Image Smoothing by Pseudo-2D Savitzky-Golay Filter

JEONG-HWAN KIM[1], GYEO-WUN JEUNG[1], KYEONG-SEOP KIM[1]
School of Biomedical Engineering[1]
Konkuk University
268 Chungwondaero Chungju-si, Chungbuk
KOREA
zindevil@kku.ac.kr, wjdrudns98@naver.com, kyeong@kku.ac.kr http://kkubme.kku.ac.kr

Abstract: - The pseudo-two dimensional Savitzky-Golay (S-G) smoothing filter was implemented by the Kronecker product of one dimensional SG filter to smooth the noisy image and simultaneously to keep the important image features. The performance of our suggested pseudo-2D S-G filter was compared with the denosing results of the conventional mean filter and Gaussian low pass filter in terms of Signal to Noise Ratio (SNR) and Minimum Square Error (MSE) index. The simulation results show that our suggested S-G filter can effectively remove the high-frequency noise and it can keep the original image features.

Key-Words: - Savitzky-Golay (S-G) filter, Noise, Signal to Noise Ratio (SNR), Minimum Square Error (MSE), Kronecker Product, Denosing

1 Introduction

In 1964 Savitzky and Golay presented a novel method for smoothing and differentiating of a given set of data in the sense of a polynomial fitting by minimizing the residual in a least-squares measure [1]. Compared with mean filter which can be implemented with simple computations, the main advantage of S-G filter is that it can effectively suppress the short-term randomness (noises) and it can simultaneously preserve Gaussian-shaped local minima or maxima in terms of the amplitude and time location. However S-G filter had some drawbacks in computational burden and data truncation problem. To overcome these glitches, the method for calculating convolution weights to least-square smoothing and differentiation had been presented by considering the Gram polynomial values and also by evaluating its derivatives [2].

The original concept of S-G filter algorithm was applied to fit a polynomial towards the odd number of data. With this aim, the convolution coefficients of the S-G filter was derived to smooth and differentiate the even number of data with specifying a specific degree of a polynomial [3].

The property of the S-G smoothing filter was initially used in analytical chemistry to measure accurate spectra [4]. The S-G smoothing property was also used to optimize edges and contours of geophysical data maps for defining the surface phosphate disturbed zone [5]. Additionally S-G filtration was applied to increase the accuracy of thickness calculation in thin film by smoothing the shape of X-ray reflectivity curve [6].

The Gaussian-shaped local-peaks preservation property of S-G filter had attracted some researchers in processing biomedical signals such as electrocardiogram (ECG) and electroencephalogram (EEG) data. For the instances, the S-G filter was combined with discrete wavelet decomposition to smooth the noise in ECG data without loss of peaks resolution [7]. To replace the conventional Pan-Tompkin's algorithm [8] for detecting QRS-fiducial features in ECG data, S-G filter was used in place of the high pass filter and differentiator [9]. The Percentage Root Mean Square Difference (PRD) and Signal to Noise Ratio (SNR) index was computed to evaluate the performance of the S-G filter in suppressing the noise in ECG signal [10]. For the segmentation of long-term EEG signal into pseudo stationary epochs, the signal is pre-processed by S-G filter to attenuate its short-term variations [11].

Concerning the image smoothing or differencing, 2D S-G filter were seldom used and consequently 2D convolution kernels were sparsely studied or documented due to the high degree of complexity in interpreting an image data into the polynomial fittings. For instance, 2D S-G digital differentiator was derived by fitting 2D vector-polynomial bases to a local sub-region of the displacement fields attained by digital image correlation operation and this 2D operator was utilized to improve the accuracy in estimating the displacement [12]. For image smoothing effort by 2D S-G filter, an image was divided into the non-overlapping blocks and the pixel data were represented in a vector format to compute the projection matrix on the column space spanned by 2D vector-polynomial bases. The smoothened target pixel was attained by projecting the data vector onto the center row of the projection matrix [13]. Our research objective presented in this study is to suggest the simpler method for deriving the pseudo-2D S-G pseudo-smoothing filter by the lower dimensional 1D S-G filter with applying Kronecker product [14]. The performance of image denoising by our suggested pseudo-2D S-G smoothing filter was evaluated by comparing with the performance of mean and Gaussian low pass filter in terms of Mean Square Error (MSE) and Signal-to-Noise Ratio (SNR).

2 1D S-G Smoothing Filter

Consider the digital signal corrupted with noise, $x[n]$ (n = 0, 1, … N - 1). The total number of samples is N and each sample is evenly time spaced. To apply polynomial fitting into the digital signal with least-square minimization, also consider the n^{th}-degree polynomial $p(n)$

$$p(n) = \sum_{r=0}^{n} a_r \cdot x^r = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n \quad (1)$$

, where a_r is the coefficients of a polynomial.

The dimension of S-G filter is to be $2M + 1$ which approximates the noisy samples with time-index between $-M$ to M. The S-G filter can be derived by fitting these signals within the window region by projecting them on the subspace which is spanned by the polynomial bases: $\{1, x, x^2, x^3, . . . , x^n\}$. The derived polynomial minimizes the mean-squared approximation error, ξ_n

$$\xi_n = \sum_{\ell=-M}^{M} \left(p(n) - x[\ell]\right)^2 = \sum_{\ell=-M}^{M} \left(\sum_{r=0}^{n} a_r \cdot x_\ell^r - x[\ell]\right)^2 \quad (2)$$

In a vector format, the noisy signals with the window region with the length of $2M + 1$ can be expressed by,

$$\mathbf{X} = \left[x[-M] \;\; x[-M+1] \;\cdots\; 0 \;\; x[1] \;\cdots\; x[M]\right]^t, \; t\text{: transpose} \quad (3)$$

The polynomial fitting coefficients can be computed by setting up a matrix equation:

$$\boldsymbol{X} = \boldsymbol{S} \cdot \boldsymbol{A} = \begin{bmatrix} 1 & x_{-M} & x_{-M}^2 & x_{-M}^3 & \cdots & x_{-M}^n \\ 1 & x_{-M+1} & x_{-M+1}^2 & x_{-M+1}^3 & \cdots & x_{-M+1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{M-1} & x_{M-1}^2 & x_{M-1}^3 & \cdots & x_{M-1}^n \\ 1 & x_M & x_M^2 & x_M^3 & \cdots & x_M^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \quad (4)$$

, where, x_i is the sampling-time index contained in the selected window region. The noisy signal for polynomial fitting contains $2M + 1$ samples and the dimension of basis (the maximum degree of polynomial) is chosen to be not larger than $2M + 1$ by linear algebra theory. If the dimension of basis is not equal to the number of samples in the considered window, the pseudo-inverse of $\boldsymbol{S}$ is computed by projecting $\boldsymbol{X}$ onto the subspace which is spanned by the polynomial bases. The fitted-polynomial coefficients $\widehat{\boldsymbol{A}}$ can be estimated by,

$$\widehat{\mathbf{A}} \approx (\boldsymbol{S}^t \cdot \boldsymbol{S})^{-1} \cdot \boldsymbol{S}^t \cdot \boldsymbol{X} \quad (5)$$

The S-G filtered signal $\widehat{\boldsymbol{X}}$ can be obtained by multiplying projection matrix, $\boldsymbol{P}$ and resolving the entries in the center row,

$$\widehat{\boldsymbol{X}} = \boldsymbol{S} \cdot \widehat{\boldsymbol{A}} = \mathbf{S} \cdot (\boldsymbol{S}^t \cdot \boldsymbol{S})^{-1} \cdot \boldsymbol{S}^t \cdot \boldsymbol{X} = \boldsymbol{P} \cdot \boldsymbol{X} \quad (6)$$

We can interpret these center-row matrix entries in $\boldsymbol{P}$ as a convolution kernel or S-G polynomial coefficients. Thus S-G filtering for the entire noisy signal can be achieved by operating the convolution operation with data samples within the same length of the window by shifting one sample to the right.

2.1 Sample S-G Coefficient (Convolution Kernel)

The dimension of polynomial basis, n should be not greater than $2M + 1$. For the simple explanation, we consider the two cases of: (i) n = 1, M = 1 (ii) n = 2, M = 2 here.

