Approximation Theorems for Solving the Common Solution for System of Generalized Equilibrium Problems and Fixed Point Problems and Variational Inequality Problems

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Abstract: In this paper, we introduce a new iterative sequence which is constructed by using the hybrid projection method for solving the common solution for a system of generalized equilibrium problems of inverse strongly monotone mappings and a system of bifunctions satisfying certain the conditions, the common solution for the families of quasi -ϕ- asymptotically nonexpansive and uniformly Lipschitz continuous and the common solution for a variational inequality problem. Strong convergence theorems are proved on approximating a common solution of a system of generalized equilibrium problems, fixed point problems for two countable families and a variational inequality problem in a uniformly smooth and 2-uniformly convex real Banach space.

Key–Words: approximation theorem; inverse-strongly monotone mapping; variational inequality problem; generalized equilibrium problem; fixed point problem.

1 Introduction

The theory of equilibrium problems, the development of an efficient and implementable iterative algorithm is interesting and important. This theory combines theoretical and algorithmic advances with novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis.

The equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization, and it has been extended and generalized in many directions. In particular, equilibrium problems are related to the problem of finding fixed points of nonexpansive mappings.

In 2008-2009, Takahashi and Zembayashi [10, 11] introduced iterative sequences for finding a common solution of an equilibrium problem and a fixed point problem for a relatively nonexpansive mapping, and established some strong and weak convergences theorems.

In 2010, Chang et al. [12] discussed the common solution of a generalized equilibrium problem and a common fixed point problem for two relatively nonexpansive mappings, and established a strong convergence theorem on the common solution problem. The frameworks of [10, 11, 12] are the uniformly smooth and uniformly convex Banach spaces. Chang et al.[9] established a strong convergence theorem for solving the common fixed point problem for a family of uniformly quasi -ϕ- asymptotically nonexpansive and uniformly Lipschitz continuous mapping in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

In 2011, Qu and Cheng [1] established a strong convergence theorem on solving common solutions for generalized equilibrium problems and fixed point problems in Banach spaces. Saewan and Kumam [2] established a new modified block iterative algorithm for finding common element of the set of common fixed point of an infinite family of closed and uniformly quasi -ϕ- asymptotically nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone mapping and the set of solution of a system of generalized mixed equilibrium problems. Zegeye and Shahzad [18] introduced an it-

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erative process which converges strongly to a common solution of finite family of variational inequality problems for $\gamma-$inverse strongly monotone mappings and fixed point of two continuous quasi-ϕ-asymptotically nonexpansive mapping in Banach spaces. Tan and Chang [19] introduced a new hybrid iterative scheme for finding a common element of the set of solutions for a system of generalized mixed equilibrium problems, set of common fixed points of a family of quasi-ϕ-asymptotically nonexpansive mappings, and null spaces of finite family of $\gamma-$inverse strongly monotone mappings in a 2-uniformly convex and uniformly smooth real Banach space. Kim [20] introduced a hybrid projection method for finding a common element in the fixed point set of an asymptotically quasi-ϕ- asymptotically nonexpansive mapping and in the solution set of an equilibrium problem. Strong convergence theorems of common elements are established in a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property. Liu [26] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problems, the set of fixed points of infinite family of quasi -ϕ- asymptotically nonexpansive mappings in Banach space by using CQ method. Zhang, Chan and Lee [27] introduced modified block iterative algorithm for finding a common element in the intersection of the set of common fixed points of an infinite family of quasi -ϕ- asymptotically nonexpansive and the set of solutions to an equilibrium problem and the set of variational inequality, he proved strong convergence theorems in 2-uniformly convex and uniformly smooth Banach space. As application, he studied the convex feasibility problem (CEP) and zero point problem of maximal monotone mappings. In this paper, Motivated and inspired by the previously mentioned above results, we introduce a new iterative sequence by the new hybrid projection method for solving the common solution problem for a system of generalized equilibrium problems of inverse strongly monotone mappings and a system of bifunctions satisfying certain the conditions, and the common solution problem for a family of uniformly quasi -ϕ- asymptotically nonexpansive and uniformly Lipschitz continuous and the common solution problem for a variational inequality problem in a uniformly smooth and 2-uniformly convex real Banach space. Then, we prove a strong convergence theorem of the iterative sequence generated by the conditions. The results obtained in this paper extend and improve several recent results in this area.

2 Definitions and Notation

Throughout this paper, we assume that $\mathbb{R}$ and $\mathbb{J}$ are denoted by the set of real numbers and the set of $\{1, 2, 3, ..., M\}$, respectively, where M is any given positive integer. Let E be a Banach space with norm $\|\cdot\|$, C be a nonempty closed and convex subset of E and let E^* denote by the dual of E. let $\{F_k\}_{k\in\mathbb{J}} : C \times C \to \mathbb{R}$ be a bifunction, and $\{B_k\}_{k\in\mathbb{J}} : C \to E^*$ be a monotone mapping. The *system of generalized equilibrium problems*, is to find $x \in C$ such that

$$F_k(x, y) + \langle y - x, B_k x\rangle \geq 0, \quad k \in \mathbb{J}, \quad \forall y \in C. \tag{2.1}$$

The set of solutions of (2.1) is denoted by $SGEP(F_k, B_k)$, that is

$$SGEP(F_k, B_k) = \{x \in C : F_k(x, y)+\langle y-x, B_k x\rangle \geq 0\}$$

$\forall y \in C, \forall k \in \mathbb{J}$.

If $\mathbb{J}$ is a singleton, then problem (2.1) reduces to the *generalized equilibrium problems*, is to find $x \in C$ such that

$$F(x, y) + \langle y - x, Bx\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

The set of solutions of (2.2) is denoted by $GEP(F, B)$, that is

$$GEP(F, B) = \{x \in C : F(x, y)+\langle y-x, Bx\rangle \geq 0, \}.$$

$\forall y \in C$.

If $B \equiv 0$ the problem (2.2) reduces into the *equilibrium problem* for F, denoted by $EP(F)$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.3}$$

If $F \equiv 0$ the problem (2.2) reduces into *variational inequality of Browder type*, denoted by $VI(C, B)$, is to find $x \in C$ such that

$$\langle y - x, Bx\rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Recall that, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Let E be a real Banach space and $\{x_n\}$ be a sequence in E. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of $\{x_n\}$, respectively. The *normalized duality mapping* J from E to 2^{E^*} is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E. \tag{2.5}$$

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By the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$.

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| \leq 1, \|y_n\| \leq 1$ and

$$\lim_{n\to\infty} \left\|\frac{x_n + y_n}{2}\right\| = 1. \tag{2.6}$$

Let U_E = $\{x \in E : \|x\| = 1\}$ be the *unit sphere* of E. Then, the Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.7}$$

exists for each $x, y \in U_E$. It is said to be *uniformly smooth* if the limit (2.7) is attained uniformly for all $x, y \in U_E$.

Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the *modulus of smoothness* of E if

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1\ ,\ \|y\| = t\right\}.$$

The space E is said to be *smooth* if $\rho_E(t) > 0$, $\forall t > 0$ and is said to be *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| \leq 1\ ,\ \|y\| \leq 1\ ;\ \|x - y\| \geq \epsilon\right\}.$$

A Banach space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Let a real number $p > 1$. Then, E is said to be $p-$ *uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, for all $\epsilon \in [0, 2]$. Observe that every p-uniformly convex space is uniformly convex. It is well-known for example (see Xu [17]) that

$$L_p(l_p) \text{ or } W_m^p\ is \begin{cases} \text{p-uniformly convex}, & \text{if } p \geq 2; \\ \text{2-uniformly convex}, & \text{if } 1 < p \leq 2. \end{cases}$$

One should note that no a Banach space is p-uniformly convex for $1 < p < 2$. It is known that a Hilbert space is uniformly smooth and 2-uniformly convex.

