Dynamical Systems With Variable Dissipation: Methods and Applications

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Abstract: This paper contains some the development of qualitative methods in the theory of nonconservative systems that arise, e.g., in such fields of science as the dynamics of a rigid body interacting with a resisting medium, oscillation theory, etc. This material can call the interest of specialists in the qualitative theory of ordinary differential equations, in rigid body dynamics, as well as in fluid and gas dynamics since the work uses the properties of motion of a rigid body in a medium under the streamline flow around conditions.

Key–Words: Dynamical Systems With Variable Dissipation, Dynamics of a Rigid Body

1 Introduction

The author obtains a full spectrum of complete integrability cases for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases of integrability, each of the first integrals is expressed through a finite linear combination of elementary functions and is a transcendental function of its variables, simultaneously. In this case, the transcendence is meant in the complex analysis case, i.e., after the continuation of the functions considered to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling limit sets in the system considered (for example, attracting and repelling foci).

It was obtains new families of phase portraits of systems with variable dissipation on lower- and higher-dimensional manifolds. He discusses the problems of their absolute or relative roughness and discover new integrable cases of the rigid body motion, including those in the classical problem of motion of a spherical pendulum placed in the accumulating medium flow too.

2 A Certain Problem of the Dynamics of a Rigid Body Interacting With a Medium

2.1 Sequence of steps in modelling

Generally speaking, the general problem of studying the body motion in the resistance force field ”is prevented” by the absence of any complete description of this force field. As is known, in principle, we can measure the positional component of the resistance force in a stationary experiment. But the component of the force field, which corresponds to the quasi-velocities of the system considered arises only under the nonstationary body motion.

Therefore, the process of describing the force field is a sequence of steps. We first study a preparatory model of the force field and construct a family of mechanical systems whose motion has different characteristics that essentially depend on model parameters such that the information about them is incomplete or does not exist at all. As a result of studying such a model, there arise questions such that the answers to them cannot be found in the framework of the accepted model. Then the elaborated objects become the subject of a detailed experimental study at the second step. Such an experiment presupposes the answers to the formulated questions and either introduces necessary corrections to the preparatory constructed model or reveals new questions, which lead to the necessity of the first step repetition but in a new level of the problem understanding.

Such an approach is related to the description of stationary motion regimes, their branching, bifurcation, stability and instability analysis, revealing surgery conditions, and appearance of regular or irregular (i.e., *chaotic*) oscillations.

Sometime, we can succeed in obtaining the answers to questions of qualitative character when discussing the traditional problem of analytic mechanics,

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the problem of existence of the full tuple of first integrals for the constructed dynamical system. At the same time, the study of the behavior of a dynamical system "as a whole" often forces us to use the numerical experiment. In this case, there arises the necessity of elaborating new computational algorithms or improving the known, as well as new qualitative methods.

In this work, we study the problem on the body motion under the condition that the line of the force applied to the body does not change its orientation with respect to the body and can only displace parallel to itself depending on the angle of attack and, possibly, on other phase variables. Such conditions arise under the plate motion with the so-called "large" angles of attack in a medium under a streamline flow (in this case, the fluid is assumed to be ideal in general, although all this are also true for fluids of a small viscosity, first of all, for the water) or under a separation flow (which is justified by an experiment completely satisfactory). Therefore, the *main objects of studying* is a family of bodies such that a part of the surface of each of which has a plane part that is flowed by a medium according to the streamline flow laws.

2.2 Physical assumptions

Assume that a rigid body of mass m executes a plane-parallel motion in a medium with quadratic resistance law and that a certain part of the exterior body surface is a plane plate being under the medium streamline flow conditions. This means that the action of the medium on the plate reduces to the force $\mathbf{S}$ (applied at the point N) whose line of action is orthogonal to the plate. Let the remained par of the body surface be situated in a volume bounded by the flow surface that goes away from the plate boundary and is not subjected by the medium action. For example, similar conditions can arise after the body entrance into the water.

2.2.1 Quasi-stationarity hypothesis and phase variables

Let us relate to the body the right coordinate system $Dxyz$ whose axis z moves parallel to itself, and for simplicity, assume that the plane Dzx is the geometric symmetry plane of the body. This ensures the fulfillment of property 2) under the motion satisfying condition 1).

To construct the dynamical model, let us introduce the following phase coordinates: the value $v = |\mathbf{v}|$ of the velocity $\mathbf{v}$ of the point D, the angle α between the vector $\mathbf{v}$ and the axis x, and the algebraic value Ω of the projection of the body absolute angular velocity on the axis z.

Assume that the value of the force $\mathbf{S}$ quadratically depend on v with nonnegative coefficient s_1 ($S = s_1 v^2$). As usual, one represents s_1 in the form $s_1 = \rho P c_x/2$, where c_x is now the dimension-free coefficient of the frontal resistance (ρ is the medium density and P is the plate area). This coefficient depends on the angle of attack, the *Struchal number*, and other quantities which are usually considered as parameters. In what follows, we also introduce the following additional phase variable of the "Struchal type": $\omega = \Omega D/v$, where D is the characteristic plate transversal size. We restrict ourselves to the dependence of c_x on the pair (α, ω) of variables, i.e., we assume that s_1 (as well as y_N) is a function of the pair (α, ω) of dimension-free variables.

Let us define (purely formally for now) the dependence of s_1 and the ordinate y_N of the point N on the phase coordinates (α, ω). The system of dynamical equations must admit a particular solution of the form $\alpha(t) \equiv 0, \quad \omega(t) \equiv 0$. Therefore, we have the condition $y_N(0,0) = 0$ for the function $y_N(\alpha, \omega)$, and in the linear case, we need to assume that $y_N = D(k\alpha - h\omega)$, where k and h are certain constants. Because the approximation is linear, we can ignore the dependence of s_1 on α and ω.

In what follows, to take into account the action direction of the force $\mathbf{S}$, we introduce the following sigh-alternating auxiliary function $s(\alpha, \omega) = s_1(\alpha, \omega) sign \cos\alpha$.

2.2.2 Key parameters

Therefore, the linearized model of the force medium action contains three parameters s, k, and h, which are determined by the plate form in the plan. As was already mentioned, the first of these parameters, the coefficient s, is *dimensional*. The parameters k and h are *dimension-free* because of the method of their introduction.

Note that the quantities s and k can be found experimentally by using weight measurements in devices of the hydro- or aerodynamic tubes type. In [4], there also is the information about the theoretical finding of these quantities for separate plate forms. This information allows us to assume that $k > 0$. As for the parameter h, even the very necessity of its introduction to the model is not a priori obvious.

2.3 Linearized equations of motion

The equations of motion of the center of masses in projections on the axes Dx and Dy of the related coordinate system and the equation of the kinetic moment variation with respect to the König axis have the following form with accuracy up to terms linear in α

and Ω (here, σ is the distance DC and I is the central moment of inertia of the body):

$$\dot{v} = -sv^2/m \tag{1}$$

$$v\dot{\alpha} - sv^2\alpha/m + v\Omega - \sigma\dot{\Omega} = 0 \tag{2}$$

$$I\dot{\Omega} = sDv^2(k\alpha - hD\Omega/v) \tag{3}$$

Assuming that $v \neq 0$, introducing the natural parameter σ_1 ($v\,dt = D\,d\sigma_1$), which is usual for such systems, and using the change $\omega = D\Omega/v$ (see above) of the variable Ω and the obvious differentiation formula $D\dot{(\,)} = vd/d\sigma_1(\,) = v(\,)'$ we arrive at the system

$$v' = -\frac{sv}{m}D$$

$$I\omega' = \omega(I - mD^2h)\frac{Ds}{m} + sD^3k\alpha \tag{4}$$

$$\alpha' = -\omega\left(1 + \frac{s\sigma D^2}{I}h\right) + sD\left(\frac{1}{m} + \frac{k\sigma D}{I}\right)\alpha \tag{5}$$

in which two latter equations are are separated from the first thus forming the independent second-order system 4, 5 and can be studied separately.

Let us transform these equations as follows: exclude α from them introducing the angle of turn ϕ by the formula $\phi' = \omega$; we obtain their linear integral in the form $\alpha - \omega(I/(kmD^2) + \sigma/D) + \phi(1 + \sigma s/m + Is/(km^2D) - hsD/km) = b = \mathrm{const}$. With account for this, the equation for the angle of turn ϕ becomes $I\phi'' + \phi' sD(D^2h - D\sigma k - 2I/m) + \phi sD^3(k + k\sigma s/m + Is/m^2D - hsD/m) = ksD^3b$. It is easy to see that it has the form of the linear pendulum equation; under certain conditions, this pendulum executes oscillations near a certain position ϕ_* determined by the value b of the linear integral.

For $h = 0$, the coefficient of the so-called linear damping is negative, whereas the coefficients of the positional component positive; this allows us to speak about the oscillatory instability of the solution $\phi = \phi_*$. On the contrary, for a sufficiently large h, the solution $\phi = \phi_*$ can be made to be stable, and, moreover, it can loose its oscillatory character, since the coefficient of the positional component becomes negative.

But the main circumstance is that because this linear pendulum is multiparameter, the oscillatory stability of the solution $\phi = \phi_*$ is possible for certain finite values of h, since both mentioned coefficient can be positive in principle.

2.4 Experiment

To describe the results and concrete properties of the body motion, in Institute of Mechanics of M. V. Lomonosov Moscow State University, V. A. Eroshin and V. M. Makarshin carried out experiments in registration of the motion of homogeneous circular cylinders in the water. Owing to the experiment, it becomes possible to find the dimension-free parameters k and h of the medium action on a rigid body.

The experiment allows one to make several important conclusions.

The first of them is as follows: the *rectilinear stationary free body drag* (*in the water*) *is unstable* at least with respect to the angle of attack and the angular velocity.

The second conclusion obtained from the carried out natural experiment is as follows: *in modelling the medium action on the body, it is necessary to take into account the additional parameter characterizing the rotational derivative of the moment with respect to the body angular velocity*. This parameter introduces the dissipation into the system. In our linear approximation, the accounting damping moment linearly depends on the body velocity.

For certain cases, the value of the damping moment coefficient under the body motion in the water was already estimated. This estimate confirms the instability of the body rectilinear motion in the water. Purely formally, increasing the value of the damping coefficient, we can attain the stability of a motion, but it is difficult to ensure this stability in realty. The rigid bode rectilinear motion is stable in certain media (for example in the clay), as the experiment shows. Possibly, this stability is attained due account for the existence of a considerable damping from the medium in the system or for the existence of forces tangent to the plate.

If the additional medium damping action on the body is of a purely dissipative character, then we can restrict ourselves to the region of positive damping moment parameter values, since a priori, its sign is not obvious. The value of this parameter is proportional to the transversal plate size, and to create the stability conditions of the motion considered, we also need to take into account the length of the moving body. Therefore, for sufficiently long bodies moving in the water, the contribution of the additional dissipation into the body orientation angle variation manifests itself only a certain inessential decrease of the exponent of the exponential related to the instability of the motion.

2.5 Beginning of nonlinear analysis

The first conclusion made from the experiment forces us to consider the class of possible body motions for small angles of attack as a supporting class for studying the class of free body drags with finite an-

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gle of attack. For different bodies, under motion in a medium and under certain conditions, the angles of attack can practically assume any value from the interval $(0, \pi/2)$, and only for the angles close to $\pi/2$, the so-called washing out of the lateral surface is inevitable. Therefore, there arises the necessity of extending the functions y_N and s to finite angles of attack, i.e., the expansion of the domain of the pair of dynamical functions up to the interval $(0, \pi/2)$. But, in fact, it is necessary to extend the dynamical functions to the whole numerical line; this is clear from the following arguments.

2.5.1 Nonlinear equations

In order to pass to a more complete description of the free body motion, let us represent the dynamics equations $mw_c = F,\ I\dot{\Omega} = M$ obtained early in the linear form (see 1-3) as follows:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = F_x/m \quad (6)$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha + \Omega v\cos\alpha - \sigma\dot{\Omega} = 0 \quad (7)$$

$$I\dot{\Omega} = y_N(\alpha,\omega)F_x, \quad \omega = D\Omega/v \quad (8)$$

As a rule, for various variants of the body motion considered below, the generalized force F_x is quadratic in velocities (v, Ω) and explicitly depends on the auxiliary sign-alternating function $s(\alpha,\omega)$ (for example, $F_x(\alpha, v, \Omega) = -s(\alpha,\omega)v^2$ in the case of the body free drag). Therefore, the class of conceptual bodies and their conceptual motions defines a certain pair of dynamical functions $(s(\alpha,\omega),\ y_N(\alpha,\omega))$ belonging to definite function classes.

2.5.2 Classes of dynamical functions

The fist stage of the complete nonlinear study of the body motion in a medium under the quasi-stationarity conditions is the study of the corresponding dynamical systems in which the damping is not taken into account (in particular, $h = 0$ in the linear case). The account for the damping is the next labor-consuming stage of studying the problem, which is presented in this work in a sufficient detail.

To begin with, we consider the case where the pair of dynamical functions (y_N, s) depends only on the angle of attack. In this case, to qualitatively describe this pair of functions, we use the experimental information about the streamline flow properties.

The classes of dynamical functions to be introduced are sufficiently wide. They consists of smooth, 2π-periodic ($y_N(\alpha)$ is odd and $s(\alpha)$ is even) functions satisfying the following conditions: $y_N(\alpha) > 0$ for $\alpha \in (0, \pi)$, and, moreover, $y'_N(0) > 0$ and $y'_N(\pi) < 0$ (the function class $\{y_N\} = Y$); $s(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2, \pi)$, and, moreover, $s(0) > 0$ and $s'(\pi/2) < 0$ (the function class $\{s\} = \Sigma$). Both y_N and s change the sign under the replacement of α on $\alpha + \pi$. Therefore, $y_N \in Y,\ s \in \Sigma$ In particular, the analytic functions

$$y_N(\alpha) = y_0(\alpha) = A\sin\alpha \in Y \quad (9)$$

$$s(\alpha) = s_0(\alpha) = B\cos\alpha \in \Sigma, \quad A, B > 0 \quad (10)$$

serve as typical representatives of the described classes and correspond to the medium interaction functions obtained by S. A. Chaplygin in studying the plane-parallel flow around of a plane infinite length by a homogeneous medium flow.

In what follows, there rises the product $F(\alpha) = y_N(\alpha)s(\alpha)$ in the dynamical systems considered. It follows from the conditions listed above that F is a sufficiently smooth odd π-periodic functions satisfying the following conditions: $F(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $F'(0) > 0$, and $F'(\pi/2) < 0$ (the function class $\{F\} = \Phi$). Therefore, $F \in \Phi$ In particular, the analytic function

$$F = F_0(\alpha) = AB\sin\alpha\cos\alpha \in \Phi \quad (11)$$

is also a typical representative of the a function class Φ arisen (and also corresponds to the S. A. Chaplygin case mentioned above).

Let us explain the necessity of a wide choice of the function classes Y and Σ. A plane plate is a geometric section of the part of the body surface that interacts with the medium and is plane. The geometric form of such a plane domain can be arbitrary. Moreover, the chord lying in the plane of the domain can differently determine the plane of the body motion itself (in the case of the plane-parallel motion). The latter circumstances allow us to refer the dynamical functions arisen to definite classes. As was noted above, sufficiently weak conditions are imposed on these function classes, and, therefore, these classes are sufficiently wide. In advance, they include admissible concrete functions take for each conceptual body and each conceptual motion.

Therefore, to study the medium flow around a plate, we use the *classes of dynamical systems* defined by a pair of dynamical functions, which considerably complicates the global analysis performance.

But certainly, it is not possible to associate a conceptual rigid body with its motion with each concrete pair of dynamical functions. Therefore, the study of this problem for sufficiently wide classes of dynamical functions allows us to speak about a relatively complete consideration of the problem of the body motion in a medium in the framework of these model assumptions under the quasi-stationarity conditions.

2.6 Principal applied question of nonlinear analysis

The instability of the rectilinear translational drag allose us to pose the principal question of nonlinear analysis in studying a finite neighborhood of such a motion. Precisely, *is it possible to find a pair of dynamical functions y_N and s for describing the conceptual body motion such that in a finite neighborhood of this stationary motion, there exist stable limit cycles?*

One of the main results of this work is a *partly negative answer* to this question, precisely, *for a quasi-static description of the interaction between the medium and the body, when the dynamical quantities y_N and s depend only on the angle of attack, for any admissible pair of obtained dynamical functions $y_N(\alpha)$ and $s(\alpha)$, in the whole range of finite angles of attack $\alpha \in (0, \pi/2)$, there are no any auto-oscillations in the system considered.*

To attain a possible *affirmative answer to the principal question of nonlinear analysis*, in modelling the interaction of a body with a medium, we use an additional damping medium action, which gives a dissipation to the system. Therefore, in principle, under certain conditions, the appearance of stable auto-oscillations is possible in the framework of the model considered, but, however, the search for a body exhibiting the necessary properties requires an additional experiment. This result is not only one of the main results of the present work, but it opens a *new direction* in analytical studying the interaction of a body with a medium with account for the medium damping actions.

2.6.1 On variable dissipation in system

After certain simplifications, the general system 6–8 reduces to second-order pendulum systems in which there is a linear dissipative force with variable coefficient alternating the sign for different angles of attack.

In this case, we therefore speak about the system with the so-called *variable dissipation*, where the term "variable" mainly refers not to the value of the dissipation coefficient but to its *sign*.

In the mean, during the period with respect to angle of attack, the dissipation can be positive, negative, or zero. In the latter case we speak about the *system with variable dissipation in the mean*.

3 Complete Integrability of Certain Classes of Nonconservative Systems

Also, we show that for homogeneous circular cylinders moving in the water, the rectilinear translational drag is not stable for any dynamical and geometric parameters of such cylinders. Probably, this is related to the motion of the cylinders in the water, when the water damping is inessential, which does not allow us to speak about the stability of the rectilinear translational damping. However, for cylinders having a hole in their interior, the attainment of the above stability is possible under certain conditions.

Therefore, under certain conditions, the account for the medium damping action on a rigid body leads to an affirmative answer to the principal question of nonlinear analysis: *under the body motion in a medium with finite angles of attack, in principle, the appearance of stable auto-oscillations is possible. Moreover, for circular cylinder, the appearance of stable and unstable auto-oscillations is possible!*

All what said above, allows us to estimate the results of the work as a whole as a *new direction in analytical mechanics of a rigid body interacting with a medium.*

The results of the presented work are appeared owing to the study of the applied problem on the rigid body motion in a resisting medium [5, 6] where complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. This circumstance allows one to perform a complete analysis of phase trajectories and show those properties of them which exhibit the *roughness* and preserve for systems of a more general form. The complete integrability of that systems is related to symmetries of latent type. Therefore, the study of sufficiently wide classes of systems having analogous latent symmetries is of interest.

As is known, the concept of integrability is sufficiently broad and indefinite in general. In its constructing, it is necessary to take into account what it means (we mean a certain criterion according to which one makes a conclusion about that the structure of trajectories of the system considered is especially "attractive and simple"), in which function classes the first integrals are sought for, etc.

In this work, we follow the approach, which takes the transcendental and, moreover, elementary functions as the function class for first integrals. Here, the *transcendence is understood not in the sense of elementary function theory* (fir example, trigonometrical functions), but in the sense that these function have essentially singular points (by the classification

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accepted in theory of functions of one complex variable, when a function has essentially singular points). Moreover, we need to formally continue them to the complex domain. As a rule, such systems are strongly nonconservative.

3.1 Variable dissipation dynamical systems and their general properties

3.1.1 General characteristic of variable dissipation dynamical systems

Generally speaking, the dynamics of a rigid body interacting with a medium is just a field, where there arise either dissipative systems or systems with the so-called *antidissipation* (energy supporting inside the system itself). Therefore, it becomes urgent to construct a methodology precisely for those classes of systems which arise in modelling body motion the contact surface of which is a plane part, the simplest part of their exterior surface.

Since in such a modelling, one uses the experimental information about the streamline flow around properties, there arises a necessity of studying the class of dynamical systems that exhibit the (relative) *structural stability* property. Therefore, it is quite natural to introduce the definitions of relative roughness for such systems. In this case, many of the system considered are turned out to be (absolutely) rough in the Andronov–Pontryagin sense.

After certain simplifications, we can reduce the system of equations for the plane-parallel motion to the second-order pendulum systems in which there is a linear dissipative force with variable coefficient whose sign alternates for different values of the periodic phase variable in the system.

As was already noted early, there are important mechanical analogies arising when comparing qualitative properties of the free body stationary motion and the pendulum equilibrium in the medium flow. Such analogies have a deep supporting meaning, since they allows us to extend the properties of the pendulum nonlinear dynamical systems to the free body dynamical systems. Both classes os systems belong to the class of the so-called *pendulum dynamical systems with zero mean variable dissipation*.

Under additional conditions, we also extend the above equivalence to the case of spatial motion, which allows us to speak about the *general character of symmetries* in a system with zero mean variable dissipation under the plane-parallel, as well as spatial motions.

3.1.2 Examples from dynamics

Below, we highlight the classes of essentially nonlinear systems of the second and third orders integrable in transcendental (in the sense of theory of functions of one complex variable) elementary functions. For example such systems are five-parametric dynamical systems including the majority of systems that are studied early in the dynamics of a rigid body interacting with a medium:

$$\begin{gathered}\dot\alpha = a\sin\alpha + b\omega + \gamma_1\sin^5\alpha + \gamma_2\omega\sin^4\alpha + \\ +\gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha \\ \dot\omega = c\sin\alpha\cos\alpha + d\omega\cos\alpha + \\ +\gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha + \\ +\gamma_3\omega^3\sin^2\alpha\cos\alpha + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha\end{gathered}$$

In this connection, it is of reason to introduce the definitions of relative structural stability (relative roughness) and relative structural instability (relative nonroughness) of various degrees.

As is known, the (purely) dissipative dynamical systems (as well as (purely) antidissipative systems), which in our case can belong to the class of systems with zero mean variable dissipation are as a rule structurally stable ((absolutely) rough), and, on the contrary, the systems with zero mean variable dissipation (which, as a rule, have additional symmetries) are either structurally unstable (non-rough) or only relatively structurally stable (relatively rough). It is difficult to prove the latter assertion in the general case. However, the introduction of the concept relative roughness (and also relative non-roughness of various degrees) allows us to present the classes of concrete systems from the rigid body dynamics that exhibit the above properties.

So, in [5], the authors studied and integrated two model variants of the body plane-parallel motion in a resisting medium, which are described by dynamical systems with zero mean variable dissipation. Such cases of motion presuppose the existence of a certain non-integrable constraint in the system considered (that is realized by a certain additional tracking force).

For example a dynamical system of the form $\dot\alpha = \Omega + \beta\sin\alpha$, $\dot\Omega = -\beta\sin\alpha\cos\alpha$, is relatively structurally stable (relatively rough) and topologically equivalent to the system describing a clamped pendulum placed in the running-out medium flow.

We can find its first integral being a transcendental (in the sense of theory of functions of one complex variable such that it has essentially singular points after continuing it into the complex domain) functions of phase variables that is expressed through a finite combination of elementary functions.

As is seen, the phase cylinder $\mathbb{R}^2\{\alpha, \Omega\}$ of quasi-velocities of the system considered exhibits an interesting topological structure of partition into trajectories.

On the cylinder, there are two domains (whose closure is just the phase cylinder) filled in by trajectories of perfectly different character.

The first domain called oscillatory or finitary (it is simply connected is entirely filled in by trajectories of the following type. Almost every such trajectory starts at the repelling point $(2\pi k, 0)$ and ended at the attracting point $((2k+1)\pi, 0)$, $k \in \mathbf{Z}$. An exception is the fixed points $(\pi k, 0)$ and separatrices that either emanate from the repelling point $(2\pi k, 0)$ and enter the saddles S_{2k} and S_{2k+1} or emanate from the saddles S_{2k+1} and S_{2k+2} and enter the attracting points $((2k+1)\pi, 0)$. Here, $S_k = (-\pi/2 + \pi k, (-1)^k \beta)$.

The second domain called rotational (it is two-connected is entirely filled in by rotational motions similar to rotations on the mathematical pendulum plane. These phase trajectories envelope the phase cylinder and are periodic on it.

Although the dynamical system considered is not conservative, in the rotational domain of its phase plane $\mathbb{R}^2\{\alpha, \Omega\}$ it admits the preservation of an invariant measure with variable density. This property characterizes the system considered as a system with zero mean variable dissipation.

Key separatrices (for example, the separatrix emanating from the point $(-\pi/2, \beta)$ and entering the point $(3\pi/2, \beta)$) are boundaries of domains in which the motion is of different character. So, in the oscillatory domain containing the repelling and attracting equilibrium points, almost all trajectories have attractors and repellers as limits sets. Hence there are no even absolutely continuous function being the density of an invariant measure in this domain.

The matter is different for the domain entirely filled in by rotational motions. As was shown early, there exists a smooth function being the density of the invariant measure in the domain entirely filled in by periodic trajectories not contracting to a point along the phase cylinder.

3.2 Systems with symmetries and zero mean variable dissipation

Let us consider systems of the form (the dot denotes the derivative in time)

$$\dot{\alpha} = f_\alpha(\omega, \sin\alpha, \cos\alpha) \qquad (12)$$

$$\dot{\omega_k} = f_k(\omega, \sin\alpha, \cos\alpha),\ k = 1, \ldots, n, \qquad (13)$$

defined on the set $\mathbf{S}^1\{\alpha \bmod 2\pi\} \setminus K \times \mathbb{R}^n\{\omega\}$, $\omega = (\omega_1, \ldots, \omega_n)$, where the functions $f_\lambda(u_1, u_2, u_3)$, $\lambda = \alpha, 1, \ldots, n$, of three variables$u_1$, u_2, and u_3 are as follows:

$$f_\lambda(-u_1, -u_2, u_3) = -f_\lambda(u_1, u_2, u_3)$$

$$f_\alpha(u_1, u_2, -u_3) = f_\alpha(u_1, u_2, u_3)$$

$$f_k(u_1, u_2, -u_3) = -f_k(u_1, u_2, u_3)$$

The set K is either empty or consists of finitely many points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$.

The latter two variables u_2 and u_3 in the functions $f_\lambda(u_1, u_2, u_3)$ depend on one parameter α, but they are distinguished in separate groups for the following reasons. First, not in the whole their domain, they are uniquely expressed from one another, and, second, the first of them is odd, whereas the second is an even function of α, which influences on the symmetry of system 12, 13 differently.

To this system, we put in correspondence the non-autonomous system

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)}$$

by the substitution $\tau = \sin\alpha$, it reduces to the form $(k = 1, \ldots, n)$

$$\frac{d\omega_k}{d\tau} = \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))}$$

$$\varphi_\lambda(-\tau) = \varphi_\lambda(\tau), \lambda = \alpha, 1, \ldots, n$$

In particular, the right-hand side of latter system can be algebraic (i.e., it can be a ration between two polynomials); sometimes, this helps to search for its first integrals in explicit form.

The following assertion immerses the class of systems 12, 13 in the class of dynamical systems with zero mean variable dissipation. The inverse embedding does not hold in general.

Lemma 1 *Systems of the form 12, 13 are dynamical system with zero mean variable dissipation.*

This proposition is proved by using the symmetries of system 12, 13 listed above.

The converse assertion is not true in general, since we can present a set of dynamical systems on a two-dimensional cylinder that are system with zero mean variable dissipation but do not satisfies the properties listed above.

In this work, we mainly consider the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ $(\lambda = \alpha, 1, \ldots, n)$ are polynomials in ω and τ.

To begin with, let us consider a certain class of autonomous systems on the two-dimensional circular cylinder $\mathbf{S}^1\{\alpha \bmod 2\pi\} \times \mathbb{R}^1\{\omega\}$. For example, to the following pendulum systems (arising in the dynamics of a rigid body interacting with a medium) with parameter $\beta > 0$:

$$\dot{\alpha} = -\omega + \beta \sin\alpha, \dot{\omega} = \sin\alpha\cos\alpha \tag{14}$$

$$\dot{\alpha} = -\omega + \beta \sin\alpha \cos^2\alpha + \beta\omega^2 \sin\alpha \tag{15}$$

$$\dot{\omega} = \sin\alpha\cos\alpha - \beta\omega\sin^2\alpha\cos\alpha + \beta\omega^3\cos\alpha \tag{16}$$

in the variables (ω, τ), we put in correspondence the following equations with algebraic right-hand side, respectively:

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + \beta\tau}, \frac{d\omega}{d\tau} = \frac{\tau + \beta\omega[\omega^2 - \tau^2]}{-\omega + \beta\tau + \beta\tau[\omega^2 - \tau^2]}.$$

In this case, systems 14 and 15, 16 are dynamical systems with zero mean variable dissipation, which is easy to verify directly.

Moreover, each of them have a first integral being a transcendental (in the sense of theory of functions of one complex variable) function that expresses through a finite combination of elementary functions [5].

For example, system 14 has a first integral of the following form (depending on the value of β, three cases are possible that corresponds to the existence of foci, nodes, or degenerate nodes in the phase portrait of the system):

$$\begin{gathered}\beta^2 - 4 < 0: \\ \{\Omega^2 + \beta\Omega\sin\alpha + \sin^2\alpha\}\times \\ \times \exp\left\{\frac{2\beta}{\sqrt{-\beta^2+4}}\arctan\frac{2\Omega+\beta\sin\alpha}{\sqrt{-\beta^2+4}\sin\alpha}\right\} = \mathrm{const}\end{gathered}$$

$$\begin{gathered}\beta^2 - 4 > 0: \\ |2\Omega + (\beta + \sqrt{\beta^2-4}\sin\alpha)|^{\sqrt{\beta^2-4}-\beta}\times \\ \times|2\Omega + (\beta - \sqrt{\beta^2-4}\sin\alpha)|^{\sqrt{\beta^2-4}+\beta} = \mathrm{const}\end{gathered}$$

$$\begin{gathered}\beta^2 - 4 = 0: \\ |2\Omega + \beta\sin\alpha|\times \\ \times\exp\left\{-\frac{\beta\sin\alpha}{2\Omega+\beta\sin\alpha}\right\} = \mathrm{const}\end{gathered}$$

The phase portrait of system 15, 16 can be of three different types depending on the values of the parameter β.

In the expression of its first integral, also three cases are possible depending on the value of the constant β and corresponding to the existence of foci, nodes, and degenerate nodes in the phase portrait of the system.

Let us represent the parameter β as the product: $\beta = \sigma^2 n_0^2$; after that, to system 15, 16 we put in correspondence a differential equation of the form

$$\frac{d\omega}{d\tau} = \frac{-n_0^2\tau + \sigma\omega[\omega^2 - n_0^2\tau^2]}{\omega + \sigma n_0^2\tau + \sigma\tau[\omega^2 - n_0^2\tau^2]},\ \tau = -\sin\alpha$$

Introduce the following notation: $C_1 = 2 - \sigma n_0,\ C_2 = \sigma n_0,\ C_3 = -2 - \sigma n_0$. Performing a number of changes by the formulas $\omega - n_0\tau = u_1,\ \omega + n_0\tau = v_1,\ u_1 = v_1 t_1,\ v_1^2 = p_1$, where $v_1 \neq 0$, we obtain the Bernoulli-type equation

$$2p_1\left[C_1 t_1 + C_2 + \frac{2\sigma}{n_0}t_1 p_1\right] = \frac{dp_1}{dt_1}[C_3 - C_1 t_1^2]$$

By the known change $p^{-1} = q_1$ for $p_1 \neq 0$, we reduce the latter equation to the form

$$\dot{q_1} = a_1(t_1)q_1 + a_2(t_1)$$

where

$$a_1(t_1) = \frac{2(C_1 t_1 + C_2)}{C_1 t_1^2 - C_3},\ a_2(t_1) = \frac{4\sigma t_1}{n_0(C_1 t_1^2 - C_3)}$$

(Here, the dot denotes the derivative in t_1.)

The general solution of the linear homogeneous equation is represented in the form

$$q_{1\,HOM}(t_1) = k(C_1 t_1^2 - C_3)Q(t_1),\ k = \mathrm{const}$$

where the function Q has the following form depending on the value of the constant C_1:

$$\begin{gathered}Q(t_1) = e^{t_1}, C_1 = 0 \\ e^{2\frac{C_2}{\sqrt{-C_1C_3}}\arctan\sqrt{-\frac{C_1}{C_3}}t_1}, C_1 > 0 \\ \left(\frac{\sqrt{-C_1}t_1 + \sqrt{-C_3}}{\sqrt{-C_1}t_1 - \sqrt{-C_3}}\right)^{\frac{C_2}{\sqrt{C_1C_3}}}, C_1 < 0\end{gathered}$$

To obtain the solution of the inhomogeneous equation, we assume that the quantity k is a function of t_1; we find it by the quadrature

$$k(t_1) = \frac{4\sigma}{n_0}\int Q^{-1}(t_1)\frac{t_1}{(C_1 t_1^2 - C_3)^2}\,dt_1$$

Therefore, the transcendental first integral of system 15, 16 becomes

$$\begin{gathered}Q^{-1}(t_1)q_1(C_1 t_1^2 - C_3)^{-1} - \\ -\frac{4\sigma}{n_0}\int_{t_0}^{t_1} Q^{-1}(\tau_1)\frac{\tau_1}{(C_1\tau_1^2 - C_3)^2}\,d\tau_1 = C^0\end{gathered}$$

where $C^0 = \mathrm{const}$.

As is seen, the final form of the first integral depends on the sign of the constant C_1, and, therefore, three variants are possible. Let us examine each of them.

FIRST VARIANT. $C_1 = 0$. After an elementary calculation, we obtain an additional integral in the form

$$e^{-\frac{u_1}{v_1}}\left(v_1^{-2} + \frac{\sigma^2}{2}\left(\frac{u_1}{v_1} + 1\right)\right) = \mathrm{const}$$

Therefore, for $C_1 = 0$, the transcendental first integral of system 15, 16 is expressed through elementary functions.

SECOND VARIANT. $C_1 > 0$. The integration leads to the function

$$-\frac{\sigma}{4n_0}e^{-2\frac{C_2}{\sqrt{-C_1C_3}}\zeta}\left(\frac{C_2}{\sqrt{-C_1C_3}}\sin 2\zeta + \cos 2\zeta\right)$$

where

$$\zeta = \arctan\sqrt{-\frac{C_1}{C_3}}t_1$$

As is seen, in the case $C_1 > 0$, the additional first integral is expressed through elementary functions.

THIRD VARIANT. $C_1 < 0$. By equivalent transformations, the integral transforms into

$$\frac{\sigma}{C_1C_2n_0}\left(2\frac{\zeta^{1-\gamma}}{\gamma-1} - 3\frac{\zeta^{-\gamma}}{\gamma} + \frac{\zeta^{-1-\gamma}}{\gamma+1}\right)$$

where

$$\gamma = \frac{C_2}{\sqrt{C_1C_3}} > 1,\ \zeta = \frac{\sqrt{-C_1}t_1 + \sqrt{-C_3}}{\sqrt{-C_1}t_1 - \sqrt{-C_3}}$$

Therefore, in the case $C_1 < 0$, the additional first integral is also expressed through elementary functions.

And so, we study the connection between the following three properties, which are independent for the first glance, but they are sufficiently harmonically combined on systems from the rigid body dynamics:

1. the distinguished class of systems 12, 13 with the above;

2. the fact that this class of systems consists of systems with zero mean variable dissipation (in the variable α), which allows us to consider them as "almost" conservative systems;

3. in certain (although lower-dimensional) cases, these systems have first integrals, which are transcendental in general.

Let us present one more important example of a higher-order system that has the properties just listed.

To the system

$$c\dot{\alpha} = -z_2 + \beta\sin\alpha \tag{17}$$

$$\dot{z}_2 = \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha} \tag{18}$$

$$\dot{z}_1 = z_1z_2\frac{\cos\alpha}{\sin\alpha} \tag{19}$$

which is distinguished considered in the three dimensional domain

$$\mathbf{S}^1\{\alpha \bmod 2\pi\} \setminus \{\alpha = 0,\, \alpha = \pi\} \times \mathbb{R}^2\{z_1, z_2\}$$

(such a system can reduce to an equivalent system on the tangent bundle of the two-dimensional sphere) and describes the spatial motion of a rigid body in a resisting medium [6], we put in correspondence the following system with algebraic right-hand side:

$$\frac{dz_2}{d\tau} = \frac{\tau - z_1^2/\tau}{-z_2 + \beta\tau}, \quad \frac{dz_1}{d\tau} = \frac{z_1z_2/\tau}{-z_2 + \beta\tau} \tag{20}$$

In this case, it is also seen that system 17–19 is a system with zero mean variable dissipation; in order to obtain a complete correspondence with the definition, it suffices to introduce the new phase variable $z_1^* = \ln|z_1|$.

Moreover, this system has two first integrals (i.e., the full list), which are transcendental functions and are expressed through a finite combination of elementary functions; as was mentioned above, this become possible after establishing its correspondence to the (non-autonomous in general) system of equations 20 with algebraic (polynomial) right-hand side.

Therefore, the systems from the rigid body dynamics presented above not only enter the class of systems (12), 13 and have the mean zero variable dissipation, but they have a full list of transcendental first integrals expressed through a finite combination of elementary functions. In this case, the integration of systems 14 and 15, 16 reduces to the integration of the corresponding equations with algebraic right-hand side.

As was noted, to seek for first integrals of the systems considered, it is better to reduce systems of the form 12, 13 to systems with polynomial right-hand sides; the possibility of integrating the initial system in elementary functions depends on their form. Therefore, we proceed as follows: we seek for sufficient conditions for integrability in elementary functions of systems of equations with polynomial right-hand sides studying systems of the most general form in this case.

4 Conclusion

The results of the presented work were appeared owing to the study the applied problem of the rigid body motion in a resisting medium, where we have obtained complete lists of transcendental first integrals expressed through a finite combination of elementary functions. This circumstance allows the author to carry out the analysis of all phase trajectories and show those their properties which have the roughness and are preserved for systems of a more general form. The complete integrability of such system is related to their symmetries of latent type. Therefore, it is of

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interest to study a sufficiently wide class of dynamical systems having analogous latent symmetries.

So, for example, the instability of the simplest body motion, the rectilinear translational drag, is used for methodological purposes, precisely, for finding the unknown parameters of the medium action on a rigid body under the quasi-stationarity conditions.

The experiment on the motion of homogeneous circular cylinders in the water carried out in Institute of Mechanics of M. V. Lomonosov State University justified that in modelling the medium action on the rigid body, it is also necessary to take into account an additional parameter that brings a dissipation to the system.

In studying the class of body drags with finite angle of attack, the principal problem is finding those conditions under which there exist auto-oscillations in a finite neighborhood of the rectilinear translational drag. Therefore, there arises the necessity of a complete nonlinear study.

The initial stage of such a study is the neglecting of the medium damping action on the rigid body. Functionally, this means the assumption that the pair of dynamical functions determining the medium action depends only on one parameter, the angle of attack. The dynamical systems arising under such a nonlinear description are variable dissipation systems. Therefore, there arises the necessity to create the methodology for studying such systems.

Generally speaking, the dynamics of a rigid body interacting with a medium is just the field where there arise either nonzero mean variable dissipation systems or systems in which the energy loss in the mean during a period can vanish. In the work, we have obtained such a methodology owing to which it becomes possible to finally and analytically study a number of plane and spatial model problems.

In qualitative describing the body interaction with a medium, because of using the experimental information about the properties of the streamline flow around, there arises a definite dispersion in modelling the force-model characteristics. This makes it natural to introduce the definitions of relative roughness (relative structural stability) and to prove such a roughness for the system studied. Moreover, many systems considered are merely (absolutely) Andronov–Pontryagin rough.

These works opens a new cycle of research works in nonlinear analysis of the body motion in a resisting medium under the quasi-stationarity conditions and with account of the medium damping.

All what was said above allows one to estimate the results of the work in to totality as a *new direction in qualitative theory of ordinary differential equations and the dynamics of a rigid body interacting with a medium*.

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