Railway Vehicle Modelling on an Isolated Track Defect of a Cosine Form

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Abstract: - During the motion of a railway vehicle on a rail, with a rail running table/surface of a random form, the rail running table imposes to the vehicle a forced oscillation. Due to different reasons –manufacturing, corrosion, deterioration etc.- the rail's running surface is not smooth but instead it comprises a lot of defects that give to it a random surface in space. Furthermore under the primary suspension of the railway vehicle there are the Non Suspended Masses (N.S.M.) which act without any dumping directly on the track panel. On the contrary the Suspended Masses (S.M.), that are cited above the primary suspension of the vehicle, act through a combination of springs and dashpots on the railway track. A part of the track mass is also added to the Non Suspended Masses, which participates in their motion. The defects with long wavelength, which play a key role, on the dynamic component of the acting loads on the railway track, and consequently –due to the principle action=reaction– on the dynamic component of the reaction of the railway track on the railway vehicle, are modelled and analyzed using the second order differential equation of motion. A parametric investigation is performed for the case of an isolated defect of cosine form.

Key-Words: - railway track; dynamic stiffness; actions; reactions; loads; deflection; subsidence; eigenperiod; forcing period.

1 Introduction

The railway track is usually modeled as a continuous beam on elastic support. Train circulation is a random dynamic phenomenon and, depending on the different frequencies of the loads it imposes, there is a corresponding response of the track superstructure. The dynamic component of the load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. The statistical probability of exceeding the calculated load -in real conditions- should be considered, so that the corresponding equations would refer to the standard deviation (variance) of the load.

2 Calculating the Static Component of the Actions on the Railway Vehicle

The most widely used theory (referred to as the Zimmermann theory) based on Winkler analysis examines the track as a beam on elastic support.

$$\frac{d^4 y}{dx^4} = -\frac{1}{E \cdot J} \cdot \frac{d^2 M}{dx^2} \tag{1}$$

where y is the deflection of the rail, M is the bending moment, J is the moment of inertia of the rail, and E is the modulus of elasticity of the rail. From the formula above it is derived that the reaction of a sleeper R_{static} (that is of each support point of the track) acting on the railway vehicle is:

$$R_{stat} = \frac{Q_{wheel}}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow \frac{R_{stat}}{Q_{wheel}} = \overline{A} = \overline{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \tag{2}$$

where Q_{wheel} the static wheel load, ℓ the distance among the sleepers, E and J the modulus of elasticity and the moment of inertia of the rail, R_{stat} the static reaction/action on the sleeper, and ρ reaction coefficient of the sleeper which is defined as: ρ=R/y, and is a quasi-coefficient of the track elasticity (stiffness) or a spring constant of the track. The track consists of a sequence of materials (substructure, ballast, sleeper, elastic pad/ fastening, rail), that are characterized by their individual coefficients of elasticity (static stiffness coefficients) ρ_i.

Hence, for the track:

$$\frac{1}{\rho_{total}} = \sum_{i=1}^{\nu} \frac{1}{\rho_i} \tag{3}$$

where ν is the number of various layers of materials that exist under the rail -including rail– elastic pad, sleeper, ballast etc. The semi-static Action/Reaction is produced by the centrifugal acceleration exerted on the wheels of a vehicle that is running in a curve with cant deficiency, given by:

$$Q_\alpha = \frac{2 \cdot \alpha \cdot h_{CG}}{e^2} \cdot Q_{wheel}$$

where α is the cant deficiency, h_{CG} the height of the center of gravity of the vehicle from the rail and e the gauge.

3 Calculating the Dynamic Component of the Actions on the Railway Vehicle

The dynamic component of the acting load consists of the action due to the Suspended Masses (SM) and the action due to the Non Suspended Masses (NSM) of the vehicle. To the latter a section of the track mass is added, that participates in its motion ([1], [2]). *Based on a cosine defect of the form*:

$$\eta = a \cdot \cos \omega t = a \cdot \cos\left(2\pi \cdot \frac{V \cdot t}{\lambda}\right) \tag{4}$$

The second order differential equation of motion is:

$$m_{NSM} \frac{d^2 z}{dt^2} + \Gamma \cdot \frac{dz}{dt} + h_{TRACK} \cdot z = -m_{NSM} \cdot a \cdot \omega^2 \cdot \cos(\omega t) \tag{5}$$

The complete solution of which using polar coordinates is ([2], p.199 and ch.3):

$$z = \underbrace{A \cdot e^{-\zeta \omega_n t} \cdot \sin\left(\omega_n t \sqrt{1-\zeta^2} - \varphi\right)}_{transient-part} + \underbrace{a \cdot B \cdot \cos(\omega t - \varphi)}_{steady-state-part} \tag{6}$$

where, the first term is the transient part and the second part is the steady state [4].

4 Modelling and Analysis of the Railway Vehicle rolling on an Isolated Defect, Forcing vs Eigen Period

We focus herein on the term from Equation 6 which represents the transient part of motion. We investigate this term for ζ=0.

$$m_{NSM} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \Rightarrow$$

$$\Rightarrow (m_{NSM} + m_{TRACK}) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \tag{7}$$

Where: g the acceleration of gravity,

$$h_{TRACK} = 2\sqrt{2} \cdot \sqrt[4]{\frac{EJ \rho_{total}^3}{\ell^3}}, \quad where \quad m_{TRACK} = 2\sqrt{2} \cdot m_0 \cdot \sqrt[4]{\frac{EJ \ell}{\rho_{total}}} \tag{8}$$

ρ_{total} the total static stiffness coefficient of the track, ℓ the distance among the sleepers, E, J modulus of elasticity and moment of inertia of the rail, for comparison of m_{TRACK} to measurements see [5]. The general solution of equation (7) is:

$$z(t) = -\frac{1}{2} \cdot \underbrace{\frac{4 \cdot \pi^2}{\tau_1^2}}_{\omega_1^2} \cdot \frac{\alpha \cdot m_{NSM}}{\underbrace{\omega_n^2 \cdot (m_{NSM} + m_{TRACK})}_{h_{TRACK}}} \cdot \frac{1}{1-\left(\frac{\omega_1}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos(\omega_1 t)}_{steady-state} - \underbrace{\cos(\omega_n t)}_{transient-part}\right] \Rightarrow$$

$$\Rightarrow z(t) = \frac{\alpha}{2} \cdot \frac{m_{NSM}}{(m_{NSM} + m_{TRACK})} \cdot \frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2} \cdot \left[\underbrace{\cos(\omega_1 t)}_{steady-state} - \underbrace{\cos(\omega_n t)}_{transient-part}\right] \tag{9}$$

where, $T_n=2\pi/\omega_n$ the period of the free oscillation of the wheel circulating on the rail and $T_1=2\pi/\omega_1$ the necessary time for the wheel to run over a defect of wavelength λ: $T_1=\lambda/V$. Consequently, $T_n/T_1=\omega_1/\omega_n$. From equation (9):

$$\left[\frac{(m_{NSM} + m_{TRACK})}{m_{NSM}} \cdot z(t) \cdot \frac{1}{\alpha}\right] = \frac{1}{2} \cdot \frac{1}{1-(n)^2} \cdot \left[\underbrace{\cos(\omega_1 t)}_{steady-state} - \underbrace{\cos(n \cdot \omega_1 t)}_{transient-part}\right] \Rightarrow$$

$$\Rightarrow \left[\frac{(m_{NSM} + m_{TRACK})}{m_{NSM}} \cdot z(t) \cdot \frac{1}{\alpha}\right] = \frac{1}{2} \cdot \frac{1}{1-(n)^2} \cdot \left[\underbrace{\cos(2\pi \cdot \mu)}_{steady-state} - \underbrace{\cos(n \cdot 2\pi \cdot \mu)}_{transient-part}\right] \tag{10}$$

where $n=\omega_n/\omega_1$. $\omega_1=\lambda/V$ and we examine values of μ·λ=0, 0.1λ, 0.2λ,...., 0.8λ, 0.9λ, λ.
for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ. In Figure 1 Left the equation (10) is depicted.