In the sequel, we shall make use of the following results.

Remark 2.1. The basic properties below hold (see Cioranescu [3] and Takahashi [13]).

1. If E is uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.
2. If E is a strictly convex reflexive Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-to-$weak^*$-continuous.
3. If E is a smooth and strictly convex reflexive Banach space, then J is single-valued, one-to-one and onto.
4. A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
5. A Banach space E is strictly convex if and only if E^* is smooth.
6. A Banach space E is smooth if and only if E^* is strictly convex.
7. If E^* is a smooth Banach space, then E is a strictly convex Banach space.
8. A Banach space E^* is a strictly convex Banach space, then E is a smooth Banach space.
9. Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $\{x_n\} \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.
10. A Banach space E is strictly convex if and only if J is strictly monotone, that is, $\langle x - y, x^* - y^*\rangle > 0$, whenever $x, y \in E,\ x \neq y\ and\ x^* \in Jx, y^* \in Jy$.
11. Both uniformly smooth Banach spaces and uniformly convex Banach spaces are reflexive.
12. If E^* is uniformly convex and J is the duality mapping of E, then J is uniformly norm-to-norm continuous on bounded sets of E., i.e., for a bounded set B of E and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta \Rightarrow \|Jx - Jy\| < \varepsilon,\ for\ x, y \in B$.

Now let E be a smooth and strictly convex reflexive Banach space. As Alber (see [4]) and Kamimura and Takahashi (see [5]) did, the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \ \forall x, y \in E.$$

It follows from Kohsaka and Takahashi (see [6]) that $\phi(x, y) = 0$ if and only if $x = y$, and that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{2.8}$$

Further suppose that C is nonempty closed convex subset of E. The *generalized projection* (Alber see [4]) $\Pi_C : E \to C$ is defined by for each $x \in E$,

$$\Pi_C(x) = \arg\min_{y\in C} \phi(x,y).$$

A mapping $A : C \to E^*$ is said to be *δ-inverse-strongly monotone*, if there exists a constant $\delta > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \delta\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

A mapping $S : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C,\ x_n \to x$ and $Sx_n \to y$ imply $Sx = y$.

Example 2.2. Let T is a nonexpansive of C into itself and I is the identity mapping of a real Banach space E. Then, a mapping $A = I - T$ is $\frac{1}{2}$-inverse-strongly monotone mapping.

Example 2.3. Let a mapping $A_k : \mathbb{R} \to \mathbb{R}$ by $A_k x = kx, \forall x \in \mathbb{R}$ and $k \in \{1,2,3,...,n\}$. Then, a mapping $A_k : \mathbb{R} \to \mathbb{R}$ is a finite family of $\frac{1}{k}$-inverse-strongly monotone.

A mapping $S : C \to C$ is said to be *quasi -ϕ- nonexpansive* (*relatively quasi-nonexpansive*) if $Fix(S) \neq \emptyset$, and

$$\phi(u,Sx) \leq \phi(u,x),\ \ \forall x \in C,\ u \in Fix(S).$$

A mapping $S : C \to C$ is said to be *quasi -ϕ- asymptotically nonexpansive* (*asymptotically relatively nonexpansive*) (see Zhou and Gao [7]) if $Fix(S) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\phi(u,S^n x) \leq k_n\phi(u,x),\ \forall x \in C,\ u \in Fix(S),\ \forall n \geq 1.$$

It is easy to see that if $A : C \to E^*$ is *δ-inverse-strongly monotone*, then A is *$\frac{1}{\delta}$-Lipschitz continuous*.

Remark 2.4. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}(0)$ is nonempty. Then, we get $J_r = (J+rA)^{-1}J$ is closed and quasi -ϕ- asymptotically nonexpansive mapping from E onto $D(A)$ and $Fix(J_r) = A^{-1}(0)$.

Remark 2.5. Let Π_c be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then, we get Π_c is closed and quasi -ϕ- asymptotically nonexpansive mapping from E onto C with $Fix(\Pi_c) = C$.

Example 2.6. Let $C := [\frac{-1}{\pi}, \frac{1}{\pi}]$ and define $T : C \to C$ by

$$Tx = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}), & \text{if } x \neq 0; \\ x, & \text{if } x = 0. \end{cases}$$

Then, T is quasi -ϕ- asymptotically nonexpansive mapping.

The class of quasi -ϕ- asymptotically nonexpansive mappings contains properly the class of relatively nonexpansive mappings (see Matsushita and Takahashi [21]) as a subclass.

Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a mapping and $Fix(T)$ be the set of fixed points of T.

A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widehat{Fix}(T)$.

A point $p \in C$ is said to be a *strong asymptotic fixed point* of T, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all strong asymptotic fixed points of T by $\widetilde{Fix}(T)$.

A mapping $T : C \to C$ is said to be *relatively nonexpansive* [21, 22], if $Fix(T) \neq \emptyset$, $Fix(T) = \widehat{Fix}(T)$ and

$$\phi(p,Tx) \leq \phi(p,x),\ \ \forall x \in C,\ p \in Fix(T).$$

A mapping $T : C \to C$ is said to be *weak relatively nonexpansive* [23], if $Fix(T) \neq \emptyset$, $Fix(T) = \widetilde{Fix}(T)$ and

$$\phi(p,Tx) \leq \phi(p,x),\ \ \forall x \in C,\ p \in Fix(T).$$

Remark 2.7. If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Remark 2.8. From the definition of a mapping. It is easy to see that

1. Each relatively nonexpansive mapping is closed.
2. Every quasi -ϕ- nonexpansive mapping is quasi -ϕ- asymptotically nonexpansive mapping with $\{k_n = 1\}$, but the converse is not true.
3. Each weak relatively nonexpansive mapping is a quasi -ϕ- nonexpansive mapping (because it does not require the condition $Fix(T) = \widetilde{Fix}(T)$, but the converse is not true.
4. Every relatively nonexpansive mapping is a weak relatively nonexpansive mappings, but the converse is not true.

5. Every countable family of weak relatively nonexpansive mappings is a countable family of of uniformly closed and quasi -ϕ- nonexpansive mappings, and so it is a countable family of uniformly closed and quasi -ϕ- asymptotically nonexpansive mappings.

Definition 2.9. (see Chang et al. [9]) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. $\{S_i\}_{i=1}^{\infty}$ is said to be a family of uniformly quasi -ϕ- asymptotically nonexpansive mappings, if $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$,

$$\phi(u, S_i^n x) \leq k_n \phi(u, x),\ \forall u \in \bigcap_{i=1}^{\infty} Fix(S_i),\ x \in C,\ \forall n \geq 1.$$

Definition 2.10. A mapping $S : C \to C$ is said to be uniformly *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|,\ \ \forall x, y \in C,\ \ \forall n \geq 1.$$

Lemma 2.11. (see Alber [4]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$. Then,*

$$\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \leq \phi(x, y), \forall x \in C,\ y \in E.$$

Lemma 2.12. (see Kamimura and Takahashi [5]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$ and $u \in C$. Then,*

$$u = \Pi_C(x) \Leftrightarrow \langle u - y, Jx - Ju\rangle \geq 0,\ \ \forall y \in C.$$

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,\ \ \forall x \in E, \forall x^* \in E^*.$$

Observe that $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. The following lemma is well-known.

Lemma 2.13. (see Alber [4]) *Let E be a smooth and strictly convex reflexive Banach space with E^*as its dual, then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.14. (see Kamimura and Takahashi [5]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

For solving the generalized equilibrium problem, let us assume that the mapping $B : C \to E^*$ is δ-inverse-strongly monotone mapping and the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0,\ \ \forall x, y \in C$;

(A3) $\limsup_{t \downarrow 0} F(x + t(z - x), y) \leq F(x, y),\ \forall x, y, z \in C$;

(A4) for any $y \in C$, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.15. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, $\{x_n\}$ and $\{y_n\}$ be two sequences of E, and $p \in E$. If $x_n \to p$ and $\phi(x_n, y_n) \to 0$, then $y_n \to p$.*

Lemma 2.16. (see Blum and Oettli [15]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1) - (A4). *Let $r > 0$ be any given number and $x \in E$ be any point. Then, there exists a $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.9}$$

Lemma 2.17. (see Chang et al. [12]) *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $B : C \to E^*$ be a δ-inverse-strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1)-(A4). *Let $r > 0$ be any given number and $x \in E$ be any point. Then, there exists a point $z \in C$ such that*

$$F(z, y) + \langle y - z, Bz\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

Lemma 2.18. (see Chang et al. [12]) *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $B : C \to E^*$ be a δ-inverse-strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1)-(A4). *Let $r > 0$ and $x \in E$. and we define a mapping $T_r^F : E \to C$ as follows: for any $x \in C$,*

$$T_r^F x = \{z \in C : F(z, y) + \langle y - z, Bz\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \} \quad \forall y \in C.$$

Then, the following conclusions hold:

(1) T_r^F *is single-valued*;

(2) T_r^F *is a firmly nonexpansive type mapping, i.e.,* $\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle, \forall x, y \in E$

(3) $Fix(T_r^F) = \widetilde{Fix(T_r^F)} = EP$;

(4) EP *is a closed and convex set of* C;

(5) $\phi(p, T_r^F x) + \phi(T_r^F x, x) \leq \phi(p, x), \forall p \in Fix(T_r^F)$;

(6) for each $n \geq 1$, $r_n > d > 0$ *and* $u_n \in C$ *with* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} T_{r_n} u_n = \overline{u}$, *we have*

$$F(\overline{u}, y) + \langle y - \overline{u}, B\overline{u}\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.19. (see Cioranescu [3]) *Let* C *be a nonempty closed and convex subset of a real uniformly smooth and strictly convex Banach space* E *with the Kadec-Klee property,* $S : C \to C$ *be a closed and quasi -*ϕ*- asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [1, \infty)$ *and* $k_n \to 1$. *Then,* $Fix(S)$ *is closed and convex in* C.

Lemma 2.20. (see Chang et al. [9]) *Let* E *be a uniformly convex Banach space,* $r > 0$ *be a positive number and* $B_r(0)$ *be a closed ball of* E. *Then, for any given sequence* $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ *and for any given* $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = 1$, *there exists a continuous, strictly increasing and convex function g :* $[0, 2r) \to [0, \infty)$ *with* $g(0) = 0$ *such that for any positive integers i , j with* $i < j$,

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

Lemma 2.21. (see Xu [17]) *Let* E *be a 2-uniformly convex real Banach space, then for all* $x, y \in E$, *we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J *is the normalized duality mapping of* E *and* $0 < c \leq 1$, *and* $\frac{1}{c}$ *is called the 2-uniformly convex constant of* E.

We note that every uniformly convex Banach space has the Kadec-Klee property. For more details on Kadec-Klee property, the reader is referred to [3, 13].

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* it for all $x, y \in C$, the mapping $F : [0, 1] \to E^*$, defined by $F(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. We define $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \quad \forall y \in C\}.$$

Lemma 2.22. (see Rockafellar [24]) *Let* C *be a nonempty, closed and convex subset of a Banach space* E *and* A *is monotone, hemicontinuous operator of* C *into* E^*. *Let* $U \subset E \times E^*$ *be an operator defined as follows:*

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then, U *is maximal monotone and* $U^{-1}(0) = VI(C, A)$.

3 Main results

In this section, we show a strong convergence theorem which solves the problem of finding a common solution of the system of generalized equilibrium problems and fixed point problems in Banach spaces.

Now, we remark that, as it is the mentioned in Zegeye and Shahzad [25] that let C be a subset of a real Banach space E and $A : C \to E^*$ be an inverse strongly monotone mapping satisfying $\|Ax\| \leq \|Ax - Ap\|$, for all $x \in C$ and $p \in VI(C, A)$, then $VI(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}$. For example the following the condition is satisfied.

Example 3.1. Let $A : \mathbb{R} \to \mathbb{R}$ by given by

$$Ax := \begin{cases} 0, & \text{if } x \leq 0; \\ 4x, & \text{if } x > 0. \end{cases}$$

Then, A is $\frac{1}{4}$-inverse-strongly monotone mapping with $VI(\mathbb{R}, A) = A^{-1}(0) = (-\infty, 0]$.

Before stat our theorem we give an example for nonlinear mappings to illustrate the theoretical results.

Example 3.2. Let $S : \mathbb{R} \to \mathbb{R}$ be given by $S := (I + rB)^{-1}$, for $r > 0$, where

$$Bx := \begin{cases} x + 1, & \text{if } x \in (-\infty, -1]; \\ 0, & \text{if } x \in (-1, 0], \\ 2x, & \text{if } x \in (0, \infty). \end{cases}$$

Then, we get that $J_r := (I + rB)^{-1} = S$ is uniformly L- Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive with $\{k_n\} = 1$ for each $n \geq 1$ and $Fix(J_r) = B^{-1}(0) = Fix(S) = [-1, 0]$.

We shall make use of this remark to prove the next theorem.

Theorem 3.3. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space* E. *Suppose that*

(B1) Let $B_k : C \to E^*$ for each $k = 1,2,3,...,M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(B2) Let $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty} : C \to C$ be countable families of uniformly closed and ω_i,μ_j-Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{k_n\},\{l_n\} \subset [1,\infty)$ and $k_n \to 1, l_n \to 1$, respectively.

(B3) Let $A_n : C \to E^*$ for each $n = 1,2,3,...,N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1,2,3,...,N\}$.

(B4) $\Omega := \Big(\bigcap_{i=1}^{\infty} Fix(T_i)\Big) \bigcap \Big(\bigcap_{j=1}^{\infty} Fix(S_j)\Big) \bigcap$

$$\Big(\bigcap_{k=1}^{M} SGEP(F_k,B_k)\Big) \bigcap \Big(\bigcap_{n=1}^{N} VI(C,A_n)\Big)$$

is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \ \textit{chosen arbitrary},\ C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} JT_i^n x_n \\ \quad + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} JS_j^n z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v,u_n) \le \phi(v,x_n) + \theta_n\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ \forall n \ge 0, \end{cases} \tag{3.1}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1,2,3,...,M$, is a mapping defined by (2.18) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z,y) + \langle y-z, B_k z\rangle + \frac{1}{r}\langle y-z, Jz-Jx\rangle \ge 0, \quad \forall y \in C,\ k = 1,2,3,...,M$, where $r_{k,n} \in [d,\infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n,l_n\}-1)\phi(p,x_n)$, $A_n = A_n(mod\ N)$, $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda_n\}$ be a sequence in* [0,1] *such that $0 < \lambda_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1. *for each $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^{\infty}\beta_{n,i}^{(2)} + \sum_{j=1}^{\infty}\beta_{n,j}^{(3)} = 1$;*
2. $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0, \forall i,j \ge 1, i \ne j$.

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

Proof. We shall complete this proof by six steps below.

Step 1. We will show that Ω and C_{n+1} are closed and convex, for each $n \ge 0$.
In fact, It follows from Lemma 2.18(4) and Lemma 2.19 that $Fix(T_i)$ and $Fix(S_j)$, for any $i,j \ge 1$ and $SGEP(F_k,B_k)$ are closed and convex subset of C. Therefore, Ω is closed and convex in C.
Clearly, $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 1$. By the assumption, $\phi(v,u_n) \le \phi(v,x_n) + \theta_n$ is equivalent to

$$\|v\|^2 - 2\langle v, Ju_n\rangle + \|u_n\|^2 \le \|v\|^2 - 2\langle v, Jx_n\rangle + \|x_n\|^2 + \theta_n.$$

So that $2\langle v, Jx_n\rangle - 2\langle v, Ju_n\rangle = 2\langle v, Jx_n - Ju_n\rangle \le \|x_n\|^2 - \|u_n\|^2 + \theta_n$. Hence, C_{n+1} is closed and convex. Therefore, $\Pi_{C_{n+1}}(x_0)$ and $\Pi_\Omega(x_0)$ are well-defined.

Step 2. We will show that $\{x_n\}$ is bounded and $\{\phi(x_n,x_0)\}$ is convergent sequence, for all $n \ge 1$.
Indeed, it follows from (3.1) and Lemma 2.11 that

$$\begin{aligned} \phi(x_n,x_0) &= \phi(\Pi_{C_n}(x_0),x_0) \\ &\le \phi(p,x_0) - \phi(p,\Pi_{C_n}(x_0)) \\ &\le \phi(p,x_0), \ \forall n \ge 0, \ p \in \Omega. \end{aligned}$$

This implies that $\{\phi(x_n,x_0)\}$ is bounded. By the virtue of (2.8). Then, the sequence $\{x_n\}$ is also bounded.
By the assumption of C_n, we have $C_{n+1} \subset C_n$, $x_n = \Pi_{C_n}(x_0)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_0)$. This implies that $x_{n+1} \in C_{n+1} \subset C_n$, and

$$\phi(x_n,x_0) \le \phi(x_{n+1},x_0), \ \forall n \ge 0.$$

Therefore, $\{\phi(x_n,x_0)\}$ is convergent sequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \phi(x_n,x_0) = d \ge 0.$$

Step 3. We will show that $\Omega \subset C_n$, for all $n \ge 0$.
By taking $K_n^j = T_{r_{j,n}}^{F_j} T_{r_{j-1,n}}^{F_{j-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$, where $j = 1,2,3,...,M$ and $K_n^0 = I$ for all $n \ge 1$. We note that $u_n = K_n^M y_n$. For $n \ge 0$, we have $\Omega \subset C = C_0$. For any given $p \in \Omega$, then by the equation (3.1), Lemma 2.12 and Lemma 2.13, we compute

$$\begin{aligned} & \phi(p,z_n) \\ =\ & \phi(p,\Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n)) \\ \le\ & \phi(p,J^{-1}(Jx_n - \lambda_n A_n x_n)) \\ =\ & V(p,Jx_n - \lambda_n A_n x_n) \\ \le\ & V(p,(Jx_n - \lambda_n A_n x_n) + \lambda_n A_n x_n) \\ & -2\langle J^{-1}(Jx_n - \lambda_n A_n x_n) - p, \lambda_n A_n x_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= V(p, Jx_n) - 2\lambda_n\langle J^{-1}(Jx_n - \lambda_n A_n x_n) - p, A_n x_n\rangle \\
&= \phi(p, x_n) - 2\lambda_n\langle x_n, -p, A_n x_n\rangle \\
&\quad -2\lambda_n\langle J^{-1}(Jx_n - \lambda_n A_n x_n) - x_n, A_n x_n\rangle \\
&\leq \phi(p, x_n) - 2\lambda_n\langle x_n - p, A_n x_n - A_n p\rangle \\
&\quad -2\lambda_n\langle x_n - p, A_n p\rangle \\
&\quad -2\lambda_n\langle J^{-1}(Jx_n - \lambda_n A_n x_n) - x_n, A_n x_n\rangle. \qquad (3.2)
\end{aligned}
$$

Because $p \in \Omega$, hence $p \in VI(C, A_n)$ and A_n is γ-inverse strongly monotone mappings, from (3.2), we get

$$
\phi(p, z_n) \leq \phi(p, x_n) - 2\lambda_n\gamma\|A_n x_n - A_n p\|^2 + 2\lambda_n\|J^{-1}(Jx_n - \lambda_n A_n x_n) - J^{-1}(Jx_n)\|\|A_n x_n\|. \qquad (3.3)
$$

From (3.3), Lemma 2.21 and the fact that $\|A_n x\| \leq \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$ and $\lambda_n < \frac{c^2\gamma}{2}$, we obtain

$$
\begin{aligned}
\phi(p, z_n) &\leq \phi(p, x_n) - 2\lambda_n\gamma\|A_n x_n - A_n p\|^2 \\
&\quad + \frac{4\lambda_n^2}{c^2}\|A_n x_n - A_n p\|^2 \\
&= \phi(p, x_n) + 2\lambda_n(\frac{2\lambda_n}{c^2} - \gamma)\|A_n x_n - A_n p\|^2 \\
&\leq \phi(p, x_n) - 2\lambda_n(\frac{2\lambda_n}{c^2} - \gamma)\|A_n x_n\|^2 \\
&\leq \phi(p, x_n). \qquad (3.4)
\end{aligned}
$$

Therefore, we have $\phi(p, z_n) \leq \phi(p, x_n)$. From (3.1) and (3.4), we compute

$$
\begin{aligned}
&\phi(p, u_n) \\
&= \phi(p, T^{F_M}_{r_{M,n}} T^{F_{M-1}}_{r_{M-1,n}} \ldots T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} y_n) \\
&= \phi(p, K_n^M y_n) \\
&\leq \phi(p, y_n) \\
&= \phi(p, J^{-1}(\beta^{(1)}_{n,0} Jx_n + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i} JT_i^n x_n \\
&\quad + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j} JS_j^n z_n)) \\
&= \|p\|^2 - 2\langle p, \beta^{(1)}_{n,0} Jx_n + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i} JT_i^n x_n \\
&\quad + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j} JS_j^n z_n\rangle \\
&\quad + \|\beta^{(1)}_{n,0} Jx_n + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i} JT_i^n x_n + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j} JS_j^n z_n\|^2 \\
&\leq \|p\|^2 - 2\beta^{(1)}_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\langle p, JT_i^n x_n\rangle \\
&\quad -2\sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\langle p, JS_j^n z_n\rangle \\
&\quad + \beta^{(1)}_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\|T_i^n x_n\|^2 + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\|S_j^n z_n\|^2 \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&= \beta^{(1)}_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\phi(p, T_i^n x_n) \\
&\quad + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\phi(p, S_j^n z_n) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&\leq \beta^{(1)}_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\phi(p, T_i^n x_n) \\
&\quad + \sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\phi(p, S_j^n z_n) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&\leq \beta^{(1)}_{n,0}\phi(p, x_n) + k_n\sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\phi(p, x_n) \\
&\quad + l_n\sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\phi(p, z_n) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&\leq \beta^{(1)}_{n,0}\phi(p, x_n) + k_n\sum_{i=1}^{\infty}\beta^{(2)}_{n,i}\phi(p, x_n) \\
&\quad + l_n\sum_{j=1}^{\infty}\beta^{(3)}_{n,j}\phi(p, x_n) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&\leq \max\{k_n, l_n\}\phi(p, x_n) - \beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|) \\
&\leq \phi(p, x_n) + \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\phi(p, x_n) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(2)}_{n,i} g(\|Jx_n - JT_i^n x_n\|) \\
&\quad -\beta^{(1)}_{n,0}\beta^{(3)}_{n,j} g(\|Jx_n - JS_j^n z_n\|)
\end{aligned}
$$

$$\begin{aligned} &= \phi(p,x_n)+\theta_n-\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_i^nx_n\|)\\ &\quad -\beta_{n,0}^{(1)}\beta_{n,j}^{(3)}g(\|Jx_n-JS_j^nz_n\|)\\ &\leq \phi(p,x_n)+\theta_n. \end{aligned} \tag{3.5}$$

It follows that

$$\phi(p,u_n)\leq\phi(p,x_n)+\theta_n, \tag{3.6}$$

where $\theta_n=\sup_{p\in\Omega}(\max\{k_n,l_n\}-1)\phi(p,x_n)$. By the assumptions of $\{k_n\}$ and $\{l_n\}$, and from (2.8), we obtain

$$\begin{aligned} \theta_n &= \sup_{p\in\Omega}(\max\{k_n,l_n\}-1)\phi(p,x_n)\\ &\leq \sup_{p\in\Omega}(\max\{k_n,l_n\}-1)(\|p\|+\|x_n\|)^2\\ &\leq \sup_{p\in\Omega}(\max\{k_n,l_n\}-1)(\|p\|+M)^2\to 0\\ &\quad (\text{as}\quad n\to\infty), \end{aligned} \tag{3.7}$$

where $M=\sup_{n\geq 0}\|x_n\|$. So, we get $p\in C_{n+1}$. This implies that $\Omega\subset C_n$ for all $n\geq 0$ and the sequence $\{x_n\}$ is well-defined.

Step 4. We will show that there exists some point $p^*\in C$ such that $x_n\to p^*$.

Since $x_n=\Pi_{C_n}(x_0)$ and $x_{n+1}=\Pi_{C_{n+1}}(x_0)\in C_{n+1}\subset C_n$, we have

$$\phi(x_n,x_0)\leq\phi(x_{n+1},x_0), \tag{3.8}$$

which implies that the sequence $\{\phi(x_n,x_0)\}$ is non-decreasing and bounded, and so $\lim_{n\to\infty}\{\phi(x_n,x_0)\}$ exists. Hence, for any positive integer m, by Lemma 2.11, we have

$$\phi(x_{n+m},x_n)=\phi(x_{n+m}\Pi_{C_n}(x_0))\leq\phi(x_{n+m},x_0)-\phi(x_n,x_0), \tag{3.9}$$

for all $n\geq 0$. Since $\lim_{n\to\infty}\{\phi(x_n,x_0)\}$ exists, we obtain

$$\phi(x_{n+m},x_n)\to 0,\quad \text{as}\ n\to\infty,\ \forall m\in\mathbb{N}. \tag{3.10}$$

By Lemma 2.14, we get $\|x_{n+m}-x_n\|\to 0$, as $n\to\infty$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed subset of Banach space E, it is complete. Hence, there exists a point $p^*\in C$ such that

$$x_n\to p^*,\quad \text{as}\ n\to\infty. \tag{3.11}$$

Step 5. We will show that $p^*\in\Omega$,

where $\Omega:=\Big(\bigcap_{i=1}^\infty Fix(T_i)\Big)\bigcap\Big(\bigcap_{j=1}^\infty Fix(S_j)\Big)\bigcap\Big(\bigcap_{k=1}^M SGEP(F_k,B_k)\Big)\bigcap\Big(\bigcap_{n=1}^N VI(C,A_n)\Big)$.

(5.1) First, we will show that $p^*\in\bigcap_{i=1}^\infty Fix(T_i)$. Since $x_{n+1}\in C_{n+1}\subset C_n$, by the definition of C_{n+1}, we have

$$\phi(x_{n+1},u_n)\leq\phi(x_{n+1},x_n)+\theta_n. \tag{3.12}$$

Again by (3.10) and Lemma 2.14, we get $\lim_{n\to\infty}\|x_{n+1}-u_n\|=0$. Since

$$\|x_n-u_n\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-u_n\|. \tag{3.13}$$

Hence,

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.14}$$

This implies that $u_n\to p^*$ as $n\to\infty$. Since E is uniformly smooth, This implies that J is uniformly continuous on bounded subset of E by remark 2.1(1), we obtain

$$\lim_{n\to\infty}\|Jx_n-Ju_n\|=0. \tag{3.15}$$

From (3.5), we have $\phi(p,u_n)\leq\phi(p,x_n)+\theta_n-\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_i^nx_n\|)-\beta_{n,0}^{(1)}\beta_{n,j}^{(3)}g(\|Jx_n-JS_j^nz_n\|)$. Hence, $\phi(p,u_n)\leq\phi(p,x_n)+\theta_n-\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_i^nx_n\|)$ and so

$$\begin{aligned} &\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_i^nx_n\|)\\ &\leq \phi(p,x_n)-\phi(p,u_n)+\theta_n\\ &= \|x_n\|^2-\|u_n\|^2+2\langle p,Ju_n-Jx_n\rangle+\theta_n\\ &\leq (\|x_n\|-\|u_n\|)(\|x_n\|+\|u_n\|)+2\langle p,Ju_n-Jx_n\rangle\\ &\quad +\theta_n. \end{aligned}$$

From (3.7), (3.14) and (3.15), we get

$$\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_i^nx_n\|)\to 0,\quad \text{as}\ n\to\infty.$$

In the view of condition, $\liminf_{n\to\infty}\beta_{n,0}^{(1)}\beta_{n,i}^{(2)}>0$, we see that

$$g(\|Jx_n-JT_i^nx_n\|)\to 0,\quad \text{as}\ n\to\infty. \tag{3.16}$$

It follows from the property of g that

$$\|Jx_n-JT_i^nx_n\|\to 0,\quad \text{as}\ n\to\infty. \tag{3.17}$$

Since $x_n\to p^*$ and J is uniformly continuous, it yield that $Jx_n\to Jp^*$. Hence, from (3.17), we have

$$JT_i^nx_n\to Jp^*,\quad \text{as}\ n\to\infty,\ \forall i\geq 1. \tag{3.18}$$

Since E^* is uniformly smooth, then J^{-1} is uniformly continuous, it follows that

$$T_i^nx_n\to p^*,\quad \text{as}\ n\to\infty,\ \forall i\geq 1. \tag{3.19}$$

Furthermore, by the assumption that for each $i \geq 1$, T_i is uniformly ω_i-Lipschitz continuous, so that we have

$$\begin{aligned} & \|T_i^{n+1}x_n - T_i^n x_n\| \\ \leq\ & \|T_i^{n+1}x_n - T_i^{n+1}x_{n+1}\| + \|T_i^{n+1}x_{n+1} - x_{n+1}\| \\ & + \|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| \\ \leq\ & (\omega_i + 1)\|x_{n+1} - x_n\| + \|T_i^{n+1}x_{n+1} \\ & - x_{n+1}\| + \|x_n - T_i^n x_n\| \\ \to\ & 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} T_i^{n+1}x_n = T_i^n x_n = p^*. \tag{3.20}$$

That is

$$\lim_{n\to\infty} T_i^{n+1}x_n = \lim_{n\to\infty} T_i T_i^n x_n = \lim_{n\to\infty} T_i p^* = p^*. \tag{3.21}$$

In view of (3.19) and the closeness of T_i, it yield that $T_i p^* = p^*$, for all $i \geq 1$. This implies that

$$p^* \in \bigcap_{i=1}^{\infty} Fix(T_i). \tag{3.22}$$

(5.2) Next, we will show that $p^* \in \bigcap_{j=1}^{\infty} Fix(S_j)$.

From (3.5), we have $\phi(p, u_n) \leq \phi(p, x_n) + \theta_n - \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} g(\|Jx_n - JT_i^n x_n\|) - \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} g(\|Jx_n - JS_j^n z_n\|)$. Hence, $\phi(p, u_n) \leq \phi(p, x_n) + \theta_n - \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} g(\|Jx_n - JS_j^n z_n\|)$, and so

$$\begin{aligned} & \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} g(\|Jx_n - JS_j^n z_n\|) \\ \leq\ & \phi(p, x_n) - \phi(p, u_n) + \theta_n \\ =\ & \|x_n\|^2 - \|u_n\|^2 + 2\langle p, Ju_n - Jx_n\rangle + \theta_n \\ \leq\ & (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) \\ & + 2\langle p, Ju_n - Jx_n\rangle + \theta_n \end{aligned}$$

From (3.7), (3.14) and (3.15), we get $\beta_{n,0}^{(1)}\beta_{n,j}^{(3)} g(\|Jx_n - JS_j^n z_n\|) \to 0, \quad \text{as } n \to \infty$. In the view of condition, $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0$, we see that

$$g(\|Jx_n - JS_j^n z_n\|) \to 0, \quad \text{as } n \to \infty. \tag{3.23}$$

It follows from the property of g that

$$\|Jx_n - JS_j^n z_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.24}$$

Since $x_n \to p^*$ and J is uniformly continuous, it yield $Jx_n \to Jp^*$. Hence, from (3.24), we have

$$JS_j^n z_n \to Jp^*, \quad \text{as } n \to \infty, \ \forall j \geq 1. \tag{3.25}$$

Since E^* is uniformly smooth, then J^{-1} is uniformly continuous, it follows that

$$S_j^n z_n \to p^*, \quad \text{as } n \to \infty, \ \forall j \geq 1. \tag{3.26}$$

From (3.4), we have

$$\phi(p, u_n) \leq \phi(p, x_n) + 2\lambda_n\left(\frac{2\lambda_n}{c^2} - \gamma\right)\|A_n x_n\|^2. \tag{3.27}$$

So that

$$2\lambda_n\left(\gamma - \frac{2\lambda_n}{c^2}\right)\|A_n x_n\|^2 \leq \phi(p, x_n) - \phi(p, u_n).$$

That is,

$$\lim_{n\to\infty} \|A_n x_n\|^2 = 0. \tag{3.28}$$

It follows from (3.1) and (3.28) that we have

$$\begin{aligned} & \lim_{n\to\infty} \|z_n - p^*\| \\ =\ & \lim_{n\to\infty} \|\Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n) - p^*\| \\ \leq\ & \lim_{n\to\infty} \|J^{-1}(Jx_n - \lambda_n A_n x_n) - p^*\| = 0. \end{aligned}$$

Furthermore, by the assumption that for each $j \geq 1$, S_j is uniformly μ_j-Lipschitz continuous, so that

$$\begin{aligned} & \|S_j^{n+1}z_n - S_j^n z_n\| \\ \leq\ & \|S_j^{n+1}z_n - S_j^{n+1}z_{n+1}\| + \|S_j^{n+1}z_{n+1} - z_{n+1}\| \\ & + \|z_{n+1} - z_n\| \\ & + \|z_n - S_j^n z_n\| \\ \leq\ & (\mu_i + 1)\|z_{n+1} - z_n\| + \|S_j^{n+1}z_{n+1} - z_{n+1}\| \\ & + \|z_n - S_j^n z_n\| \\ \to\ & 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} S_j^{n+1}z_n = S_j^n z_n = p^*. \tag{3.29}$$

That is

$$\lim_{n\to\infty} S_j^{n+1}z_n = \lim_{n\to\infty} S_j S_j^n z_n = \lim_{n\to\infty} S_j p^* = p^*. \tag{3.30}$$

In view of (3.19) and the closeness of S_j, it yield that $S_j p^* = p^*$, for all $j \geq 1$. This implies that

$$p^* \in \bigcap_{j=1}^{\infty} Fix(S_j). \tag{3.31}$$

(5.3) Next, we will show that $p^* \in \bigcap_{k=1}^{M} SGEP(F_k, B_k)$.

Putting $K_n^M = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{m-1}} \dots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$ and $K_n^0 = I$ for all $n \in \mathbb{N}$. For any $p \in \Omega$, we have

$$\phi(K_n^M y_n, K_n^{M-1} y_n)$$

$$\begin{aligned}
&\le \phi(p, K_n^{M-1}y_n) - \phi(p, K_n^M y_n) \\
&\le \phi(p, y_n) - \phi(p, K_n^M y_n) \\
&\le \phi(p, x_n) + \theta_n - \phi(p, K_n^M y_n) \\
&\le \phi(p, x_n) + \theta_n - \phi(p, u_n).
\end{aligned}$$

It follows from (3.16) that $\lim_{n\to\infty} \phi(K_n^M y_n, K_n^{M-1}y_n) = 0$. Since E is uniformly smooth and 2-uniformly convex Banach space and $\{z_n\}$ is bounded, we have

$$\lim_{n\to\infty} \|K_n^M y_n - K_n^{M-1}y_n\| = 0. \tag{3.32}$$

Since $u_n \to p^*$, and $u_n = K_n^M y_n$, so that $K_n^M y_n \to p^*$ as $n \to \infty$. and

$$\lim_{n\to\infty} \|K_n^M y_n - K_n^{M-1}y_n\| = \lim_{n\to\infty} \|u_n - K_n^{M-1}y_n\| = 0. \tag{3.33}$$

That is $K_n^{M-1}y_n \to p^*$. By induction, the conclusion can be obtained. Since J is uniformly continuous on bounded subset of E, we get

$$\lim_{n\to\infty} \|JK_n^M y_n - JK_n^{M-1}y_n\| = 0, \tag{3.34}$$

and from the condition $r_{k,n} \in [d, \infty)$ for some $d > 0$, we have

$$\lim_{n\to\infty} \frac{\|JK_n^M y_n - JK_n^{M-1}y_n\|}{r_{M,n}} = 0. \tag{3.35}$$

Consider

$$\begin{aligned}
& \phi(u_n, y_n) \\
\le\; & \phi(K_n^M y_n, y_n) \\
\le\; & \phi(p, y_n) - \phi(p, K_n^M y_n) \\
=\; & \phi(p, y_n) - \phi(p, u_n) \\
\le\; & \phi(p, x_n) + \theta_n - \phi(p, u_n) \\
=\; & \|p\|^2 - 2\langle p, x_n\rangle + \|x_n\|^2 + \theta_n - \|p\|^2 \\
& +2\langle p, Ju_n\rangle - \|u_n\|^2 \\
=\; & \|x_n\|^2 - \|u_n\|^2 + \theta_n \\
& +2\langle p, Ju_n - Jx_n\rangle \\
\le\; & (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) + \theta_n \\
& +2\|p\|\|Ju_n - Jx_n\| \\
\to\; & 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Using lemma 2.15, we get $y_n \to p^*$. Since $F_k(K_n^M y_n, y) + \langle y - K_n^M y_n, B_k K_n^M y_n\rangle + \frac{1}{r_{M,n}}\langle y - K_n^M y_n, JK_n^M y_n - Jy_n\rangle \ge 0, \quad \forall y \in C, k = 1, 2, ..., M$. By condition (A2), we have $\langle y - K_n^M y_n, B_k K_n^M y_n\rangle + \frac{1}{r_{M,n}}\langle y - K_n^M y_n, JK_n^M y_n - Jy_n\rangle \ge -F_k(K_n^M y_n, y) \ge F_k(y, K_n^M y_n)$. From $K_n^M y_n \to p^*$ and $y_n \to p^*$, we have

$$\langle y - p^*, B_k p^*\rangle \ge F_k(y, p^*), \forall y \in C.$$

For any $0 < t < 1$, $y \in C$ and setting $y_t = ty + (1 - t)p^*$, we have $y_t \in C$ and so

$$\langle y_t - p^*, B_k p^*\rangle \ge F_k(y_t, p^*), \forall y \in C.$$

In view of the convexity of ϕ it yield

$$t\langle y_t - p^*, B_k p^*\rangle \ge F_k(y_t, p^*), \forall y \in C.$$

It follows from (A1) and (A4) that

$$\begin{aligned}
0 = F_k(y_t, y_t) &\le tF_k(y_t, y) + (1 - t)F_k(y_t, p^*) \\
&\le tF_k(y_t, y) + (1 - t)t\langle y_t - p^*, B_k p^*\rangle.
\end{aligned}$$

Let $t \to 0$, from the condition (A3), we obtain

$$F_k(p^*, y) + \langle y_t - p^*, B_k p^*\rangle \ge 0, \quad \forall y \in C, k = 1, 2, ..., M.$$

This implies that $p^* \in \bigcap_{k=1}^M SGEP(F_k, B_k)$.

(5.4) Last, we will show that $p^* \in \bigcap_{n=1}^N VI(C, A_n)$.

From Lemma 2.12, we note that $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n)$ if and only if

$$\langle y - z_n, A_n x_n\rangle + \langle y - z_n, \frac{Jz_n - Jx_n}{\lambda_n}\rangle \ge 0, \quad \forall y \in C. \tag{3.36}$$

Let $\{n_j\}_{j\ge1} \subset \mathbb{N}$ be an increasing sequence of natural numbers such that $A_{n_j} = A_1$ for all $j \in \mathbb{N}$. From (3.36), we get

$$\langle y - z_{n_j}, A_1 x_{n_j}\rangle + \langle y - z_{n_j}, \frac{Jz_{n_j} - Jx_{n_j}}{\lambda_{n_j}}\rangle \ge 0, \quad \forall y \in C. \tag{3.37}$$

Put $y_t = ty + (1 - t)p^*$ for any $0 < t < 1$ and $y \in C$. Consequently, we get that $y_t \in C$. From (3.37), it follows that

$$\begin{aligned}
& \langle y_t - z_{n_j}, A_1 y_t\rangle \\
\ge\; & \langle y_t - z_{n_j}, A_1 y_t\rangle - \langle y_t - z_{n_j}, A_1 x_{n_j}\rangle \\
& -\langle y_t - z_{n_j}, \frac{Jz_{n_j} - Jx_{n_j}}{\lambda_{n_j}}\rangle \\
=\; & \langle y_t - z_{n_j}, A_1 y_t - A_1 z_{n_j}\rangle + \langle y_t - z_{n_j}, A_1 z_{n_j} - A_1 x_{n_j}\rangle \\
& -\langle y_t - z_{n_j}, \frac{Jz_{n_j} - Jx_{n_j}}{\lambda_{n_j}}\rangle.
\end{aligned}$$

By the continuity of A_1 and the fact that $z_{n_j}, x_{n_j} \to p^*$ and $Jz_{n_j}, Jx_{n_j} \to p^*$ as $k \to \infty$, we obtain that $A_1 z_{n_j} - A_1 x_{n_j} \to p^*$ and $\frac{Jz_{n_j} - Jx_{n_j}}{\lambda_{n_j}} \to p^*$ as $k \to \infty$. Since A_1 is monotone, we also have $\langle y_t - z_{n_j}, A_1 y_t - A_1 z_{n_j}\rangle \ge 0$. Hence, it follows that

$$0 \le \lim_{k\to\infty} \langle y_t - z_{n_j}, A_1 y_t\rangle = \langle y_t - p^*, A_1 y_t\rangle,$$

and so

$$\langle y - p^*, A_1 y_t \rangle \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, we obtain

$$\langle y - p^*, A_1 p^* \rangle \geq 0, \quad \forall y \in C.$$

This implies that $p^* \in VI(C, A_1)$. Similarly we obtain that $p^* \in VI(C, A_n)$. for $n = 2, 3.4, ..., N$. So that $p^* \in \bigcap_{n=1}^{N} VI(C, A_n)$. From (5.1) to (5.4), we can conclude that $p^* \in \Omega$.

Step 6. Finally, we will show that $x_n \to p^* = \Pi_\Omega(x_0)$.
Let $w = \Pi_\Omega(x_0)$. Since $w \in \Omega \subset C_n$ and $x_n = \Pi_{C_n}(x_0)$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0.$$

This implies that

$$\phi(p^*, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.38}$$

In view of the definition of $\Pi_\Omega(x_0)$, from (3.38) we have $p^* = w$. Therefore, $x_n \to p^* = \Pi_\Omega(x_0)$. This completes the proof of Theorem 3.1. □

If we change the condition (*B2*) in Theorem 3.3 as follows : $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty$ are quasi -ϕ- nonexpansive mappings. Since every quasi -ϕ- nonexpansive mappings is quasi -ϕ- asymptotically nonexpansive mappings. Then, we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(C1) Let $B_k : C \to E^*$ for each $k = 1, 2, 3, ..., M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(C2) Let $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty : C \to C$ be countable families of uniformly closed and quasi -ϕ- nonexpansive mappings.

(C3) Let $A_n : C \to E^*$ for each $n = 1, 2, 3, ..., N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1, 2, 3, ..., N\}$.

(C4) $\Omega := \Big(\bigcap_{i=1}^\infty Fix(T_i)\Big) \bigcap \Big(\bigcap_{j=1}^\infty Fix(S_j)\Big) \bigcap \Big(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\Big) \bigcap \Big(\bigcap_{n=1}^{N} VI(C, A_n)\Big)$ *is a nonempty and bounded in C.*

Let $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by

$$\begin{cases} x_0 \in C \ \textit{chosen arbitrary}, \ C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^\infty \beta_{n,i}^{(2)} JT_i x_n \\ \quad + \sum_{j=1}^\infty \beta_{n,j}^{(3)} JS_j z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ \forall n \geq 0, \end{cases} \tag{3.39}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1, 2, 3, ..., M$, is a mapping defined by (2.18) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z, y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C, \ k = 1, 2, 3, ..., M$. $r_{k,n} \in [d, \infty)$, for some $d > 0$, $A_n = A_n$(mod N), $\|A_n x\| \leq \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda_n\}$ be a sequence in* [0,1] *such that $0 < \lambda_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1. *for each $n \geq 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^\infty \beta_{n,i}^{(2)} + \sum_{j=1}^\infty \beta_{n,j}^{(3)} = 1$;*
2. $\liminf_{n\to\infty} \beta_{n,0}^{(1)} \beta_{n,i}^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}^{(1)} \beta_{n,j}^{(3)} > 0, \forall i, j \geq 1, i \neq j$.

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

Proof. Since $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty$ are countable families of uniformly closed and quasi -ϕ- nonexpansive mappings, By Remark 2.8 (ii), it is countable families of uniformly closed and quasi -ϕ- asymptotically nonexpansive mapping with $\{k_n = 1\}$ and $\{l_n = 1\}$. So $\theta_n = \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\phi(p, x_n) = 0$. Therefore, the conditions appearing in Theorem 3.3 : "Ω is bounded subset in C. and for each $i \geq 1$", ($\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty : C \to C$ be countable families of uniformly closed and ω_i,μ_j-Lipschitz continuous and nonexpansive mappings) are no use here. Therefore, all conditions in Theorem 3.3 are satisfied. The conclusion of Corollary 3.4 can be obtained from Theorem 3.3 immediately. □

If we change the condition (*C2*) in Corollary 3.4 as follows : $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty$ are weak relatively nonexpansive mappings. Since every weak relatively nonexpansive mappings is quasi -ϕ- nonexpansive mappings and every quasi -ϕ- nonexpansive mappings is quasi -ϕ- asymptotically nonexpansive mappings. Then, we obtain the following corollary.

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Corollary 3.5. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(D1) Let $B_k : C \to E^*$ for each $k = 1,2,3,...,M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(D2) Let $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty} : C \to C$ be countable families of uniformly closed and weak relatively nonexpansive mappings.

(D3) Let $A_n : C \to E^*$ for each $n = 1,2,3,...,N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1,2,3,...,N\}$.

(D4) $\Omega := \Big(\bigcap_{i=1}^{\infty} Fix(T_i)\Big) \bigcap \Big(\bigcap_{j=1}^{\infty} Fix(S_j)\Big) \bigcap \Big(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\Big) \bigcap \Big(\bigcap_{n=1}^{N} VI(C, A_n)\Big)$ *is a nonempty and bounded in C.*

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \;\; \text{chosen arbitrary}, \; C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} JT_i x_n \\ + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} JS_j z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \;\; \forall n \ge 0, \end{cases} \tag{3.40}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1,2,3,...,M$, is a mapping defined by (2.18) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z,y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C, \; k = 1,2,3,...,M$. $r_{k,n} \in [d, \infty)$, for some $d > 0$, $A_n = A_n$(mod N), $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda_n\}$ be a sequence in* [0,1] *such that $0 < \lambda_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1. *for each $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} = 1$;*
2. *$\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ and $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0, \forall i,j \ge 1, i \ne j$.*

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

Proof. Since $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty}$ are countable families of uniformly closed and weak relatively nonexpansive mappings, By Remark 2.8 (v) and (ii), it is countable families of uniformly closed and quasi -ϕ- nonexpansive mappings, and it is countable families of uniformly closed and quasi -ϕ- asymptotically nonexpansive mappings. Therefore, all conditions in Corollary 3.4 are satisfied. The conclusion of Corollary 3.5 can be obtained from Corollary 3.4 and it can be obtained from Theorem 3.3 immediately. $\square$

If $T_i = T$, $S_j = S$, $F_k = F$, $B_k = B$ and $A_n = A$ where $\forall i,j \in \mathbb{N}$, $k = 1,2,3,...,M$ and $\forall n = 1,2,3,...,N$ in Theorem 3.3, then the Theorem 3.3 is reduced to the following corollary.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(E1) Let $B : C \to E^*$ be a δ-inverse-strongly monotone mappings, and let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(E2) Let T and $S : C \to C$ be two uniformly closed and ω,μ-Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and $k_n \to 1, l_n \to 1$, respectively.

(E3) Let $A : C \to E^*$ be a γ-inverse strongly monotone mappings.

(E4) $\Omega := Fix(T) \bigcap Fix(S) \bigcap GEP(F, B) \bigcap VI(C, A)$ *is a nonempty and bounded in C.*

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \;\; \text{chosen arbitrary}, \; C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n) \\ y_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n z_n) \\ u_n = T_{r_n}^{F} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \theta_n\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \;\; \forall n \ge 0, \end{cases} \tag{3.41}$$

where $T_{r_n}^{F} : E \to C$, is a mapping defined by (2.18) *and it is the solutions to the following a generalized equilibrium problem:*

$$F(z,y) + \langle y - z, Bz\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

$r_n \in [d, \infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\phi(p, x_n)$, $\|Ax\| \le \|Ax - Ap\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda\}$ be a sequence in [0,1] *such that $0 < \lambda < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1. *for each $n \ge 0$, $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;*

2. $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $p^* = \Pi_\Omega(x_0)$.

If we set $\beta_n^{(1)} = \beta_n$ and set $\beta_n^{(2)} = 0$ in Corollary 3.6, then $\beta_n^{(2)} JT^n x_n = 0$ and so $\beta_n^{(3)} = 1 - \beta_n$, thus the Corollary 3.6 is reduced to the following corollary.

Corollary 3.7. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(F1) Let $B : C \to E^*$ be a δ-inverse-strongly monotone mappings, and let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(F2) Let $S : C \to C$ be closed and uniformly μ-Lipschitz continuous and uniformly quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{l_n\} \subset [1,\infty)$ and $l_n \to 1$.

(F3) Let $A : C \to E^*$ be a γ-inverse strongly monotone mappings.

(F4) $\Omega := Fix(S) \bigcap GEP(F,B) \bigcap VI(C,A)$ is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrary},\ C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n) \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n) \\ u_n = T_{r_n}^F y_n \\ C_{n+1} = \{v \in C_n : \phi(v,u_n) \le \phi(v,x_n) + \theta_n\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0),\ \ \forall n \ge 0, \end{cases} \tag{3.42}$$

where $T_{r_n}^F : E \to C$, is a mapping defined by (2.18) *and it is the solutions to the following a generalized equilibrium problem:*

$$F(z,y) + \langle y-z, Bz\rangle + \frac{1}{r}\langle y-z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

$r_n \in [d,\infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n,l_n\} - 1)\phi(p,x_n)$, $\|Ax\| \le \|Ax - Ap\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda\}$ be a sequence in [0,1] *such that $0 < \lambda < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_n\}$ be a sequence in* [0,1] *such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.*

Taking $B_k = 0, F_k = 0$, which $k = 1,2,3,...,M$ in Corollary 3.5, we can obtained the following corollary.

Corollary 3.8. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(G1) Let $B_k : C \to E^*$ for each $k = 1,2,3,...,M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(G2) Let $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty : C \to C$ be countable families of uniformly closed and weak relatively nonexpansive mappings.

(G3) Let $A_n : C \to E^*$ for each $n = 1,2,3,...,N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1,2,3,...,N\}$.

(G4) $\Omega := \left(\bigcap_{i=1}^\infty Fix(T_i)\right) \bigcap \left(\bigcap_{j=1}^\infty Fix(S_j)\right) \bigcap \left(\bigcap_{n=1}^N VI(C,A_n)\right)$ is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrary},\ C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} J x_n + \sum_{i=1}^\infty \beta_{n,i}^{(2)} J T_i x_n \\ + \sum_{j=1}^\infty \beta_{n,j}^{(3)} J S_j z_n) \\ C_{n+1} = \{v \in C_n : \phi(v,y_n) \le \phi(v,x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0),\ \ \forall n \ge 0, \end{cases} \tag{3.43}$$

where $A_n = A_n(mod\, N)$, $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{\lambda_n\}$ be a sequence in [0,1] *such that $0 < \lambda_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1. *for each $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^\infty \beta_{n,i}^{(2)} + \sum_{j=1}^\infty \beta_{n,j}^{(3)} = 1$;*

2. $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0, \forall i,j \ge 1, i \ne j$.

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

4 Conclusions

In this work we have introduced a new iterative sequence by the new hybrid projection method for solving the common solution problem for a system of generalized equilibrium problems of inverse strongly monotone mappings and a system of bifunctions satisfying certain the conditions, and the common solution

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problem for a family of uniformly quasi -ϕ- asymptotically nonexpansive and uniformly Lipschitz continuous and the common solution problem for a variational inequality problem in a uniformly smooth and 2-uniformly convex real Banach space. Then, we also obtained a strong convergence theorem of the iterative sequence generated by the conditions. The results obtained in this paper extend and improve several recent results in this area in the following remark.

Remark 4.1. Theorem 3.3 and Corollaries 3.4, 3.5, 3.6, 3.7 and 3.8 improve and extend the corresponding results in [1], [2], [9], [10], [11], [12], [14], [18], [19], [20], [21], [22], [23] in the following aspect:

(a) For the solution of the classical equilibrium problem to the system of generalized equilibrium problems.

(b) For the mapping, we extend the mappings from nonexpansive mappings, quasi -ϕ- nonexpansive mappings, relatively nonexpansive mappings, weak relatively nonexpansive mappings and a closed quasi -ϕ- nonexpansive mappings to more general than the countable families of uniformly closed and quasi -ϕ- asymptotically nonexpansive mappings.

(c) For the frame work of the spaces, we extend the space form a uniformly smooth and uniformly convex real Banach space or a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property to more general than a uniformly smooth and 2-uniformly convex real Banach space.

(d) For the algorithm, we propose a new hybrid iterative algorithms which are different from ones given in above and others.

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