Extension the Consistent Mass Matrices of Beam Elements for Vibration Problems of Rectangular Plates on Winkler Foundation

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Abstract: In many engineering structures assessment of stress conditions created by vertical or horizontal forces to the foundation is a frequent problem of design. In engineering practice, beside static case often dynamic effects have to be taken into consideration to the plate design problems. It will be necessary to describe the governing equation of motion of plates in a general mathematical form for such cases. In this study, it is proposed to extend analytical solutions of the discrete one-dimensional beam elements resting on one-parameter elastic foundation for solution of plate vibration problems. In order to observe the influences of foundation, some graphical comparisons have been done.

Key-Words: Consistent mass matrices, Finite grid solution, Grillage of beams, Vibration, Winkler foundations

1 Introduction

Plates on elastic foundations have received considerable attention due to their wide applicability in many engineering disciplines. Since the interaction between structural foundations and supporting soil has a great importance in many engineering applications, a considerable amount of research has been conducted on plates on elastic foundations. Many studies have been done to find a convenient representation of physical behaviour of a real structural component supported on a foundation. There are several realistic foundation models as well as their proper mathematical formulations, e.g. Selvadurai [1] and Scott [2]. Introducing the finite element method in 1960s and the developments in computers have a great importance for the developments in applied mechanics. A broad range of the engineering problems has been solved by computer-based methods such as finite element, boundary elements methods etc.

Omurtag and Kadıoğlu [3] studied a functional and a plate element capable of modelling the Kirchhoff type orthotropic plate resting on elastic foundation are given and numerical results of a free vibration analysis is performed by using the Gateaux Differential Method (GDM) that successfully applied to various structural problems such as space bars, plates and shells by Omurtag and Aköz [4]. Natural angular frequency results of the orthotropic plate are justified by the analytical
expressions present in the literature and some new problems for orthotropic plates on elastic foundation (Winkler and Pasternak type foundation) are solved. The Pasternak foundation, as a special case, converges to Winkler type foundation if shear layer is neglected. Tameroğlu [5] studied a different solution technique for free vibrations of rectangular plates resting on elastic foundations with clamped boundaries and subjected to uniform and constant compressive, unidirectional forces in the midplane. The applied method is based on the use of a nonorthogonal series expansion consisting of some specially chosen trigonometric functions for the deflection surface of the plate. The orthogonalization of the series and other calculations are performed using Fourier expansion of Bernoulli polynomials under some realistic approximations for the limiting values of the boundary conditions.

The usual approach in formulating problems of beams, plates, and shells continuously supported by elastic media is based on the inclusion of the foundation reaction in the corresponding differential equation of the beam, plate, or shell. In case of Winkler foundation under the combined action of transverse load and vibration the governing differential equation of the plates can be obtained as;

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + k_1 w - \bar{m} \frac{\partial^2 w}{\partial t^2} = q(x, y)$$

where $w = w(x, y, t)$ is the transverse deflection of the plate, $k_1$ is Winkler parameter with the unit of force per unit area/per unit length (force/length$^3$), $q(x,y)$ is the external loads, $D$ is flexural rigidity of plate and $\bar{m}$ is the mass of the plate per unit area.

This study will be oriented toward the development of finite grid element, on a Winkler foundation. The aim is to investigate an improved finite grid solution for vibration problems of plates on a one-parameter elastic foundation. The solution can be stated as an extension of the so-called discrete parameter approach where the physical continuous domain is broken down into discrete sub-domains, each endowed with a response suitable for the purpose of mimicking problem at hand. Conceptually, it is similar to the finite element method, except that each discrete element utilized is equipped with an exact solution.

## 2 Theory and Formulation

In engineering practice, beside static case dynamic effects have to be taken into consideration to the plate design problems. It will be necessary to describe the governing equation of motion of plates in a general mathematical form for such cases. This can be achieved by inserting both of the inertia force due to the lateral translation and vibration simultaneously, in an appropriate way, into the governing differential equation for static case.

Dynamic problems of the plates with arbitrary contours and arbitrary boundary condition are very difficult or often impossible to solve by the classical methods based on the plate governing equations. In some respects dynamic behaviour of plates resembles to that of beams. By representing the plate with assemblage of individual beam elements interconnected at their neighboring joints, the system cannot truly be equal to the continuous structure, however sufficient accuracy can be obtained similar to the static case. Therefore the plates can be modeled as an assemblage of individual beam elements interconnected at their neighboring joints can be represented as;

Fig. 1. The Representation of a modal plate by parallel sets of one-dimensional elements replaced by the continuous surface.