5 A Long-Wavelegth Defect and the Implied Actions on a High Speed Railway Vehicle

The first term in the bracket of equation (10) is depicted on the vertical axis while on the horizontal axis the percentages of the wavelength μ·λ are shown. We observe that z(x) has its maximum value for T_1/T_n=0,666667=2/3, equal to 1,465:

$$z(t) = \left[\frac{m_{NSM}}{(m_{NSM} + m_{TRACK})}\right] \cdot \alpha \cdot 1{,}465 \tag{11}$$

for x=0,91λ

The relation T_1/T_n represents the cases for short and long wavelength of the defects. For T_1/T_n=2-2,5 the wavelength is long and for T_1/T_n the wavelength is short ([3], p.49). The second derivative of z(x) from equation (11), that is the vertical acceleration that gives the dynamic overloading due to the defect, is calculated:

$$z'(t)=\frac{\alpha}{2}\cdot\frac{m_{NSM}}{\left(m_{NSM}+m_{TRACK}\right)}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{-\omega_1\cdot\sin\left(\omega_1 t\right)}_{steady-state}+\underbrace{\omega_n\cdot\sin\left(\omega_n t\right)}_{transient-part}\right] \quad (12a)$$

$$z''(t)=-\frac{\alpha}{2}\cdot\frac{m_{NSM}}{\left(m_{NSM}+m_{TRACK}\right)}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\omega_1^2\cdot\cos\left(\omega_1 t\right)}_{steady-state}-\underbrace{\omega_n^2\cdot\cos\left(\omega_n t\right)}_{transient-part}\right] \quad (12b)$$

for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ, and T_n=0,0307 sec as calculated above. The maximum value of z is given in Table 1 below, as it is –graphically– measured in Figure 1 (the damping was omitted).

Table 1: Maximum Values of ζ:
$\zeta=[(m_{NSM}+m_{TRACK})/m_{NSM}]\cdot[z_{max}/\alpha]$

T_1/T_n	2,5	2	1,5	1	0,8	0,6667	0,6	0,5
ζ	0,18	0,335	0,65	1,205	1,415	1,47	1,43	1,34
where: $\zeta=[(m_{NSM}+m_{TRACK})/m_{NSM}]\cdot[z_{max}/\alpha]$								

As a case study we use a ballasted track, for high speed, equipped with rail UIC60, sleepers B70 type, W14 fastenings, ballast fouled after 2 years, subgrade for high speed lines, h_{TRACK}=8539,6t/m, m_{TRACK} = 0,426 t, m_{NSM}=1 t.

$$z_{static}=\frac{Q_{wheel}}{2\sqrt{2}}\cdot\sqrt[4]{\frac{\ell^3}{EJ\rho_{total}^3}}=\frac{112.500N}{2\sqrt{2}}\cdot\sqrt[4]{\frac{600^3mm^3}{210.000\frac{N}{mm^2}\cdot 3{,}06\cdot 10^7mm^4\cdot 85.396^3\frac{N^3}{mm^3}}}\Rightarrow$$

$$\Rightarrow z_{static}=\frac{112.500N}{2\sqrt{2}}\cdot 1{,}524228617\cdot 10^{-5}\frac{mm}{N}=0{,}606mm$$

For α=1 mm, the dynamic increment of the static deflection is equal to (0,133/ 0,606)=21,9% of the static deflection (for every mm of the depth of the defect).

If we examine the second derivative (vertical acceleration) a percentage of g:

$$\left[\frac{\left(m_{NSM}+m_{TRACK}\right)}{m_{NSM}}\cdot\frac{z''(t)}{\alpha}\right]=-\frac{1}{2}\cdot\frac{\omega_1^2}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\cos\left(\frac{2\pi V}{\lambda}\cdot\frac{\mu\cdot\lambda}{V}\right)}_{steady-state}-\underbrace{\frac{\omega_n^2}{\omega_1^2}\cdot\cos\left(n\cdot\frac{2\pi V}{\lambda}\frac{\mu\cdot\lambda}{V}\right)}_{transient-part}\right]\Rightarrow$$

(13)

$$\left[\frac{\left(m_{NSM}+m_{TRACK}\right)}{m_{NSM}}\cdot\frac{z''(t)}{\alpha}\right]=-\left(\frac{2\pi}{n\cdot T_n}\right)^2\cdot\frac{1}{g}\cdot\left[\frac{1}{2\cdot\left[1-(n)^2\right]}\cdot\left[\underbrace{\cos(2\pi\mu)}_{steady-state}-\underbrace{(n)^2\cdot\cos(2n\pi\mu)}_{transient-part}\right]\right] \quad [\%g]$$

Equation (13) is plotted in Figure 2.

6 Conclusion

For a defect of wavelength λ and sagitta of 1 mm (depth of the defect), the dynamic increase of the acting load both on railway track and on the railway vehicle –compared to the static wheel load– is equal to 9,24%. When the speed increases, the period T_1 decreases and the supplementary sagitta (depth of the defect) increases. The increase of the dynamic component of the load (action and reaction) increases faster since it is dependent on the square of the speed $(\omega_1)^2$. When the dynamic stiffness coefficient h_{TRACK} increases, T_n decreases, T_1/T_n increases, the supplementary sagitta decreases (for the same V), and the dynamic component of the action (and the reaction) decreases also.

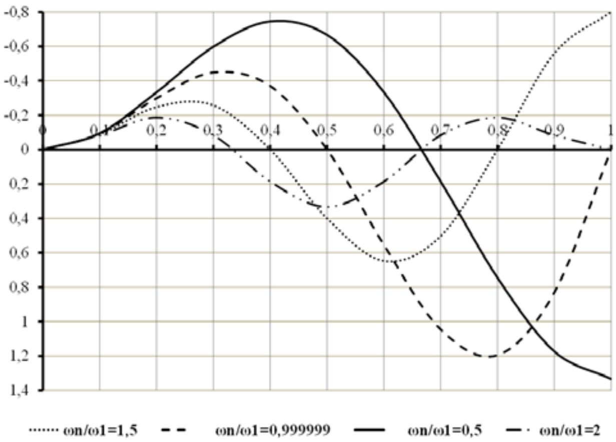

Figure 1. Mapping of Equation (9), in the vertical axis the first term of equation (10), in the horizontal axis the percentage of the wavelength λ of the defect are depicted.

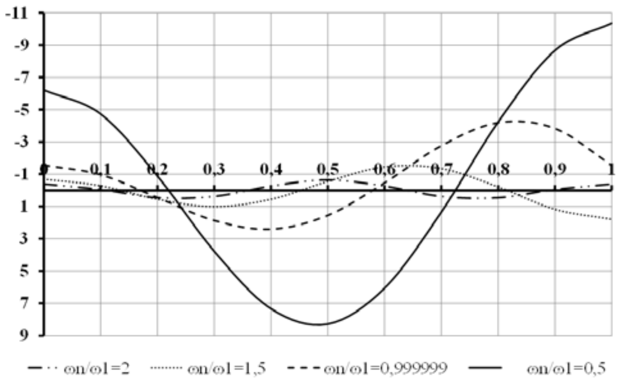

Figure 2. Mapping of Equation (13), in the vertical axis the first term of equation (10), in the horizontal axis the percentage of the wavelength λ of the defect are depicted.

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