Analytical solution of one phase Stefan problem

We consider solution in the form of combination of Heat polynomials and Integral Error Functions

\[ u(x,t) = \sum_{n=0}^{\infty} C_n \sum_{m=0}^{\infty} x^{2n-2m+1} \beta_{2m+1} + \sum_{n=0}^{\infty} A_n (2\alpha \sqrt{t})^n \left[ i^n \text{erfc} \left( -\frac{x}{2\alpha \sqrt{t}} \right) + i^n \text{erfc} \left( \frac{x}{2\alpha \sqrt{t}} \right) \right] \]  

Taking k times derivatives at t=0 of (25) or equating coefficients at same powers of t we have

\[ C_k = -\frac{1}{2\lambda \beta_{2m+1}} \rho^{(k)} \]  

making substitution \( \sqrt{t} = \tau \) into the Heat polynomial of (28) we have

\[ \sum_{n=0}^{\infty} C_n \sum_{m=0}^{\infty} \left[ \alpha(\tau) \right] \tau^{2m} \beta_{2m+1} = \sum_{n=0}^{\infty} C_n \sum_{m=0}^{\infty} \left( 2n - 2m + 1 \right) \left( s_1, s_2, ..., s_k \right) \cdot \alpha_1^s \alpha_2^s ... \alpha_k^s \tau^{(n_1 + n_2 + ... + k_1 + 2m)} \]

To find \( A_n \) coefficients we utilize Leibniz rule for k-th derivative of product and Faa Di Bruno formula for k-th derivative of composite function, thus

\[ \left[ \sum_{n=0}^{\infty} A_n (2\alpha \tau)^n \left[ i^n \text{erfc} \left( \pm \frac{\alpha(\tau)}{2\alpha \tau} \right) \right] \right]^{(k)} = \sum_{n=0}^{\infty} \frac{2^k k!}{(k-n)!} \left[ i^n \text{erfc} \left( \pm \delta \right) \right]^{(k-n)} \]

\[ = \sum_{n=0}^{\infty} \frac{2^k k!}{(k-n)!} \sum_{m=0}^{n} \left[ i^n \text{erfc} \left( \pm \delta \right) \right]^{(m)} \cdot \cdot B_{k-n,m} \left( \left( \pm \delta \right)^1, \left( \pm \delta \right)^2, ..., \left( \pm \delta \right)^{k-n} \right) \]

where \( B_{k-n,m} \) is Bell’s polynomial

\[ B_{k-n,m} = \sum_{j_1 + j_2 + ... + j_{k-n-m+1} = m} \frac{(k-n)!}{j_1! j_2! ... j_{k-n-m+1}!} \cdot \left( \pm \delta_1^j \right)^{j_1} \left( \pm \delta_2^j \right)^{j_2} ... \left( \pm \delta_{k-n-m+1}^j \right)^{j_{k-n-m+1}} \]

where \( \delta = \frac{\alpha(\tau)}{2a\tau} \) and \( \delta_n = \frac{\alpha_n}{2a} \)

\[ j_1 + j_2 + ... + j_{k-n-m+1} = m, \]

\[ j_1 + 2j_2 + ... + (k-n-m+1)j_{k-n-m+1} = k - n, \]

\[ \left[ \text{erfc}(\pm \delta) \right]^{(n)} \left[ n=0 \right] = \left( \pm 1 \right)^{\frac{m}{2}} \frac{\Gamma \left( \frac{n-m+1}{2} \right)}{(n-m)! \sqrt{n}} \]

Ultimately, taking k-times derivative of (26) at \( \tau = 0 \) we have

\[ \left[ \sum_{n=0}^{\infty} A_n (2\alpha \tau)^n \left[ i^n \text{erfc} \left( -\delta(\tau) \right) + i^n \text{erfc} \left( \delta(\tau) \right) \right] \right]^{(k)} \]

\[ = \left[ u_m - \sum_{n=0}^{\infty} C_n \sum_{m=0}^{\infty} \left( \alpha(\tau) \right)^{2n-2m+1} \tau^{2m} \beta_{2m-2m+1} \right]^{(k)} \]

