On the Critical Curves of a Degenerate Parabolic Equation with Multiple nonlinearities and Variable Density

Mersaid Aripov, Zafar Rakhmonov
National University of Uzbekistan, Tashkent, 100174, Uzbekistan

mirsaidaripov@mail.ru, zraxmonov@inbox.ru

Abstract: - In this paper concerned the critical curves of a degenerate parabolic equation with a source and nonlinear boundary flux. We establish the critical global existence curve and critical Fujita curve by constructing various self-similar supersolutions and subsolutions. The asymptotic of a solution near the free boundary at critical values of numerical parameters are investigated.

Key-Words: - Critical global existence curve, Critical Fujita curve, Blow-up, Multiple nonlinearities, Asymptotic, numerical solution.

Introduction
Consider the following polytrophic filtration equation with variable density

\[ \rho(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial u^n}{\partial x} - \frac{\partial u^m}{\partial x} \right) + \rho(x) u^\theta, \quad (x,t) \in \mathbb{R} \times (0, +\infty), \]

subject to a nonlinear boundary flux

\[ \frac{\partial u^n}{\partial x}(0, t) = u^q(0, t), \quad t \in (0, +\infty) \]

and initial value condition

\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \]

where \( \rho(x) = (1 + x)^\alpha, \) \( m > 0, \) \( p > 1 + 1/m, \) \( q > 0, \) \( n \in \mathbb{R} \) and \( u_0(x) \) is a nontrivial, nonnegative, bounded and appropriately smooth function.

Equations (1) arise in some physical models such as population dynamics, chemical reactions, heat transfer, etc. In particular, problem (1)-(3) can be thought of as a model to describe heat propagation with a gradient-dependent thermal conductivity in a medium with chemical reaction and a nonlinear radiation law at the boundary (see [1, 22, 23]). The nonlinear boundary condition (2) appears also in combustion problems when the reaction happens only at the boundary of the container, for example because of the presence of a solid catalyst, see [22] for a justification. Equation (1) is called a parabolic equation with variable density [13, 14].

The problem (1) - (3) for different values of the parameters has been studied by many authors [3-23]. In [23], the conditions of existence of blow-up and global solutions by constructing self-similar supersolutions and subsolutions studied. Equation (1) due to the degeneration for \( u = 0 \) is may cannot have a classical solution. Therefore, it is natural to understand the solution in the weak sense of

\[ 0 \leq u, \quad \frac{\partial u^n}{\partial x}, \frac{\partial u^m}{\partial x} \in C^0(\mathbb{R} \times (0, +\infty)) \]

and satisfies the equation (1) in an integral way.

In [6] Galaktionov and Levine studied the problem (1)-(3) for \( p = 2, n = 0 \) and for \( m = 1, n = 0 \) in the case without a source They proved...
that for the problem (1)-(3) the critical global exponent is \(q_0 = \frac{(m+1)}{2}\) and the critical Fujita exponent is \(q_c = m+1\) (for \(p = 2\), \(n = 0\)), while for the \(m = 1\), \(n = 0\) critical global exponent is \(q_0 = 2(p-1)/p\) and the critical Fujita exponent is \(q_c = 2(p-1)\).

Wang and Yin [4], Li and Mu [7] studied problem (1)-(3) without a source in the cases of a slow diffusion and a fast diffusion, respectively. They showed that the critical global existence exponent and critical Fujita exponent are \(q_0 = \frac{1}{2}m + p\) and \(q_c = m + p\) (for \(2p = 0\), \(n = 0\)). For the \(m = 0\), \(n = 0\) critical global exponent is \(q_0 = \frac{2(1-p)}{p}\) and the critical Fujita exponent is \(q_c = 2(p-1)\).

Jiang and Zheng [8] studied the following problem:

\[
\begin{cases}
t_t = \Delta u_t - u_t^\rho (u_t)^k, & x > 0, 0 < t < T, \\
-u_t^\rho (u_t)(0,t) = u_t^p(0,t), & 0 < t < T, \\
u_t(x,0) = u_0(x), & x > 0
\end{cases}
\]

where \(m \geq 1\), \(p > 0\), \(\beta > 0\). They obtained the critical global existence exponent \(p_0 = \frac{2}{(\beta + m + 1)}\) and the critical Fujita exponent \(p_c = 2\beta + m + 1\).

Many authors intensively studied the Cauchy problem to the equation (1) in particular value of parameters. For instance Qi [9] studied the following Cauchy problem

\[
\begin{cases}
u_t = \Delta u_t + |x|^\rho u_t^p, & x \in \mathbb{R}^N, \ t > 0 \\
u(x,0) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]

and established the critical Fujita exponent \(p_c = m + (2 + \sigma)/N\) for \(m > (N - 2)/N\).

In [18], some Fujita-type results extended to

\[
|x|^\nu u_t = \Delta u_t^k + |x|^\rho u_t^p, \ x \in \mathbb{R}^N, \ t > 0,
\]

with \(q > k \geq 1\) and \(0 < m \leq n < qm + q - 1\) and the critical Fujita exponent was given as \(q_c = k + (2 + n)/(N + m)\).

The Cauchy problem of another nonlinear diffusive equation of the form

\[
u_t = \text{div}\left(|\nabla u_t|^{p-2}\nabla u_t\right) + |x|^\rho u_t^q, \quad (4)
\]

where \(p > (N + 1)/(2N), \ q > 1\) was also considered by some authors. For the problem (4) with \(p > 2\) and \(n = 0\), Qi [10, 11] obtained that \(q_c = p-1 + p/N\) is the critical Fujita exponent of (4) and \(q_c\) belongs to the blow-up case. If \(n \neq 0\) in (4), Qi and Wang [12] proved that the critical Fujita exponent is \(q_c = p-1 + (p + n)/N\) for \((N + 1)/(2N) < p < 2\).


Regular property of the Cauchy problem for the equations

\[
s(x)\frac{\partial u_t}{\partial t} = \text{div}\left(|Du_t|^{p-1}Du_t\right), \ (x, t) \in \mathbb{R}^{N+1}
\]

where \(s(x) = |x|^{l}\), \(l \geq 0\), \(Du \equiv \left(\frac{\partial u_t}{\partial x_1}, \ldots, \frac{\partial u_t}{\partial x_N}\right)\), was considered by authors of the work [20].

Martynenko and Tedeev [13, 14], studied the Cauchy problem for the following two equations with variable coefficients:

\[
u_t = \text{div}(u_t^m|\nabla u_t|^{l-1}\nabla u_t) + u_t^p, \quad x \in \mathbb{R}^N, \ t > 0
\]

and

\[
u_t = \text{div}(u_t^m|\nabla u_t|^{l-1}\nabla u_t) + \rho(x)u_t^p, \quad x \in \mathbb{R}^N, \ t > 0
\]

where \(\lambda > 0\), \(m + \lambda - 2 > 0\), \(p > m + \lambda - 1\), \(\rho(x) = |x|^{-n}\) or \(\rho(x) = (1 + |x|)^{-n}\). It was shown that under some restrictions on the parameters, any nontrivial solution to the Cauchy problem blows up in a finite time. Moreover, the authors established a sharp universal estimate of the solution near the blow-up point.
Authors of the work [17] investigated properties of weak solutions of the Cauchy problem for the following equation with variable coefficients:

$$\begin{align*}
\rho(x)u_t &= \text{div} [ |x|^\alpha u^{m-1} \nabla u^{\lambda - 1} \nabla u] + \rho(x)u^p, \\
&\quad x \in \mathbb{R}^N, \quad t > 0
\end{align*}$$

where $\lambda > 0$, $m + \lambda - 2 > 0$, $p > m + \lambda - 1$, $\rho(x) = |x|^{\beta - 2}$.

The first works devoted to the problem with distributed parameters were considered by Kurdumov S. P., Kurkina E. S., Malinetskii G. G. [21] and the study the problem of eigenfunctions for the equation (5) when $\lambda = 1$, $n = 0$, $\rho(x) = |x|^{\beta - 2}$.

Zhongping Li, Chunlai Mu and Li Xie [22] studied the problem for $m = 1$, $n = 0$. They have critical global exponent $\beta_g = 1$, $q_0 = 2(p - 1)/p$ and critical Fujita exponent $\beta_f = 2(p - 1) + 1$, $q = 2(p - 1)$ respectively.

Properties of the more general equation (1) with nonlocal nonlinear boundary condition were considered in [16] where were established the critical exponent of the Fujita type and second critical value.

The main purpose of this paper is to find conditions for the existence and non-existence results for global solutions of the problem (1)-(3) on the basis of the self-similar analysis and the method of standard equations [2]. The critical exponent of the global existence of solutions and the critical exponent of Fujita type are obtained. Establish the asymptotic of a solution near the free boundary at critical values of numerical parameters. The choice an appropriate initial approximation for the iteration process is a main problem in the numerical studies of the solution of the problem (1)-(3). Provides method a choosing an appropriate initial approximation for the iteration process in numerical studies of the solution of the problem (1) - (3). Using the asymptotic formula as the initial approximation for the iterative process the numerical calculation carried out.

**Estimates of solutions**

Let’s formulate the results on global solvability or nosolvability of the problem (1)-(3).

Introduce notation

$$\begin{align*}
\beta_0 &= \frac{p(mn + 1) - mn}{(p + n)}, \\
\beta_f &= (m + 1)(p - 1) + 1; \\
q_0 &= \frac{(m + 1)(p - 1) + n}{(p + n)}, \\
q_f &= (m + 1)(p - 1) + n.
\end{align*}$$

**Theorem 1.** If $0 < \beta \leq \beta_0$, $0 < q \leq q_0$, then each solution of the problem (1)-(3) exists globally.

**Proof.** Let

$$u_+(x, t) = e^{\xi t} g(\xi), \quad \xi = (1 + x)e^\beta$$

with

$$g(\xi) = M \left( K + e^{-\xi} \right)^{\beta_0^{\alpha}},$$

$$L = \frac{(p - 1)(p + n)mM^{m(p - 1) - 1}K^{1 + m}}{mk(p + n) - (m(p - 1))e^{-\gamma}}, \quad K > \|u_0\|_L,$$

$$J = \frac{m(p - 1) - 1}{p + n} L,$$

$$M = e^{m(p - 1)\gamma} \left( K + e^{-\gamma} \right)^{\frac{q_f}{m(p - 1) - 1}}.$$

A direct calculation yields

$$- \left[ \frac{\partial u_+^{p - 1}}{\partial x} \frac{\partial u_+^m}{\partial x} \right]_{x = 0} = -e^{(p - 1)(m + \lambda)\gamma} \left( g^m \right)^{p - 2} \left( g^m \right)'(1) = M^m e^{p - 1}(m + \lambda)\gamma e^{-(p - 1)},$$
Theorem 4. In the region
\[ F = \{ \beta > \beta_0, \ q > q_0 \} \], there exist solution (1)-(3) that blow up in finite time as well as global solutions.

Theorem 5. If \( \beta \leq \beta_0, \ q > q_c \), then the compactly supported solution of problem (1)-(3) has the asymptotic
\[
 u(x,t) = t^\alpha \varphi(\xi), \ \xi = (1 + x) t^{-\gamma},
\]
in \( \xi \to (a/b)^{(p-1)/p} \), where
\[
 \varphi(\xi) = \left( a - b \xi^{p-1} \right)^{\frac{p-1}{m(p-1)-1}},
\]
\[
 b = \frac{m(p-1)-1}{m(p+n)} \gamma^{-\frac{1}{\gamma-1}}, \ \alpha = \frac{1-\gamma n}{1-\beta},
\]
\[
 \gamma = \frac{p-1-\beta}{p(1-\beta)+n(m(p-1)-\beta)}.
\]

Theorem 6. If \((p+n)q < (p-1)(\beta+1)\), \(q > q_c\), then the compactly supported solution of problem (1)-(3) has the asymptotic
\[
 u(x,t) = t^\alpha f(\xi), \ \xi = (1 + x) t^{-\gamma},
\]
in \( \xi \to (a/d)^{(p-1)/p} - h \), where
\[
 f(\xi) = \left( a - d(\xi + h)^{p-1} \right)^{\frac{p-1}{m(p-1)-1}},
\]
\[
 d = \frac{p-2}{p} \gamma^{-\frac{1}{\gamma-1}}, \ h > 0,
\]
\[
 \alpha = \frac{p-1}{q(p+n)-(p-1)(m(n+1)+1)},
\]
\[
 \gamma = \frac{q-m(p-1)}{p-1} \alpha.
\]

Note that because there is no uniqueness solution arises many difficult cases in the numerical study of the problem(1)-(3). Therefore, the question arises of selecting a good initial approximation preserving properties of nonlinearity. Depending on the parameters of the equation, this difficulty is
overcome by appropriate choice of initial approximations, which are taken as established above asymptotic formulas. On the basis of the above qualitative studies were carried out the numerical calculations. The numerical results show quickly convergence of the iterative process due to the successful choice of the initial approximation. Below are given some results of numerical experiments for different values of the numerical parameters.

Fig 1. $m = 1, \ p = 2.5, \ \beta = 2, \ q = 2.85$.

Fig 2. $m = 1, \ p = 2.45, \ \beta = 2, \ q = 4$.

References


[14].Martynenko A.V. and TedeevA. F. On the behavior of solutions to the Cauchy problem