2.1.1 S-G Coefficient (*n* = 1, *M* = 1)

The first-degree 1D-polynomial is used for fitting the equation on the three samples, $\boldsymbol{X} = [x[-1] \; x[0] \; x[1]]^t$

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$$\boldsymbol{X} = \boldsymbol{S} \cdot \boldsymbol{A} \quad (7)$$

$$\begin{bmatrix} x[-1] \\ x[0] \\ x[1] \end{bmatrix} = \begin{bmatrix} 1 & x_{-1} \\ 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \quad (8)$$

The S-G coefficients can be resolved by the entries in the center row of $\boldsymbol{P}_{3\times3}$

$$\boldsymbol{P}_{3\times3} = \begin{bmatrix} 1 & x_{-1} \\ 1 & x_0 \\ 1 & x_1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 1 & 1 \\ x_{-1} & x_0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_{-1} \\ 1 & x_0 \\ 1 & x_1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ x_{-1} & x_0 & x_1 \end{bmatrix} \quad (9)$$

It turns out that these entries are same as the kernel of a moving average filter due to the first degree-polynomial approximation.

2.1.2 S-G Coefficient (*n* = 2, *M* = 2)

The 2nd degree 1D-polynomial is used for fitting the equation on the five samples, $\boldsymbol{X} = [x[-2]\ x[-1]\ x[0]\ x[1]\ x[2]]^t$

$$\begin{bmatrix} x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1 & x_{-2} & x_{-2}^2 \\ 1 & x_{-1} & x_{-1}^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad (10)$$

The S-G convolution kernel can be resolved by the entries in the center row of $\boldsymbol{P}_{5\times5}$

$$\boldsymbol{P}_{5\times5} = \begin{bmatrix} 1 & x_{-2} & x_{-2}^2 \\ 1 & x_{-1} & x_{-1}^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_{-2} & x_{-1} & x_0 & x_1 & x_2 \\ x_{-2}^2 & x_{-1}^2 & x_0^2 & x_1^2 & x_2^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_{-2} & x_{-2}^2 \\ 1 & x_{-1} & x_{-1}^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_{-2} & x_{-1} & x_0 & x_1 & x_2 \\ x_{-2}^2 & x_{-1}^2 & x_0^2 & x_1^2 & x_2^2 \end{bmatrix} \quad (11)$$

3 Pseudo-2D S-G Filter

For the image processing applications by the S-G filter including image denoising, 2D-vector polynomial should be considered due to the two-dimensional spatiality of an image data. With assumption of wide-sense stationary in a sub-region of original (clean) image in which the cardinality is decided by the pre-selected window size (for instance 5×5), we propose pseudo-2D S-G filter for image smoothing by operating the Kronecker product on 1-D S-G filters.

Kronecker product, denoted by, $\otimes$, is the generalization of the outer product and it is a useful operation in generating high-order matrix from low-order matrices. If $\boldsymbol{C}$ is a $M_1 \times M_2$ matrix and $\boldsymbol{D}$ is a $N_1 \times N_2$ matrix, then the Kronecker product is defined as a $M_1 \times M_2$ block matrix of basic dimension $N_1 \times N_2$.

$$\mathbf{C} \otimes \mathbf{D} = \{c(m,n) \cdot \boldsymbol{D}\} = \begin{bmatrix} c(1,1)\boldsymbol{D} & \cdots & c(1,M_2)\boldsymbol{D} \\ \vdots & & \vdots \\ c(M_1,1)\boldsymbol{D} & \cdots & c(M_1,M_2)\boldsymbol{D} \end{bmatrix} \quad (12)$$

This product is a quite effective operator to interpret the various image transforms such as Discrete Fourier transform, Discrete Cosine transform and Discrete Hadamard transform as a fast form, respectively. For our experimental simulations for image smoothing application, our pseudo-2D S-G filter is implemented by the Kronecker product of the S-G convolution kernel that resolved by collecting the entries in the center row of $\boldsymbol{P}_{5\times5}$.

4 Experimental Simulations for Image Denoising

The performance of image denoising by our suggested pseudo-2D S-G smoothing filter was compared with the filterd results from mean and Gaussian low pass filter in terms of Mean Square Error (MSE) and Signal-to-Noise Ratio (SNR) in dB scale, respectively.

$$\text{SNR} = 10log_{10}\left(\frac{\sigma_s^2}{\sigma_n^2}\right) \quad (13)$$

,where σ_s^2 is a variance of the filtered image and σ_s^2 is a variance of the selected Region Of Interest (ROI) for estimating noise power.

The MSE is defined by the mean-squared difference between the pixels-intensites of a clean image and the filtered image. Fig.1 shows the convolution kernels for the mean filter, Gaussian low pass filter and pseudo-2D S-G filter, respectively.

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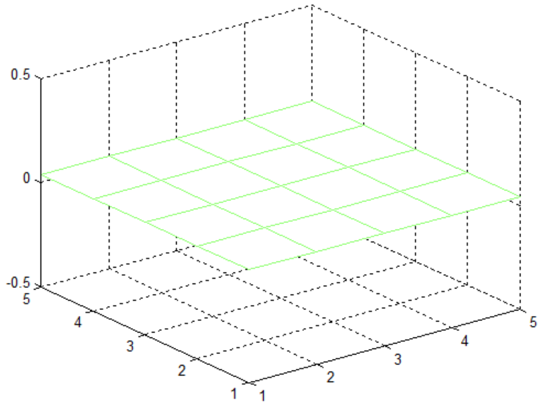
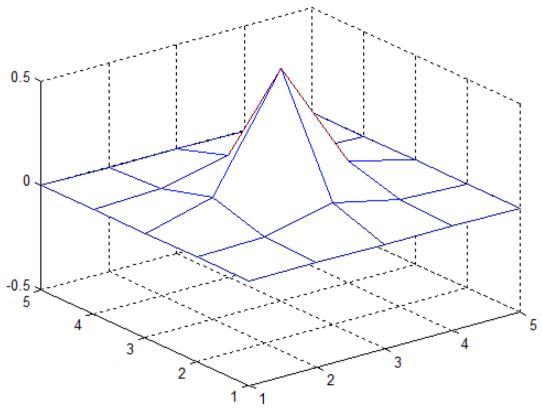
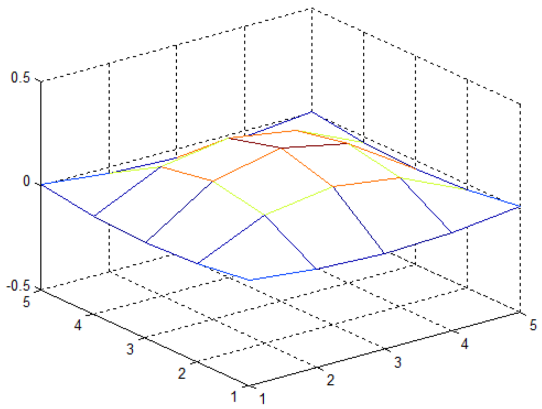


(a)



(b)



(c)

Fig.1 Surface-mesh plot for the convolution kernels of (a) mean, (b) Gaussian low pass and (c) pseudo-2D S-G filter, respectively.

Here, the pseudo-2D S-G filter can be implemented by Kronecker product of the vector itself in which its elements are the entries of the center row in $\boldsymbol{P}_{5\times5}$. For experimental simulations for image denoising application, we consider the noisy image, $\boldsymbol{g}$ which is corrupted by the additive Gaussian-distribution random noise, $\boldsymbol{n}$

$$\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{n} \tag{14}$$

where $\boldsymbol{f}$ is the original (clean) image. Fig.2 displays "Lenna" image contaminated with the additive Gaussian-distribution random noise (mean = 0.0108, variance = 163.3824).

Fig.2 "Lenna" image corrupted with the additive random noise.

The image smoothing operations can be achieved by the applying 2D convolution operation using the 2D kernels as shown in Fig.1, respectively. Fig.3 shows the filtered results with displaying ROI by drawing a rectangular polygon. This local area is selected to compute the filtered-noise variance after applying the filtering operation.

(a)

(b)

(c)

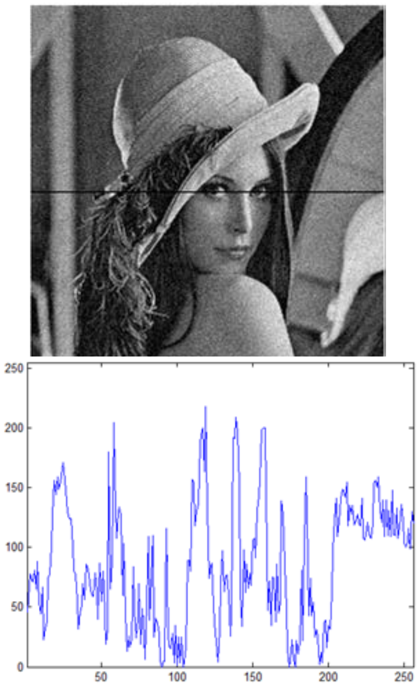
Fig.3 The filtered (cleaned) image after applying (a) mean filter (b) Gaussian low pass filter (c) pseudo-2D S-G filter.

Table 1 summarizes the performance indexs of the low pass filters in terms of SNR and MSE.

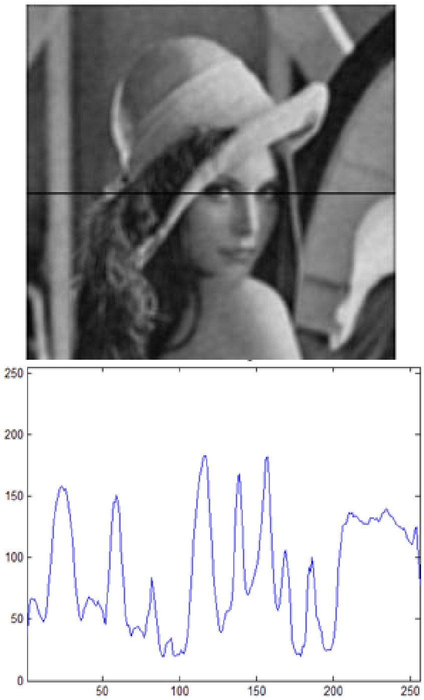
Table 1. The calculated performance index: SNR and MSE

	SNR (dB)	MSE (dB)
Noisy image	13.8117	22.0507
Mean filtered image	24.7324	24.2827
Gaussian low pass filterd image	16.4813	19.4380
Pseudo-2D S-G filtered image	17.6622	20.9742

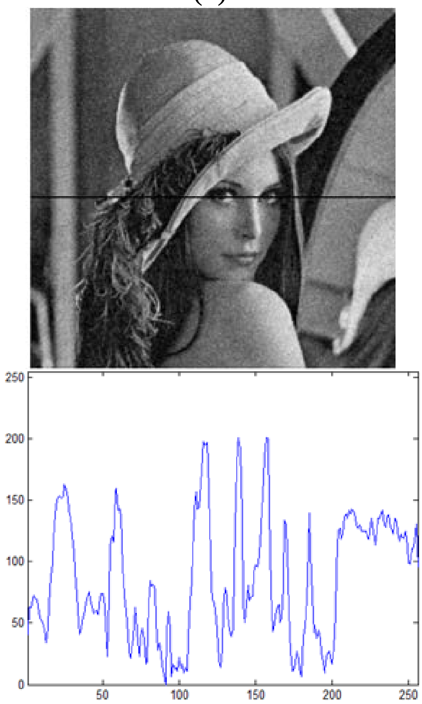
To describe the local variations in the pixel intensities and the efftiveness of suppressing noises by processing the filters, a line profile and the graylevels of the pixels are also illustrated in Fig.4.



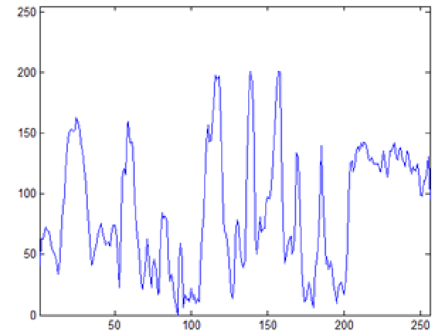
(a)



(b)



(c)



(d)

Fig.4 A line profile and the intensity variations along the line in (a) noisy image (b) mean filtered image (c) Gaussian low pass filtered image (d) pseudo-2D S-G filtered image.

The pseudo-2D S-G smoothing filter can be also attained by Kroncker product of the vector elements deduced by $\boldsymbol{P}_{5\times5}$ with a 1D Gaussian kernel [15] as shown in Fig.5.

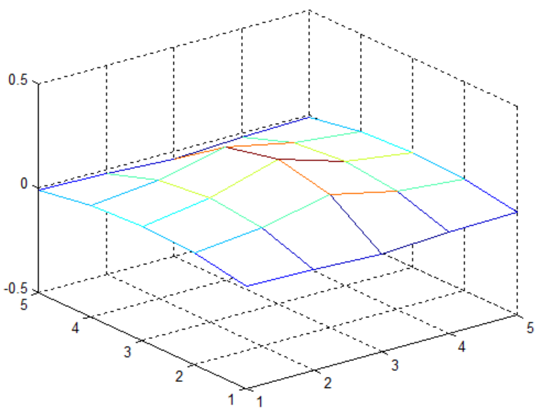


Fig.5 Surface-mesh plot for the convolution coefficient obtained by Kroncker product of the vector elements in $\boldsymbol{P}_{5\times5}$ and an 1D Gaussian kernel.

Fig.6 shows the smoothed image by this composite filter and SNR and MSE are calculated as 20.0469 dB and 21.6721 dB, respectively.

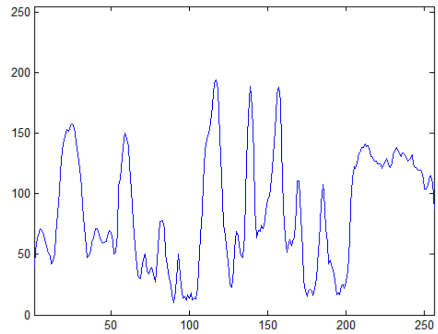


Fig.6 A line profile and the intensity variations along the line in the composite-filtered image.

Thus we can find the fact that SNR is improved with the loss of MSE with apply the composite filter.

Consider the "Lenna" image again with increasing the additive-noise power from 163.3814 to 655.5272 to lower SNR. Table 2 shows the performance indexs of the low pass filters in terms of SNR and MSE.

Table 2. The calculated performance index: SNR and MSE

	SNR (dB)	MSE (dB)
Noisy image	7.7781	27.8867
Mean filtered image	19.8053	24.6230
Gaussian low pass filterd image	10.8358	24.4312
Pseudo-2D S-G filtered image (using $\boldsymbol{P}_{5\times5}$)	12.1446	23.6897
Pseudo-2D S-G filtered image (using $\boldsymbol{P}_{5\times5}$ and Gaussian kernel)	14.7998	23.1816

Fig.7 shows filtered images and their profiles after applying the various smoothing filters.

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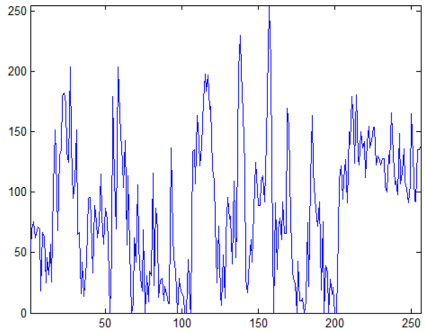
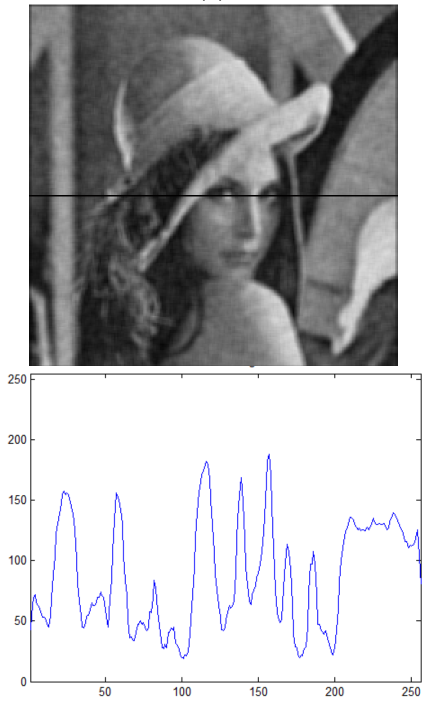
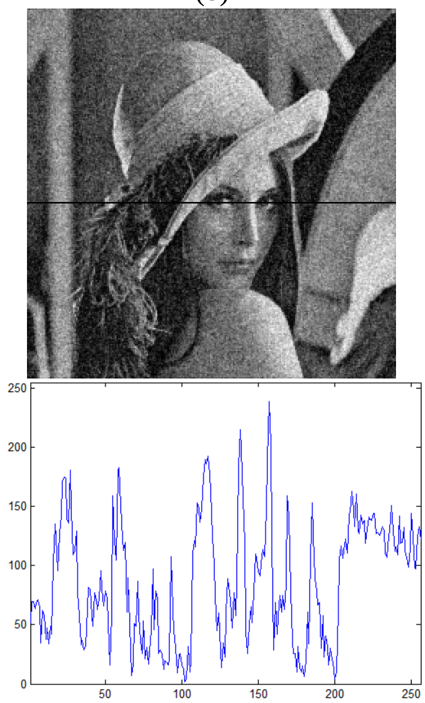