By representing the plate shown in Fig. 1 with individual beam elements the problem can be reduced to one-dimensional. Then the governing equation of plate resting on Winkler foundation Eqn. (1) can be reduced in the form one dimensional beam element as;

$$D \frac{d^4 w}{dx^4} + k_1 w + \bar{m} \frac{d^2 w}{dt^2} = 0$$

(2)

The main advantage of the reduction is that the exact geometric stiffness matrix can be determined.
for the beam elements. These matrices can be used as a basis of assembling the plate problems in a proper way as Kazarin [6]. Then dynamic problems of the plates resting on Winkler foundation with arbitrary loading and boundary conditions could be solved approximately. Networks of beam elements that have no limitations for loading and boundary conditions can represent the plates which have no rigorous solution except in the form of infinite Fourier series. The properties of beam elements resemble that of plates resting on elastic foundations is denoted to be a very useful tool to solve complicate plate problems.

For one-parameter foundation case it is possible to evaluate mass influence coefficients of a structural element with the procedures similar to that obtaining the element stiffness matrix by making the use of finite element concept such [7-10]. The consistent mass matrix of beam elements resting on elastic foundations can also be evaluated by the same procedures.

The degrees of freedom of the element are the torsion, rotation and translation at each end. Since the angular displacements are obtained from the pure torsion member, the torsional DOF’s are independent. Then it can be assumed that the displacements within the span are defined again by the same interpolation functions those already derived for obtaining the element stiffness matrices.

Consider the beam element shown in Fig.2 having a mass distribution $m(x)$. If it is subjected to a unit angular acceleration at point $a$, the acceleration would be developed along its length as follow:

$$\ddot{w}(x) = \psi_2(x)\ddot{w}_2$$

By d’Alembert’s principle, the inertial force due to this acceleration is:

$$f_3(x) = m(x)\ddot{w}(x) = m(x)\psi_2(x)\ddot{w}_2$$

(3)

![Fig. 2. The Representation of a Beam Element subjected to a unit real acceleration and virtual translation at the left side.](image)

By the principle of virtual displacements the mass influence coefficients associated with this acceleration as the nodal inertial forces can be evaluated. As an example, it is possible to evaluate the vertical force $p_a$, equating work done by the external force due to virtual displacement, to the work done on the distributed inertial forces $f_I(x)$.

That is,

$$\int_0^L p_a \delta w(x) dx = \int_0^L f_3(x) \delta w(x) dx$$

(4)

Substituting the vertical virtual displacement in terms of the shape functions into the equation then,

$$m_{23} = \int_0^L m(x)\psi_2(x)\psi_3(x) dx$$

(5)

By this analogy, this equation can be extended to evaluate for the other degrees of freedoms such as;

$$m_{ij} = \int_0^L m(x)\psi_i(x)\psi_j(x) dx$$

(6)

By using the proper shape functions for conventional beam or beam element resting on one parameter elastic foundations, this equation lets us to evaluate all of the mass matrix terms. Computing the mass coefficients by the same shape functions with same procedures as done for determining the stiffness matrices is called consistent-mass matrices.
3 Consistent Mass Matrix for Winkler Foundation

The corresponding shape functions derived by Karasin [11] can be substituted into Eqn. (6) for leading to evaluate the consistent mass matrix for the beam elements resting on one-parameter elastic foundations. After evaluating the necessary integrations and introducing the constant mass distribution \( m(x) = \mu \) as uniform mass per unit length, the mass matrix terms will be;

\[
m_{11} = \frac{\mu L}{3} \quad \text{and} \quad m_{44} = \frac{\mu L}{6} \times \left[ 8p\cos[2p] - \cosh[2p] + \frac{4\sin[2p] - \sin[4p]}{16p} \right] \]

\[
m_{22} = \frac{\mu L^2}{16p} \left[ \frac{4 - 4(1 - \cosh[2p])(1 - p\sin[2p])}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

\[
m_{23} = \frac{\mu L^2}{8p} \left[ \frac{2p\cos[p]\cosh[3p] - \cos[p]\cosh[3p] - 4\cos[p]\sin[p]}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

\[
m_{26} = \frac{\mu L^2}{4p} \left[ \frac{p\cosh[p]\sin[3p] - 2\sin[p] - p\cosh[3p]\sin[p]}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

\[
m_{33} = \frac{\mu L^2}{8p} \left[ \frac{8p(1 - \cosh[2p])(1 - \cos[2p]) + 3\sin[4p]}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

\[
m_{36} = \frac{\mu L^2}{4p} \left[ \frac{12\sin[p]\cosh[p] - 3\sin[p]\cosh[3p] - 3\sin[3p]\cosh[p]}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

\[
m_{55} = \frac{\mu L^2}{4p} \left[ \frac{2p\sin[p]\sin[3p] - 3\sin[3p]\cos[p]}{(\cosh[2p] + \cos[2p] - 2)^2} \right] \]

where \( \mu \) is mass per unit length and

\[ p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}} L \]

When foundation parameter \( k_1 \) tends to zero (or \( p \to 0 \)), the terms in the equations reduce to the conventional beam consistent mass terms obtained by Hermitian functions. The correctness of the terms is verified that the terms reduce to the following conventional terms in matrix form.

\[
\lim_{p \to 0} \left[ \begin{array}{cccc}
140 & 0 & 0 & 70 & 0 \\
0 & 4L^2 & -22L & 0 & -3L^2 - 13L \\
0 & -22L & 156 & 0 & 13L & 54 \\
70 & 0 & 70 & 0 & 0 \\
0 & -3L^2 & 13L & 0 & 4L^2 & 22L \\
0 & -13L & 54 & 0 & 22L & 156
\end{array} \right]
\]

The normalized terms represent the influence of the foundation parameter \( k_1 \) on the mass matrix terms given in the consistent mass equations and corresponding terms of the matrix are portrayed in Figs. 3 to 8 as follow;

**Fig. 3.** Influence of one-parameter foundation on the normalized consistent mass term \( m_{22} \)

**Fig. 4.** Influence of one-parameter foundation on the normalized consistent mass term \( m_{23} \)
From the figures the consistent mass stiffness terms related to beams on one-parameter elastic foundations are very sensitive to variation of foundation parameters for $p>1$ values. After obtaining the consistent mass matrices of each one-dimensional element it is possible to obtain free vibration frequencies of a total structure. Firstly using a proper numbering shame to collect all displacements for each nodal point in a convenient sequence the consistent mass matrix of the system for rectangular grids can be generated as follow:

$$M_{sys} = \sum_{i=1}^{NE} M_i$$

where $i$ is the individual element number, $NE$ is the number of elements depending on boundary conditions, $a_i$ is the individual rotation element matrix, $M_i$ is the proper element consistent mass matrix for a beam conventional resting on one-parameter elastic foundation and $M_{sys}$ is the system consistent mass matrix. Then the equations of motion for a system in a free vibration as an eigenvalue problem may be written as:

$$(K_{sys} - \omega^2 M_{sys})\mathbf{w} = 0$$

where the quantities $\omega^2$ are the eigenvalues indicting the square of free vibration frequencies that satisfy the above equation, while the corresponding displacement vector $\mathbf{w}$ express the fitting shapes of the vibrating system as the eigenvectors of mode shapes and $K_{sys}$ is the stiffness matrix of the total structure.
4 Conclusion

The solution of free vibration problems for rectangular plates resting on Winkler foundations is considered to be too complex. In many cases there is apparently no analytical solution other than simple cases. A grid work analogy called the Finite Grid Solution involving discretized plate properties mapped onto equivalent beams with adjusted parameters and matrix displacement analysis are used to develop a more general simplified numerical approach for such complicated problems. It is shown that after obtaining solutions of the governing differential equations of beam elements, the derived exact shape functions (interpolation functions) have extended to determine consistent mass matrices by finite element method. Then the discretized plate element reassembled by the matrix displacement method in a proper way. Than the consistent mass matrices of the total structure is generated by using a proper numbering shame to collect all displacements for each nodal point in a convenient sequence.

It is noted that the shape functions and stiffness terms related to beams on one-parameter elastic foundations are very sensitive to variation of foundation parameters. It can be concluded that the finite grid solution as a combination of finite element method, lattice analogy and matrix displacement analysis of grid works is a useful tool to improve the solution for free vibration of plate problems.

References: