Estimation for Burr-X model based on progressively censored with random removals: Bayesian and non-Bayesian Approaches

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Abstract: This paper considers the estimation problem for the Burr type-X, when the lifetimes are collected under Type-II progressive censoring with random removals, where the number of units removed at each failure time follows a binomial distribution. We use the methods of maximum likelihood as well as the Bayes procedure to derive both point and interval estimators of the parameters. The expected test time to complete the test is computed and analyzed for different censoring schemes. The effect of the binomial parameter $p$ on the expected test time under progressive censoring and the relative expected test time over the complete sample are investigated. Monte Carlo simulations are performed to compare the performance of the different methods and for the expected termination time of the test. Furthermore, an example is presented for illustrative purposes.

Key–Words: Burr Type-X model; Maximum Likelihood estimator; Bayes estimator; Type-II progressive censoring with random removals; Life testing; Expected test time.

1 Introduction

In many life test studies, it is common that the lifetimes of test units may not be able to record exactly. An experimenter, may be terminate the life test before all units fail in order to save time or cost. Therefore, the test is considered to be censored in which data collected are the exact failure times on those failed units and the running times on those non-failed units. A generalization of Type-II censoring is the Type-II progressive censoring which is useful when the loss of live test units at points other than the termination point is unavoidable. For the theory methods and applications of progressive censoring, one can refer to the monograph by Balakrishnan and Aggarwala [3] and the recent survey paper by Balakrishnan [2]. Recently, the Type-II progressively censoring scheme has received considerable interest among the statisticians. In this scheme, the number of units (i.e. $n$), the number of observed failure times (i.e. $m$) and the numbers of withdrawn unite are all prefixed. However, in many practical applications, test units may have to be removed during test due to excessive pressure although they have not yet failed completely; furthermore, due to the reduction of budget and facility. In some industrial experiments, it may be too dangerous to carry on the testing on some test units, or some units have to be taken for other analysis. Therefore, these units have to be removed from the test even though these units have not failed completely. Thus, the number of removals cannot be prespecified and is random pending on the outcome of the experiment.

Inference, sampling design and generalization based on progressively censored samples were studied by many authors. However, there is very little work introduced in the Bayesian context. Amin [1] considered the Bayes estimation and Bayes prediction problems for the Pareto distribution based on the Type-II progressive censoring with random removals. Also, in Bayesian setting, Sarhan and Abuammoh [8] discussed some statistical inference for the exponential distribution using progressively censoring sample with random removals. Wu [11] has studied estimation for the Pareto distribution under progressive censoring with uniform removals. A uniform removal pattern may not seem very realistic as it assumes that each removal event occurs with an equal probability regardless of the number of units removed. A more realistic alternative to describe the number of occurrences of an event out of $n$ trials is the binomial distribution as suggested by Tse et al. [10].

The rest of this paper is organized as follows, the model formulation and the corresponding likelihood function under Type-II progressive censoring with binomial removals are discussed in section 2. In section 3, the procedures of obtaining the maximum likelihood estimates of the parameters $\theta$ and $p$ are discussed. Both point and interval estimations of the parameters are derived. Point and interval estimations
using Bayesian procedures are presented in sections 4. In section 5, we discussed the expected test time under Type-II progressive censoring with the effect of various p. Illustrative examples and the results from simulation studies assessing the performance of our proposed method are included in section 6. Finally, we conclude the paper in section 7.