yields

\[ \sum_{n=0}^{\infty} A_n \frac{2^k k!}{(k-n)!} \sum_{m=0}^{n} (-1)^m \left[ \frac{\Gamma \left( \frac{n-m+1}{2} \right)}{(n-m)! \sqrt{n}} \right] \cdot \sum_{j_1 + j_2 + ... + j_{k-n-m+1} = m} \left( \pm \delta_1^j \right)^{j_1} \left( \pm \delta_2^j \right)^{j_2} ... \left( \pm \delta_{k-n-m+1}^j \right)^{j_{k-n-m+1}} \]

\[ = \sum_{n=0}^{\infty} C_n \sum_{m=0}^{n} \sum_{j_1 + j_2 + ... + j_{k-n-m+1} = m} \left( 2n - 2m + 1 \right) \cdot \cdot B_{k-n,m} \left( \left( \pm \delta \right)^1, \left( \pm \delta \right)^2, ..., \left( \pm \delta \right)^{k-n} \right) \]

(30)

Thus \( A_k \) can be determined from the formula (30). Taking both sides of (27) at \( \tau = 0 \), in the same manner, we determine \( \alpha_k \) coefficients from following recurrent formula

\[ \alpha_{k+1} = \left[ \frac{\lambda}{L} \cdot \frac{\partial u}{\partial \alpha(\tau)} \right]^{(k)} \cdot k = 0, 1, 2, ... \]  

(31)
3.2 Convergence

Let $\alpha(t_0) = \alpha_0$ for any time $t = t_0$. Then the series

$$
\sum_{n=0}^{\infty} A_n \left(2a\sqrt{t_0}\right)^n \left[ i^n \text{erfc} \left( -\frac{\alpha_0}{2a\sqrt{t_0}} \right) + i^n \text{erfc} \left( \frac{\alpha_0}{2a\sqrt{t_0}} \right) \right]
$$

should be converged because $u = U_m$ on the interface. Therefore there exist a constant $C_1$, independent of $n$, such that

$$
|A_n| < C_1 \left(2a\sqrt{t_0}\right)^n \left[ i^n \text{erfc}(-\delta_0) + i^n \text{erfc}\delta_0 \right],
$$

$$
\delta_0 = \frac{\alpha_0}{2a\sqrt{t_0}}
$$

The function $i^n \text{erfc}(-\delta) + i^n \text{erfc}\delta$ is a monotonically increasing positive function, therefore

$$
i^n \text{erfc}(-\delta) + i^n \text{erfc}\delta < i^n \text{erfc}(-\delta_0) + i^n \text{erfc}\delta_0,
$$

$0 < \delta < \delta_0$

Thus

$$
\sum_{n=0}^{\infty} \left(2a\sqrt{t_0}\right)^n \left[ i^n \text{erfc}(-\delta) + i^n \text{erfc}\delta \right] <
$$

$$
< C_1 \sum_{n=0}^{\infty} \frac{t}{t_0} \left(2a\sqrt{t_0}\right)^n \frac{i^n \text{erfc}(-\delta_0) + i^n \text{erfc}\delta_0}{i^n \text{erfc}(-\delta_0) + i^n \text{erfc}\delta_0} < C_1 \sum_{n=0}^{\infty} \left(\frac{t}{t_0}\right)^{n/2}
$$

These are geometric series and the series for $u(x,t)$ converges for all $x < \alpha_0$ and $t < t_0$.

The series for $\alpha(t)$ can be estimated similarly.

This means that $u(x,t)$ is bounded, thus is the series for $\alpha(t)$ converges for all $t < t_0$.

4 Conclusion and discussion

Thus, coefficients of series (28) $C_1$, $A_i$ and $\alpha_i$ are obtained from (29), (30) and (31) respectively.

Successful applications of represented method induce the question: is it possible to find and to use similar functions for solution of other equations? It can be shown that Degenerate Hypergeometric functions and their linear combinations satisfy the equation

$$
\frac{\partial^2 \theta}{\partial t^2} = a^2 \left( \frac{\partial^2 \theta}{\partial z^2} + \frac{\nu}{z} \frac{\partial \theta}{\partial z} \right)
$$

Where $\nu = 0, 1, 2$ are plane, cylindrical and spherical cases which is the subject of further investigations.

References:


