# About Signorini's principle of the gyroscopic effect

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*Abstract:* In this work we show that Signorini's principle of the gyroscopic effect can be expressed by means of a vectorial equation of the type $\boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b}$, whose solutions lets us immediately obtain the motion of the gyroscope in many usual applications.

*Key–Words:* Gyroscope, Signorini, gyroscopic effect
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## 1 Principle of the gyroscopic effect

Usually a gyroscope is defined as a rigid system of particle, symmetrical about an axis and rotating about that axis (spinning wheel or disc, see [1, § 4.2 p.165]).
Now we want to remember a more precise definition of gyroscope (see [2, 3]): a rigid body $\mathcal{G}$ has a *gyroscopic structure* about one of its points $O$, when two of its principal moments of inertia relative to $O$ are equal; in this case its ellipsoid of inertia relative to $O$, $\mathcal{E}_O$, is an ellipsoid of revolution (oblate or prolate spheroid or sphere).
In particular if the rigid body has gyroscopic structure about its centre of mass $G$, the body is said to be a *gyroscope* and the rotation (or revolution) axis of its central ellipsoid of inertia is the so–called *gyroscope axis*.

Since in the gyroscopic phenomena the active forces are usually applied to the points of the gyroscope axis, it generally results

$$M_z^{(a)} = 0 \tag{1}$$

and the third scalar Euler's equation reduces to

$$C\dot{r} = 0 \tag{2}$$

where $C$ is the central moment of inertia related to the gyroscope axis, usually chosen as body axis $z$, and $r$ is the spin of the gyroscope with respect to the gyroscope axis. Consequently we obtain the integral of motion

$$r = r_0 = \text{const} . \tag{3}$$

In the case of a gyroscope in rapid rotation around its axis, the research of the solutions is very simplified by using the *principle of the gyroscopic effect*, enunciated by Signorini [4] and justified by Stoppelli [5]:

**Principle 1.** *If at a certain instant $t_0$ a gyroscope is spinning very rapidly around its axis and the active net force has null resultant moment about the gyroscope axis, it is possible to approximate the angular momentum theorem for a rigid body with the following equation*

$$Cr_0 \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = \boldsymbol{M}_O^{(a)} , \tag{4}$$

*where $\boldsymbol{k}$ is the versor of the gyroscopic axis.*

## 2 Vectorial equation of motion of a gyroscope

Let $\mathcal{G}_O$ be a gyroscope with a fixed point $O$ and let $T_O \equiv O\xi\eta\zeta$ be a fixed coordinate system with origin at the fixed pont $O$ and with versors $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$; we choose as body frame an *inertia principal frame* $\mathcal{T}_O$ [1] with origin at the fixed point $O$, whose $z$ axis is coincident with the gyroscope axis, and whose versors are $\boldsymbol{\imath}, \boldsymbol{\jmath}, \boldsymbol{k}$.
We give the gyroscope $\mathcal{G}$ a rapid initial rotation around its axis $z$ and leave the axis in an arbitrary direction. By means of the principle of the gyroscopic effect 1, we can approximate the equation

[1] That is a set of three cartesian axes fixed in $\mathcal{G}_O$, which are principal axes.

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of motion of the gyroscope with the equation (4), then by using Poisson's formula

$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{\omega} \times \boldsymbol{k} \,, \tag{5}$$

we can change it into the finite vectorial equation [2]

$$\boldsymbol{\omega} \times \boldsymbol{k} = \frac{1}{Cr_0} \boldsymbol{M}_O^{(e,a)} \,. \tag{6}$$

Its compatibility condition

$$\underbrace{\boldsymbol{k} \cdot \boldsymbol{M}_O^{(e,a)}}_{M_z^{(e,a)}} = 0 \tag{7}$$

is verified, because the priciple of the gyroscopic effect requires the hypothesis

$$M_z^{(e,a)} = 0 \,.$$

Consequently equation (6) admits the infinite solutions:

$$\boldsymbol{\omega} = \frac{1}{Cr_0} \boldsymbol{k} \times \boldsymbol{M}_O^{(e,a)} + \lambda \boldsymbol{k} \tag{8}$$

where $\lambda \in (-\infty, +\infty)$.
The factor $\lambda$ can be calculated by means of the integral of motion (3):

$$\boldsymbol{\omega} \cdot \boldsymbol{k} = r_0$$

and consequently $\lambda = r_0$. So equation (8) becomes

$$\boldsymbol{\omega} = \frac{1}{Cr_0} \boldsymbol{k} \times \boldsymbol{M}_O^{(e,a)} + r_0 \boldsymbol{k} \,. \tag{9}$$

Equation (8) seems rather formal, however it lets us obtain immediately the solutions of the motion in many cases of application of the principle of the gyroscopic effect 1, in particular when the motion is (or approximates) a precession.

## 3 Gyroscope subject to a force applied to a point of its axis

A very common case in gyroscopic motions arises when the active external forces are reducible to their own resultant $\boldsymbol{R}$, applied to a point $C$ of the gyroscope axis:

$$\Sigma^{(e,\,a)} \mapsto \{(C, \boldsymbol{R})\} \,.$$

[2] It is a vectorial equation of the type $\boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b}$: if the compatibility condition $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ is verified, the most general solution of the vectorial equation is $\boldsymbol{x} = \dfrac{\boldsymbol{a} \wedge \boldsymbol{b}}{\boldsymbol{a}^2} + \lambda \boldsymbol{a}$ (see [7, §I.31 Example 17 p.25]).

Now let $\mathcal{G}$ be a gyroscope, freely rotating around a fixed point $O$, not coincident with the point $C$, and let's choose the gyroscope axis as the principal axis $z$, oriented as the vector $C - O = z_C \boldsymbol{k}$, where $z_C$ is the $z$–coordinate of $C$, and the moment of the active external forces about $O$ is

$$\boldsymbol{M}_O^{(e,\,a)} = (C - O) \times \boldsymbol{R} = z_C \boldsymbol{k} \times \boldsymbol{R} \,.$$

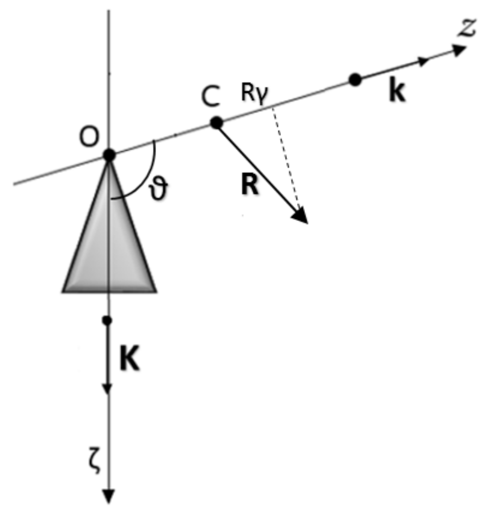


Figure 1: Gyroscope subject to a force applied to a point of its axis

We want to study the motion of the gyroscope $\mathcal{G}$, after an initial very rapid rotation, with spin $r_0$, around the gyroscope axis $z$. In this case the solution (8) of the vectorial equation (6) becomes

$$\begin{aligned} \boldsymbol{\omega} &= \frac{1}{Cr_0}\boldsymbol{k} \times \boldsymbol{M}_O^{(e,a)} + r_0\boldsymbol{k} = \frac{z_C}{Cr_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{R}) + r_0\boldsymbol{k} \\ &= -\frac{z_C}{Cr_0}\boldsymbol{k} \times (\boldsymbol{R} \times \boldsymbol{k}) + r_0\boldsymbol{k} \end{aligned} \tag{10}$$

For the continued vector product theorem we have

$$\boldsymbol{k} \times \boldsymbol{R} \times \boldsymbol{k} = \boldsymbol{R} - R\gamma \boldsymbol{k} \,,$$

where $\gamma$ is the direction cosinus of $\boldsymbol{R}$ with respect to the gyroscope axis, [3] so equation (10) becomes

$$\boldsymbol{\omega} = -\frac{z_C}{Cr_0}\boldsymbol{R} + \frac{z_C R}{Cr_0}\gamma \boldsymbol{k} + r_0 \boldsymbol{k}$$

[3] In this case the continued vector product is independent from the order of products because the first and the third vector are equal (and so parallel), and it provides the orthogonal component $\boldsymbol{R}^*$ of $\boldsymbol{R}$ with respect to $\boldsymbol{k}$, $\boldsymbol{R}^* = \boldsymbol{k} \times \boldsymbol{R} \times \boldsymbol{k} = \boldsymbol{R} - \boldsymbol{k}\boldsymbol{k} \cdot \boldsymbol{R} = \boldsymbol{R} - \underbrace{R_z}_{R\gamma} \boldsymbol{k}$.

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that we can also write as

$$\boldsymbol{\omega} = \left( r_0 + \frac{z_C R}{C r_0} \gamma \right) \boldsymbol{k} - \frac{z_C}{C r_0} \boldsymbol{R} \, . \tag{11}$$

Let's observe that, because of the hypothesis of rapid rotation, if the term $\dfrac{z_C R}{C r_0}\gamma$ is neglectable compared to $r_0$, the motion reduces, with good approximation, to the simple expression

$$\boldsymbol{\omega} = r_0 \boldsymbol{k} - \frac{z_C}{C r_0} \boldsymbol{R} \, . \tag{12}$$

Now consider the case of a force $\boldsymbol{R}$ with constant direction (and in particular the usual case of a constant force, just like the weight force): if we choose as fixed axis $\zeta$ the axis passing through $O$, parallel ed equally orientated with $\boldsymbol{R}$, so that $\boldsymbol{R} = R\boldsymbol{K}$, it results $\gamma = \cos(-\theta) = \cos\theta$, where $\theta$ is the nutation angle (i.e. the angle between the fixed versor $\boldsymbol{K}$ and the mobile versor $\boldsymbol{k}$), and equation (11) represents a precession [4]

$$\boldsymbol{\omega} = \underbrace{\left( \frac{z_C R}{C r_0} \cos\theta_0 + r_0 \right) \boldsymbol{k}}_{\boldsymbol{\omega}_f} \underbrace{- \frac{z_C R}{C r_0} \boldsymbol{K}}_{\boldsymbol{\omega}_p} \, . \tag{13}$$

If the term $\dfrac{z_C R}{C r_0} \cos\theta_0$ is neglectable compared to $r_0$ (because of the hypothesis of very rapid rotation), the equation (13) becomes, with good approximation,

$$\boldsymbol{\omega} = r_0 \boldsymbol{k} - \frac{z_C R}{C r_0} \boldsymbol{K} \, . \tag{14}$$

If in addition we suppose constant the norm of the force too, then $\boldsymbol{R} = R\boldsymbol{K} = \mathbf{const}$, so the precession rate is constant and the equations (13),(14) both represent a regular precession.
Finally we observe that, in the particular case $\theta_0 = \dfrac{\pi}{2}$ , the equation (13) turns, this time without any approximations, into (14), that in this case describes a degenerate precession.

As an example we consider an heavy gyroscope, subject only to the weight force $m\boldsymbol{g}$, whose axis, oriented as the vector $G - O$, is free to rotate around its fixed point $O$, distinct from the centre of mass $G$; we choose as usual the vertical axis, passing through $O$ and oriented downwards, as the fixed axis $\zeta$, so that $m\boldsymbol{g} = mg\boldsymbol{K}$.

[4] Since in equation (13) the angular velocity $\boldsymbol{\omega}$ is decomposed into the sum of two vectors, the first $\boldsymbol{\omega}_f$ with the direction of the $z$–axis, fixed in $\mathcal{G}$, the second $\boldsymbol{\omega}_p$ with the direction of the fixed $\zeta$–axis, the motion is a precession and so the nutation angle must be constant, $\theta = \text{const} = \theta_0$ (cfr. for example [3, § III.14 p.140]).

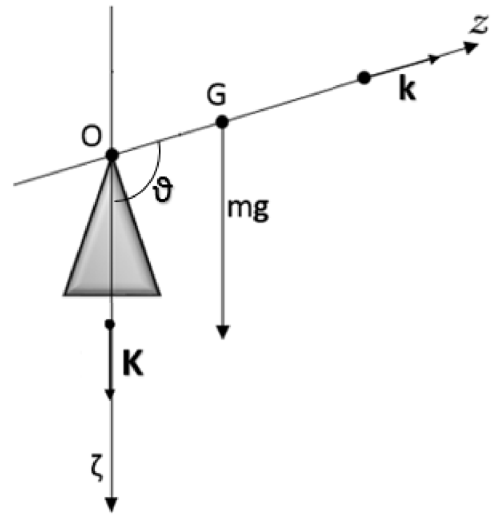


Figure 2: Heavy gyroscope

Once given the gyroscope $\mathcal{G}$ an initial rapid rotation of spin $r_0$, around the gyroscope axis $z$, the equation (13) becomes

$$\boldsymbol{\omega} = \left( \frac{mgz_G}{C r_0} \cos\theta_0 + r_0 \right) \boldsymbol{k} - \frac{mgz_G}{C r_0} \boldsymbol{K} \, , \tag{15}$$

where $\theta$ is as usual the nutation angle (i.e. the angle that the solidal versor $\boldsymbol{k}$ describes with the fixed versor $\boldsymbol{K}$).
If the term $\dfrac{mz_G}{C r_0} \cos\theta$ is neglectable compared to $r_0$ (because of the hypothesis of very rapid rotation), equation (15) will represent, with good approximation, a regular precession with rotation rate

$$\boldsymbol{\omega_f} = r_0 \boldsymbol{k} \tag{16}$$

and precession rate

$$\boldsymbol{\omega_p} = - \frac{mgz_G}{C r_0} \boldsymbol{K} \, . \tag{17}$$

This case occurs, for example, in the case of Fessel's gyroscope (see [3, § VII.7 p.319]).
Finally if $\theta = \dfrac{\pi}{2}$, the equation (15) turns into a degenerate regular precession

$$\boldsymbol{\omega} = r_0 \boldsymbol{k} - \frac{mgz_G}{C r_0} \boldsymbol{K} \, . \tag{18}$$

This case occurs when a bicycle wheel, is set on a rapid rotation in a vertical plane around the gyroscope axis, fixed in a frictionless point $O$.

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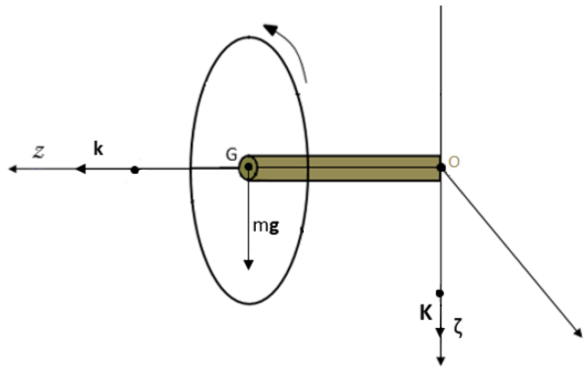


Figure 3: Gyroscopic effect for a spinning wheel.

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