2 The Model

Let the lifetime of a particular unit have a Burr type-X distribution with probability density function (pdf)

\[ f(x; \theta) = 2\theta x \exp(-x^2)(1 - \exp(-x^2))^{\theta-1}, \]

where \( x > 0, \theta > 0. \)

The corresponding cumulative distribution function (cdf) is

\[ F_X(x) = (1 - \exp(-x^2))^\theta, \quad x > 0, \quad \theta > 0. \]  

Let \((X_1, R_1), (X_2, R_2), \ldots, (X_m, R_m)\) denotes a progressively type II censored sample, where \(X_1 < X_2 < \cdots < X_m\) with predetermined number of removals, say \(R_1 = r_1, R_2 = r_2, \ldots, R_m = r_m,\) the conditional likelihood function can be written as in [5]

\[ L(\theta; x \mid R = r) = c \prod_{i=1}^{m} f(x_i) [1 - F(x_i)]^{r_i}. \]  

where \( c = n \prod_{i=1}^{m} (n-r_i-1) \cdots (n-r_i-r_{i-1}-m+1), \) and \( r_i \) can be any integer value between 0 and \((n-m-r_i-\cdots-r_{i-1})\) for \( i = 1, 2, 3, \ldots, m-1, \) substituting (1) and (2), into (3) we get

\[ L(\theta; x \mid R = r) = c \prod_{i=1}^{m} f(x_i) \theta^m \sum_{j=0}^{r_i} \cdots \sum_{j_m=0}^{r_m} G \times \exp\left( \theta \sum_{i=1}^{m} (ji + 1) \ln U_i \right). \]  

Where

\[ T(x) = \prod_{i=1}^{m} 2x_i \exp(-x_i^2)(U_i)^{-1}, \]

\[ U_i = (1 - \exp(-x_i^2)) \quad \text{and} \quad G = (-1)^{j_1+\cdots+j_m}. \]  

Suppose that an individual unit being removed from the test at the \( i^{th} \) failure, \( i = 1, 2, \ldots, m - 1, \) is independent of the others but with same probability \( p. \) Then, the number \( R_i \) of units removed at the \( i^{th} \) failure, \( i = 1, 2, \ldots, m - 1, \) follows a binomial distribution with parameters \( n - m - \sum_{i=1}^{m-1} r_i \) and \( p. \) Therefore,

\[ P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1 - p)^{n-m - r_1}, \]  

and for \( i = 1, 2, 3, \ldots, m - 1 \)

\[ P(R_i = r_i \mid R_{i-1} = r_{i-1}, \ldots, R_1 = r_1) = \binom{n-m-\sum_{i=1}^{m-1} r_i}{r_i} p^{r_i}(1 - p)^{n-m-\sum_{i=1}^{m-1} r_i}, \]  

where \( 0 \leq r_i \leq n-m \) and \( 0 \leq r_i \leq n-m-\sum_{i=1}^{m-1} r_i \) for \( i = 1, 2, 3, \ldots, m - 1. \) Furthermore, suppose that \( R_i \) is independent of \( x_i \) for all \( i. \) Then the likelihood function takes the following form,

\[ L(\theta, p; x \mid R = r) = L(\theta; x \mid R = r) P(R = r), \]  

where

\[ P(R = r) = \frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1} r_i)! \prod_{i=1}^{m-1} r_i! \times \sum_{i=1}^{m-1} r_i (1 - p)^{(m-1)(n-m)-\sum_{i=1}^{m-1} (m-i)r_i}}. \]  

Using (4), (8) and (10), the full likelihood function take the following form

\[ L(\theta, p; x, r) = c^r T(x) L_1(\theta) L_2(p), \]  

where

\[ L_1(\theta) = \theta^m \sum_{j=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \exp\left( \theta \sum_{i=1}^{m} (ji + 1) \ln U_i \right). \]
It is obvious that $L_m$arnations of the parameters are derived.

3 Maximum Likelihood Estimation

This sections discuss the procedures of obtaining the Maximum likelihood estimates of the parameters $\theta$ and $p$ based on progressively Type-II censoring data with binomial removals. Both point and interval estimations of the parameters are derived.

3.1 Point Estimation

It is obvious that $L_1$ in Eq. (12) does not involve $p$. Thus the maximum likelihood estimate (MLE) of $\theta$ can be derived by maximizing Eq. (12) directly. On the other hand, $L_2$ in Eq. (13) does not depend on the parameter $\theta$, then the MLE of $p$ can be obtained directly by maximizing Eq. (13). In particular, after taking the logarithms of $L_1(\theta)$ and $L_2(p)$, The MLE’s of $\theta$ and $p$ can be found by solving the following equations

$$
\frac{\partial \log L_1(\theta)}{\partial \theta} = \frac{m}{\theta} + \sum_{j_i=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G \sum_{i=1}^{m} (j_i + 1) \ln U_i = 0,
$$

3.2 Interval Estimation

3.2.1 Bootstrap Confidence Intervals

In this subsection, we use the parametric bootstrap percentile method suggested by [6] to construct confidence intervals for the parameters. The algorithms for estimating confidence intervals of the parameters $\theta$ and $p$ are illustrated below.

1. From the original data $X = X_1, X_2, \cdots, X_m$ with the corresponding values $R = r_i, i = 1, 2, \cdots, m$ compute the ML estimates $\hat{\theta}$ and $\hat{p}$ of the parameters using equations (16) and (17).

2. Use $\hat{p}$ to generate a bootstrap sample $R^* = r_i^*, i = 1, 2, \cdots, m$ using binomial distribution, where $r_i^*$ follows the bin$(n - m, \hat{p})$ distribution and the variables $r_i^*, r_2^*, \cdots, r_{m-1}^*$ follow the bin$(n - m - \sum_{j=1}^{i-1} r_j^*, \hat{p})$ distributions for $i = 2, 3, \cdots, m - 1$.

3. Use $\hat{\theta}$ in step 1, with the binomial progressive censoring scheme obtained in step 2, we generate a bootstrap sample $X^* = X_1^*, X_2^*, \cdots, X_m^*$ using algorithm presented in [4].
4. As in step 1, based on $X^*$ compute the bootstrap sample estimates of $\theta$ and $p$, say $\hat{\theta}^*$ and $\hat{p}^*$.

5. Repeat steps 2-4 $N$ BOOT times.

6. Arrange all $\hat{\theta}^*$’s and $\hat{p}^*$’s, in an ascending order to obtain the bootstrap sample $(\varphi_1^{[1]}, \varphi_2^{[2]}, ..., \varphi_l^{[N]})$, $l = 1, 2$ (where $\varphi_1 \equiv \hat{\theta}^*, \varphi_2 \equiv \hat{p}^*$).

Let, $G(z) = P(\varphi_l \leq z)$ be the cumulative distribution function of $\varphi_l$. Define $\varphi_{lboot} = G^{-1}(z)$ for given $z$. The $100(1-\delta)\%$ approximate bootstrap confidence interval of $\varphi_l$ is given by

$$[\varphi_{lboot}(\gamma/2), \varphi_{lboot}(1 - \gamma/2)]. \quad (18)$$

### 4 Bayes Estimation

It is well known that choice of loss function is an integral part of Bayesian estimation procedures. In this section we obtain the Bayes estimation of the parameters using both symmetric and asymmetric loss functions, namely (squared error(SE) and LINEX loss functions), for details see [9].

#### 4.1 Point Estimation

We assume that the parameters $\theta$ and $p$ behave as independent random variables. We use gamma prior distribution with known parameters $\alpha, \beta$ for $\theta$, which is given by

$$\pi_1(\theta \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\beta\theta), \quad (19)$$

where $\theta > 0, \alpha > 0, \beta > 0$.

While $p$ has a $\text{Beta}(\gamma, \lambda)$ prior distribution given by

$$\pi_2(p) = \frac{1}{B(\gamma, \lambda)}p^{(\gamma-1)}(1-p)^{(\lambda-1)}, \quad (20)$$

where $0 < p < 1, \gamma, \lambda > 0$.

The joint prior (pdf) of $(\theta; p)$ is

$$\pi(\theta; p) = \pi_1(\theta)\pi_2(p) = \frac{\beta^\alpha}{B(\gamma, \lambda)\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\beta\theta)p^{(\gamma-1)}(1-p)^{(\lambda-1)},$$

where $\theta > 0, 0 < p < 1$.

Therefore the joint posterior distribution of $\theta$ and $p$ is

$$\pi^*(\theta, p \mid x, r) = \frac{p^{(\gamma-1)}(1-p)^{(\lambda-1)}}{B(\gamma, \lambda)\Gamma(m + \alpha)} \sum_{j_1=0}^{r_1} \sum_{j_m=0}^{r_m} G^{m+\alpha-1} \exp\{-q_j\theta\}$$

$$\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G^{j_m-m-\alpha}, \quad (22)$$

where

$$q_j = \beta - \sum_{i=1}^{m} (j_i + 1) \ln U_i,$$

$$\gamma^* = \gamma + \sum_{i=1}^{m-1} r_i,$$

and $\lambda^* = \lambda + (m - 1)(n - m) - \sum_{i=1}^{m-1} (m - i)r_i. \quad (23)$

Therefore, the marginal posterior distributions of $\theta$ and $p$ are given by

$$\pi_1^*(\theta \mid x, r) = \frac{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G^{m+\alpha-1} \exp(-\theta q_j)}{\Gamma(m + \alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G^{j_m-m-\alpha}}, \quad (24)$$

$$\pi_2^*(p \mid x, r) = \frac{1}{B(\gamma^*, \lambda^*)}\Gamma(\gamma^*-1)(1-p)^{\lambda^*-1}. \quad (25)$$

We notes that the posterior distribution of $\theta$ is Gamma with parameters $(m + \alpha)$ and $q_j$, while the posterior distribution of $p$ is Beta with parameters $\gamma^*$ and $\lambda^*$.

#### 4.1.1 Symmetric Bayes Estimation

**SE loss function:** Under $SEL$ function (symmetric), the estimator of the parameter is the posterior mean. Thus, using the posterior densities (24) and (25), the Bayes estimators $\bar{\theta}_{BS}$ and $\bar{p}_{BS}$ of the parameters $\theta$ and $p$ are

$$\bar{\theta}_{BS} = \int_0^{\infty} \theta \pi_1^*(\theta \mid x, r) d\theta,$$

$$\bar{p}_{BS} = \int_0^{\gamma^*-1} \pi_2^*(p \mid x, r) dp,$$

$$\gamma^* = \gamma + \sum_{i=1}^{m-1} r_i,$$

$$\lambda^* = \lambda + (m - 1)(n - m) - \sum_{i=1}^{m-1} (m - i)r_i. \quad (23)$$

$$\pi_1^*(\theta \mid x, r) = \frac{\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G^{m+\alpha-1} \exp(-\theta q_j)}{\Gamma(m + \alpha) \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} G^{j_m-m-\alpha}}, \quad (24)$$

$$\pi_2^*(p \mid x, r) = \frac{1}{B(\gamma^*, \lambda^*)}\Gamma(\gamma^*-1)(1-p)^{\lambda^*-1}. \quad (25)$$

We notes that the posterior distribution of $\theta$ is Gamma with parameters $(m + \alpha)$ and $q_j$, while the posterior distribution of $p$ is Beta with parameters $\gamma^*$ and $\lambda^*$. 
and

\[ p_{\tilde{\theta}S} = \int_0^\infty p \pi_2^*(p \mid x, r)dp = \frac{\gamma^r}{\gamma+r} \]  

(27)

4.1.2 Asymmetric Bayes estimation

**LINEX loss function:** The Bayes estimate \( \tilde{\theta}_{BL} \) of the parameter \( \theta \) relative to LINEX loss function using (24) is

\[
\tilde{\theta}_{BL} = -\frac{1}{a} \log \left[ \int_0^\infty \exp(-a\theta)\pi_1^*(\theta \mid X)d\theta \right],
\]

\[
= -\frac{1}{a} \log \left[ \sum_{j_1=0}^{r_1} \sum_{j_m=0}^{r_m} G_{q_j}^{-(m+\alpha)} \right] \tag{28}
\]