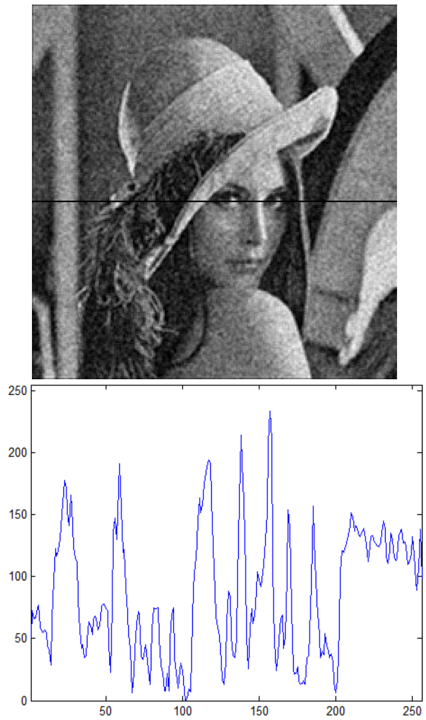
(a)



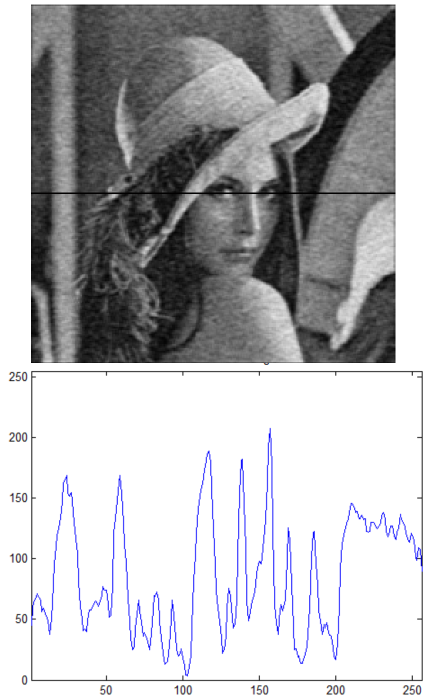
(b)



(c)



(d)



(e)

Fig.7 A line profile and the intensity variations along the line in (a) noisy image (b) mean filtered image (c) Gaussian low pass filtered image (d) pseudo-2D S-G filtered (using $\boldsymbol{P}_{5\times5}$) image (e) pseudo-2D S-G filtered (using $\boldsymbol{P}_{5\times5}$ and Gaussian kernel) image.

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5 Conclusion

This research introduces the polynomial fitting algorithm to smoothe the noisy image and simultaneoulsy preserve the main features in it by implementing pseudo-two dimensional S-G filter based on Kronecker products of the lower dimensional 1D S-G filters. Computer simulations illustrated in Fig.3-4 show that our suggested S-G filter can effectively remove high-frequency noise and it can keep the important image features. As summarized in Table 1, pseudo-2D S-G filter maintains the intermediate SNR and MSE levels comparing with the performance of mean filter and Gaussian low pass filter.

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