On Crack Behavior in the Composite Piece-wise Homogeneous Body

TEIMURAZ DAVITASHVILI, ARCHIL PAPUKASHVILI, ZURAB VASHAKIDZE
Faculty of Exact and Natural Sciences
Iv. Javakhishvili Tbilisi State University
I.Vekua Institute of Applied Mathematics, 2 University Str., 0186, Tbilisi, Georgia
GEORGIA
tedadavitashvili@gmail.com    http://www.viam.science.tsu.ge

Abstract: - In the present article the problem for composite (piece-wise homogeneous) body weakened by crack when the crack intersect an interface or penetrate it at rectangular angle is studied. The problem is reduced to the singular integral equation (when crack spreads to the interface) and system of singular integral equations (when crack intersects the interface) with respect to the unknown characteristic function of disclosing of cracks containing an immovable singularity. First time behavior of solutions in the neighborhood of the crack endpoints is studied by a method of discrete singularity in the both uniform, and non-uniformly cases of the knots arrangement. A general scheme of the approximate solution of the task by collocation method is presented. Some results of numerical investigations are presented.

Key-Words: Crack, composite body, singular integral equations, collocation method

1 Introduction
Study of composite bodies weakened by cracks has a great theoretical and practical significances in many branches of industries[4,9,19]. Mathematical simulation represents the cheapest and rather reliable mean for investigation of cracks behavior in the composite bodies [1-3]. That is why many scientific works have been dedicated to the study of composite bodies weakened by cracks [1-7,11-30]. For instance boundary value problem of the elasticity theory for a composite plane with a semi-infinite cut perpendicular to the interface is studied in [29]. The problems on stress distribution for plates and shells near the cracks for bodies weakened by cracks are studied in [26,28]. Elastic half-plane with a crack perpendicular to the boundary is studied in [28]. Problems on bending of plates weakened by cracks is studied in [14]. A mixed problem for composite plane with a crack on the interface are presented in [1,14,28]. Using the theory of analytical functions and Fourier transformation an effective solving of the first main problem of the elasticity theory for piece-wise homogeneous orthotropic plane, when one half-plane has a rectilinear cut of finite length is presented in [2]. A 2D model of elastic body with the elastic inclusion and non-linear boundary conditions at the thin crack faces with existence and uniqueness of the variational and differential formulations and proofing of the convergence of solutions is proposed in [7]. Study of the planar mixed mode crack propagation in the brittle materials by a Regularization of the Principle of Local Symmetry proposed in [13]. The existences of a Lipschitz path of a $BV$-parametrization for the approximated stress intensity factors has been proved in [13]. There are several deferent approache studies denoted to the investigation of the dynamic crack propagation in the elastic domains [4-6,11,12,23,24,29]. For example dynamic of crack propagation without any a priori assumptions regarding to the crack path in finite and infinite elastic domains in conjunction with a time-stepping scheme by collocation method is studied in [22]. Simulation of dynamic of crack propagation by boundary element method, under arbitrary loading conditions has been studied in [24]. A hybrid numerical-analytical method for investigation of linear elastic bodies containing many straight cracks in various configurations was investigated in [21]. Studies on fracture mechanic problems in media possessing coupled piezoelectric, piezomagnetic and magnetoelastic effects, have been carried out by numerous researchers [4,12,25,30]. For instance exact treatments on the crack problems in magnetoelastic solids are presented in [4,25]. In [6] the mixed-boundary-value problem of the interface crack was reduced to solving dual integral equations, which further was expressed in terms of
the second kind Fredholm integral equations and then the stress intensity factor near the crack tip had been calculated. The anti-plane problem of a magnetoelectroelastic strip sandwiched between elastic layers has been studied in [5]. The results obtained in those studies are very useful for the safety and reliability design of the magnetoelectroelastic composites. In the present article the problems of composite bodies when cracks intersect an interface or penetrate into magnetoelectroelastic strip sandwiched between orthotropic bodies  occupies complex variable \( z = x + iy \) which is cut on the line \( L = \{ z : Rez = 0, x \notin L_1 = [0; 1] \} \) and \( L_2 = \{ z : Rez \leq 0, x \notin L_2 = [-1; 0] \} \), which are welded on the axis \( y \). Define by index \( k \), \( k = 1, 2 \) and functions connected with \( \Omega_k \). The problem is to find the function \( w_k(x, y) \), which satisfies differential equation:

\[
\frac{\partial^2 w_k(x, y)}{\partial x^2} + \lambda_k^2 \frac{\partial^2 w_k(x, y)}{\partial y^2} = 0, \ (x, y) \in \Omega_k. \tag{1}
\]

and boundary conditions:

a) on the boundary of the crack tangent stresses are given:

\[
b_{14}^{(k)} \frac{\partial w_k(x, 0)}{\partial y} = q_k^{(+)}, \quad x \in L_k \tag{2}
\]

b) on the axis \( y \) the condition of continuity is fulfilled:

\[
w_1(0; y) = w_2(0; y), \quad y \in (-\infty; +\infty), y \neq 0, \quad \tag{3}
\]

\[
b_{15}^{(1)} \frac{\partial w_1(0; y)}{\partial x} = b_{25}^{(2)} \frac{\partial w_2(0; y)}{\partial x}, \quad \tag{4}
\]

where \( \lambda_k^2 = \frac{b_{14}^{(k)}}{b_{15}^{(k)}} \), \( \lambda_k^2 = \frac{b_{14}^{(k)}}{b_{15}^{(k)}} \) are elastic constants, \( q_k(x) \) is a function of Holder’s class, \( k = 1, 2 \); In particular, if we have isotropic case \( b_{14}^{(k)} = b_{15}^{(k)} = \mu_k, \lambda_k^2 = 1 \), where \( \mu_k \) is module of displacement, \( k = 1, 2 \); We consider a case of symmetric load \( q_k^{(+)}(x) = q_k^{(-)}(x) = q_k(x) \), then we have an avoided singularity and can use method of integral equations. With using the theory of analytical functions (in particular, we use formulas of definition of piecewise holomorphic functions for given leap), also boundary value Sokhotski-Plémeur formula of Cauchy type integral[10], from boundary conditions (2)-(4) the system of singular integral equations with respect to leaps \( \rho_k(x) \) [16-18]

\[
\int_0^1 \left( \frac{1}{t-x} - \frac{a_1}{t+x} \right) \rho_1(t) dt + b_1 \int_{-1}^0 \frac{\rho_1(t) dt}{t-x} = 2\pi f_1(x), \quad x \in [0; 1), \tag{5}
\]

\[
b_2 \int_0^1 \frac{\rho_1(t) dt}{t-x} + \int_{-1}^0 \left( \frac{1}{t-x} - \frac{a_2}{t+x} \right) \rho_2(t) dt = 2\pi f_2(x), \quad x \in (-1; 0),
\]

where \( \rho_k(x), f_k(x) \) unknown and given and given real functions, respectively, \( a_k, b_k \) constants,

\[
a_k = \frac{1-\gamma_k}{1+\gamma_k}, \quad b_k = \frac{2}{1+\gamma_k}, \quad \gamma_1 = 1/\gamma_2, \quad \gamma_2 = \frac{b_{15}^{(2)}}{b_{25}^{(2)}}, \quad f_k(x) \in H^+, \quad \rho_k(x) \in H^+, \quad k = 1, 2.
\]

Explanation of behavior of solutions near the ends of the boundary presents a special interest. The solutions of the system (5) of the integral equations can be presented in the following way:

\[
\rho_1(t) = \frac{x_1(t)}{t^{a_1(1+c_1)}}, \quad \rho_2(t) = \frac{x_2(t)}{t^{a_2(1+c_2)}}, \tag{6}
\]

where \( a_k, \beta_k \) are unknown constants \( 0 < a_k, \beta_k < 1 \), and \( \chi_k(t) \) are functions, which belong to Holder’s class, \( k = 1, 2 \). In the point \( t = \pm 1 \) we obtain correspondingly \( \beta_1 = \beta_2 = \frac{1}{2} \). In the considered case there is no peculiarity in the point \( t = 0 \) [16-18]. In a partial case when one half-plane has a rectilinear cut of finite length, which is perpendicular to the boundary, and one end of which is located on the boundary. We have one singular integral equation containing an immovable singularity. In a partial case, when a crack reaches the boundary of separation, we get that an order of peculiarity in the point \( t = 0 \) depends on elastic constants of material and belongs to \( 0 < \alpha < 1 \). Let \( \rho_2(x) \equiv 0, \rho_1(x) \neq 0 \), then integral of the system (5) we have one integral equation:

\[
\int_0^1 \left( \frac{1}{t-x} + \frac{a_1}{t+x} \right) \rho(t) dt = 2\pi f_1(x), \quad x \epsilon [0; 1], \tag{7}
\]
3 Problem Solution

3.1 Approximate solution of one singular integral equation by collocation method

Antiplane problems of elasticity theory, composed by orthotropic plane and weakened with cracks, are reduced to the following integral equation containing an immovable singularity [16]:

\[
\frac{1}{\pi} \int_{0}^{1} \left( \frac{1}{t-x} + \frac{\varepsilon}{t+x} \right) u(t) dt = f(x),
\]

where \( u(t) \in H^m([0,1]), \varepsilon \in [-1,1], f(x) \in H_{\mu}[0,1], 0 < \mu \leq 1. \)

Analysis of the above-mentioned integral equation and study of their exact and approximate solving methods are accompanied with some specific complexities due to the fact, that the solution has a composite asymptotic, which can be considered only in certain cases introducing weight functions. If we have square root type singularity on both ends of the integral, then Chebyshev orthogonal polynomials can be used. The solution of equation (8) we can represent in the following form:

\[
u(t) = u_0(t) / t^{\alpha} \sqrt{1-t},
\]

where \( u_0(t) \in H([0,1]), \alpha \) depends on material elasticity constants, \( 0 < \alpha < 1. \) Integral equation (3.1) can be solved by three approximate methods: spectral, collocation and asymptotic ones. In the work we use the collocation method. We consider cases as regular intervals, so nonregular located knots in relation to the integrated variable. In the first case the integral is replaced with the quadrature formula of open type, and in the second case the quadrature formula of the higher accuracy. We take roots of polynomials Chebishev of the first sort as knots. For approximate solution of the above-mentioned problem we form the program. The program is examined with testing problem. Several numerical experiments gave the satisfactory results. As it was mentioned above, with respect to the variable \( t \) we get

\[
j = 1,2, \ldots, 2n + 1, \quad h_1 = \frac{1-2\varepsilon_1}{2n}, \quad \varepsilon_1 \neq 0 \text{ is a small parameter}, 2n - \text{even number of the interval division}. \]

We use trapezoid and Simpson’s quadrature formulas. The variable \( x \) takes in turn the values:

\[0 < x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} < 1;\]

to determine the values of required functions we get a linear algebraic equation system.

We can consider nodes \( x_i \) of different form, such as

a) \( x_i = \varepsilon_1 + (i - 1)h_2, \quad h_2 = \frac{(1-2\varepsilon_2)}{2n}, \quad \varepsilon_2 \neq \varepsilon_1, \quad i = 1,2, \ldots, 2n + 1;\)

b) \( x_i = \sin(\varepsilon_3 + (i - 1)h_3), \quad h_3 = \frac{\pi / \varepsilon_2}{2n}, \quad i = 1,2, \ldots, 2n + 1.\)

or algorithm realization by the main condition is that factor that it is necessary to elimination of especial cases in underintegral functions. For definition of values of unknown function in netgrids we receive system of the linear algebraic equations of an order \( 2n + 1. \) In the second case the set integral is replaced by Gauss quadratur formulas of the higher order. We take as knots roots of Chebishev polynomials of the first sort \( t_j = \cos \frac{2j}{2n+1}, \quad j = 1,2, \ldots, n. \) We apply following Gauss formula for singular integrals

\[
\int_{-1}^{+1} \varphi(t) \frac{dt}{t-x} \approx \frac{\pi}{n} \sum_{j=1}^{n} \varphi(t_j) \frac{1 - T_{n-1}(t_j) U_{n-1}(x)}{t_j - x},
\]

\[-1 \leq x \leq +1,\]

which is exact for polynomials of degree no more than \( 2n. \) We take as knots also roots of Chebyshev polynomial of the first sort \( x_i = \cos \frac{2 \pi}{2n+1}, \quad i = 1,2, \ldots, n \text{ on a variable } x. \) As well as in the previous case we receive system of the linear algebraic equations of an order \( n \text{ in knots of a grid for definition of values of unknown function.}\)

3.2 An Approximate Solution of One System of the Singular Integral Equations

Let’s consider the system of singular integral equations (5) containing an immovable
singularity[16]. (5) is solved by a collocation method, in particular, a method discrete singular [3] in cases both uniform, and non-uniformly located knots. Let us first of all consider an algorithm of uniform division. Solution of (5) has such view [16]

\[ \rho_1(t) = \frac{\rho_1^*(t)}{\sqrt{1-t^2}}, \rho_2(t) = \frac{\rho_2^*(t)}{\sqrt{1+t^2}}, \]

Let’s enter such distribution of knots for variables of integration and account points accordingly

\[ t_{1i} = 0 + i h, \quad t_{2i} = -1 + i h, \quad i = 1, 2, ..., n; \]
\[ x_{1j} = t_{1j} - \frac{h}{2}, \quad x_{2j} = t_{2j} + \frac{h}{2}, \quad j = 1, 2, ..., n; \]
\[ h = 1/(n + 1). \]

The pair of the equations (5) probably to present as follows with the help quadrature formulas[21]

\[ \sum_{i=1}^{n} \left( \frac{h}{t_{1i} - x_{1j}} - \frac{a_1 h}{t_{1i} + x_{1j}} \right) \rho_1(t_{1i}) + b_1 \sum_{i=1}^{n} \left( \frac{h}{t_{2i} - x_{1j}} \right) \rho_2(t_{2i}) = 2\pi f_1(x_{1j}), j = 1, 2, ..., n; \]
\[ b_2 \sum_{i=1}^{n} \left( \frac{h}{t_{1i} - x_{2j}} \right) \rho_1(t_{1i}) + \sum_{i=1}^{n} \left( \frac{h}{t_{2i} - x_{2j}} - \frac{a_2 h}{t_{2i} + x_{2j}} \right) \rho_2(t_{2i}) = 2\pi f_2(x_{2j}), \quad (11) \]

Thus, we have 2n equations with 2n unknowns. The received system of the linear equations it is possible to solve with the help to one of direct method, for example, by Gauss modified method.

4 Results and Discussion

The system of the linear algebraic equations and corresponding graphics were solved and constructed by programming system Matlab. The base system of integral equations (5) is solved by a discrete singular method when the coefficients of the term containing unmov ing specifics \( a_1 = a_2 = 0 \), on the both sides of the body there are equal quality materials \( b_1 = b_2 = 0 \), then for the functions \( f_1(x) \) and \( f_2(x) \) we are studing the following three cases:

a) \( f_1(x) = 1, \quad f_2(x) = 1 \) (variant 1),
b) \( f_1(x) = 1, \quad f_2(x) = 2 \) (variant 2),
c) \( f_1(x) = 2, \quad f_2(x) = 1 \) (variant 3).

At the initial stage the base system of singular integral equations (5) is solved by a discrete singular method in the interval [-1, 1] when the number of the crack splitting were \( n = 10 \) or \( n = 20 \). The performed experimental calculations had shown: a) on what party of body loading was more, con formably the functions of disclosing of cracks at that ends of the crack have been quickly growing. b) Behaviour of the characteristic functions of the crack have significantly depended on a ratio of loadings on the both sides of the crack. Namely when these loadings on the both sides of a crack were equal, then graphs of characteristic functions of a crack were asymmetrical (skew-symmetric) and when a ratio of loadings on the both sides of a crack was equal to two, then there were observed neither symmetry nor anti-symmetry graphs which corresponds to the physical nature of the task. c) Increment of the number of knots (as well usage of algorithm of crack non-uniform splitting) has promoted augmentation of a diapason of stability of the characteristic function of the crack and it is important to remark that the fast increments of the characteristic function of the crack were observed only at the end points of the crack. The base system of singular integral equations (4) is solved by a discrete singular method in the interval [-1, 1] when the number of the crack splitting are \( n = 10, \quad n = 20 \). On the Fig.1 and Fig.2 are given grasps of the desired functions at \( n = 10, \quad n = 20 \) using an algorithm of uniform splitting of a crack while on the Fig.3 and Fig.4 are given grasps of the desired functions at \( n = 10, \quad n = 20 \) using an algorithm of non-uniform splitting of the crack.

Fig. 1 Results of calculation for the values \( n = 10 \) with algorithm of crack uniform splitting.

Fig. 2 Results of calculation for the values \( n = 10 \) with algorithm of crack uniform splitting.
Fig. 3 Results of calculation for the values n=10 with algorithm of crack non-uniform splitting.

![Graph](image1)

Fig. 4 Results of calculation for the values n=10 with algorithm of crack non-uniform splitting.

![Graph](image2)

On the left half of the area of the Fig.1-Fig.4 are presented graphs of $\rho_2(t)$ and on the right sides graphs of $\rho_1(t)$ which are the functions of disclosing of cracks. It is possible to make also such conclusion from the Fig.1-Fig.4: as knots increment as well usage of algorithm of crack non-uniform splitting increases diapason of stability of the characteristic function of the crack. From the Fig.2 and Fig.3 it is visible that there are fast increments of the characteristic function of the crack only at the end points of the crack.

4 Conclusion

Equations of the anti-plane elasticity theory for composite bodies weakened by cracks can be used as an initial approximation for the mathematical models investigated cracks problems in concave bodies. From the theoretical and practical point of view the cases when cracks intersect an interface or penetrate the interface at right angle were reduced to the singular integral equation (when crack reaches to the interface) and the system (pair) of singular integral equations (when crack intersects the interface). Integral equations in both cases have contained an immovable singularity, with respect to the unknown characteristic functions of disclosure of cracks. Behavior of solutions in the neighborhood of the crack endpoints have been studied by a method of discrete singularity with uniform division of an interval by knots. For both cases have been constructed and realized corresponding algorithms. The results of calculations obtained by the given algorithms were in a good consonance with theoretical results and so they give possibility to make hypothetical prediction of the cracks penetration into the body.

References:


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