Similarly, the Bayes estimate \( \tilde{p}_{BL} \) of the parameter \( p \) relative to LINEX loss function using (25) is

\[
\tilde{p}_{BL} = -\frac{1}{a} \log \left[ \int_0^1 \exp(-ap)\pi_2^*(p \mid x, r)dp \right]
\]

\[
= -\frac{1}{a} \log \left[ \frac{1}{B(\gamma^*, \lambda^*)} \times \int_0^1 \exp(-ap)^{(\gamma^*+1)}(1-p)^{(\lambda^*-1)}dp \right]. \tag{29}
\]

One can use a numerical integration technique to get the integration in (29).

4.2 Interval Estimation

**highest posterior density interval (HPDI)**

In general, the Bayesian method to interval estimation is much more direct than the maximum likelihood method. Now, having obtained the posterior distribution \( p(\omega \mid Data) \) we ask, “How likely is it that the parameter \( \omega \) lies within the specified interval \( [\omega_L, \omega_U] \)?”. Bayesian call this interval based on the posterior distribution a ‘credible interval’. The interval \( [\omega_L, \omega_U] \) is said to be a \( (1-\delta)100\% \) credible interval if

\[
\int_{\omega_L}^{\omega_U} p(\omega \mid Data)d\omega = 1 - \alpha . \tag{30}
\]

For the shortest credible interval, we have to minimize the interval \( [\omega_L, \omega_U] \) subject to the condition (30) which requires

\[
p(\omega_L \mid Data) = p(\omega_U \mid Data). \tag{31}
\]

As interval \( [\omega_L, \omega_U] \) which simultaneously satisfies (30) and (31) is called the ‘shortest’ \( (1-\delta)100\% \) credible interval. A highest posterior density interval (HPDI) is such that the posterior density for every point inside the interval is greater than that for every point outside of it. For a unimodal, but not necessarily symmetrical, posterior density the shortest credible and the HPD intervals are identical.

We now proceed to obtain the \( (1-\delta)100\% \) HPDI intervals for the parameters \( p \) and \( \theta \). Consider the posterior distribution of \( \theta \) in (24), the \( (1-\delta)100\% \) HPDI \( [\theta_L, \theta_U] \) for the parameter \( \theta \) is given by the simultaneous solution of the equations

\[
\int_{\theta_L}^{\theta_U} \pi_1^*(\theta \mid x, r) = (1-\delta)
\]

and

\[
\pi_1^*(\theta_L \mid x, r) = \pi_1^*(\theta_U \mid x, r). \tag{32}
\]

Similarly, using the posterior pdf of \( p \) in (25), the \( (1-\alpha)100\% \) HPDI \( [p_L, p_U] \) for the parameter \( p \) is given by the simultaneous solution of the equations

\[
\int_{p_L}^{p_U} \pi_2^*(p \mid x, r) = (1-\delta)
\]

and

\[
\pi_2^*(p_L \mid x, r) = \pi_2^*(p_U \mid x, r). \tag{33}
\]

5 Expected test time

In practical applications, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to the cost. Under a Type-II censoring plan, the time required to complete a test is the time to observe the \( m \)th failure in a sample of \( n \) test units. The time is given by \( X_m \), which denotes the \( m \)th order statistics in a sample of size \( n \). Similarly, under Type-II progressive censoring sampling plan with random or binomial removals conditioning on \( R \), the expected value of \( X_m \) (see [3]) is given by
The expected termination point for progressively type II censoring with binomial removals is evaluated by taking expectation on both sides (34) with respect to the \( R \). That is

\[
E[X_m \mid R = r] = \begin{cases} 
\sum_{l_1=0}^{r_1} \cdots \sum_{l_m=0}^{r_m} \frac{(-1)^{l_1} \cdots (-1)^{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \times \int_0^\infty x f(x) F^{h(l_i)-1}(x) \, dx & \text{for } i = 1, \ldots, m-1, \\
\sum_{l_1=0}^{r_1} \cdots \sum_{l_m=0}^{r_m} \frac{(-1)^{l_1} \cdots (-1)^{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \times 2 \theta C(r) \times \sum_{k=0}^{\theta h(l_m)-1} (-1)^k \binom{\theta h(l_m) - 1}{k} \left( \frac{\sqrt{\pi}}{4(k + 1)^{3/2}} \right) & \text{for } i = m.
\end{cases}
\]

(34)

where

\[
al = \sum_{i=1}^{m} l_i, \quad C(r) = n(n - r_1 - 1) \times (n - r_1 - r_2 - 2) \cdots [n - \sum_{i=1}^{m-1} (r_i - 1)],
\]

and \( h(l_i) = l_1 + l_2 + \ldots + l_i + i \).

Furthermore, the expected time of a type II censoring test without removals can be found by setting the \( r_i = 0 \) for all \( i = 1, \ldots, m - 1 \) and \( r_m = n - m \) in equation (34). It is given by

\[
E[X_m^*] = 2m \theta \binom{n}{m} \times \sum_{k=0}^{m-1} (-1)^k \binom{m \theta - 1}{k} \left( \frac{\sqrt{\pi}}{4(k + 1)^{3/2}} \right).
\]

(36)

Similarly, the expected time of a complete sampling case with \( n \) test units can also be obtained by setting \( m = n \) and \( r_i = 0 \) for all \( i = 1, \ldots, m \) in equation (34). It is given by

\[
E[X_m^*] = 2n \theta \sum_{k=0}^{n-1} (-1)^k \binom{n \theta - 1}{k} \left( \frac{\sqrt{\pi}}{4(k + 1)^{3/2}} \right).
\]

(37)

The expected termination point for progressively type II censoring with binomial removals is evaluated by taking expectation on both sides (34) with respect to the \( R \). That is

\[
\begin{align*}
E[X_m] &= E_R[E[X_m \mid R = r]] \\
&= \sum_{r_1=0}^{g(r_1)} \cdots \sum_{r_m=0}^{g(r_m-1)} P(R, r) E[X_m \mid R = r],
\end{align*}
\]

(38)

where \( g(r_1) = n - m \), \( g(r_i) = n - m - r_1 - \cdots - r_{i-1} \), \( i = 2, \ldots, m - 1 \), and \( P(R, r) \) is given in equation (10).

The ratio of the expected time under different schemes to the expected time under complete sampling namely, ratio of expected experiment times (REET) is

\[
\text{REET} = \frac{E_{\text{Eetuds}}}{E_{\text{Eetuds}}},
\]

(39)

where \( E_{\text{Eetuds}} \) (Expected experiment time under different schemes) and \( E_{\text{Eetuds}} \) (Expected experiment time under complete sample).

Suppose that an experimenter wants to observe the failure of at least \( m \) complete failures when the test is anticipated to be conducted under different schemes. Then the REET provides important information in determining whether the experiment time can be shortened significantly if a much larger sample of \( n \) test units is used and the test is stopped once \( m \) failures are observed.

To compare equations (37) and (38), we use:

\[
\text{REET} = \frac{E[X_m^*]}{E[X_m^*]},
\]

which define the ratio the expected termination point under type II progressive censoring with binomial removals and the expected termination point for complete sample. It is clear that when the value of REET closer to 1, the termination point is closer to the complete sample.

6 Illustrative Example

Example 1: simulated data

Consider a life test where 20 units of lifetimes follow the same Burr-X distribution given in (1) are put in test simultaneously. The test is terminated at the time of the thirteenth failure. The number of surviving items removed from the experiment at the failure of each units denoted by \( r_i \) are generated from the binomial distribution as follows: \( r_1 \) from \( \text{bin}(7, p = 0.3) \) distribution and the variables \( r_1, r_2, \ldots, r_{i-1} \) from \( \text{bin}(7 - \sum_{j=1}^{i-1} r_j, 0.3) \) distributions for \( i = 2, 3, \ldots, 12 \). We set \( r_m \) according to the following relation: \( r_m = n - m - \sum_{l=1}^{i-1} r_l \) if \( n - m - \sum_{l=1}^{i-1} r_l > 0 \) and \( r_m = 0 \), otherwise.
The observed failure times of the first 13 units measured in an informative experiment with the corresponding values of \( r_i \) are:

\[
(x_i, r_i) = (0.0857, 1), (0.1141, 0), (0.1253, 3), \\
(0.2024, 0), (0.2691, 0), (0.4149, 2), (0.4924, 1), \\
(0.5100, 0), (0.6897, 0), (0.8949, 0), (0.9530, 0), \\
(1.1227, 0), (1.4008, 0).
\]

Data for this example were generated from Burr-X Equation (1) with \( \theta = 0.6 \) , \( \tau = 0.3 \) , \( n = 20 \) and \( m = 13 \) using algorithm presented in [4]. We use our results of the previous sections and the above simulated data to derive different estimates of the parameters \( \theta \) and \( p \). The Bayes point estimates are derived under the non-informative prior relative to symmetric and asymmetric loss function. The Bayes point estimates relative to squared error, LINEX and general entropy loss functions are denoted respectively by: \( (\cdot)_BL \), \( (\cdot)_BL \), \( (\cdot)_BG \). The result of different point estimates are shown in Table 1. Also, we compute 95\% approximate confidence interval (ACI), 95\% Bootstrap confidence interval (BCI) and 95\% highest posterior density interval (HPDI) for the parameters \( \theta \) and \( p \). The results are reported in Table 2.

Table 1: Different point estimates for \( \theta \) and \( p \)

<table>
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<th>Parameters ( (\cdot) )</th>
<th>( \theta )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\cdot)_ML )</td>
<td>0.5840</td>
<td>0.2443</td>
</tr>
<tr>
<td>( (\cdot)_Boot )</td>
<td>0.6126</td>
<td>0.2643</td>
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<tr>
<td>( (\cdot)_BS )</td>
<td>0.5810</td>
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<td>( (\cdot)_BL )</td>
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<td>1</td>
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<td></td>
<td>2</td>
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<td>( (\cdot)_BG )</td>
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<tr>
<td></td>
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Table 2 : 95\% confidence intervals for \( \theta \) and \( p \)

<table>
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<th>Parameters</th>
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<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>[0.0856,0.3971]</td>
</tr>
<tr>
<td>BCI Length</td>
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<td>[0.2069,0.5833]</td>
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<tr>
<td>HPDI Length</td>
<td>[0.3525,0.8705]</td>
<td>[0.1186,0.4110]</td>
</tr>
</tbody>
</table>

7 Conclusion

The purpose of this paper is to develop Bayesian and ML estimations of the Burr type-X distribution when data are collected under Type-II progressive censoring with binomial removals. We investigate both point and interval estimations of the parameters and the expected time to complete the test. The results show that the MSE for different estimators of the parameter \( \theta \) is decreasing when the removal probability \( p \) increasing, on the other hand, the corresponding time required to complete the test increases significantly. We also computed the expected termination time for Type-II progressive censoring with binomial removals. Illustrative Example and simulation study was conducted to examine the performance of the ML and the Bayes estimators. Finally we discussed some numerical results concerning the expected test time. In summary, results of the numerical examples demonstrate that, when data are collected under Type-II progressive censoring with binomial removals, the test time is most influenced by the removal probability \( p \). From Tables users can decide the censoring number in their lifetest under the consideration of expected termination point. Generally, the results would provide important information to experimenters in planning a life test.

References:


