An Efficient Numerical Technique for Solving of Certain Classes of Functional Differential Equations

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Abstract: In the paper, we present an efficent semi-analytical approach for functional differential equations (FDEs) with constant delays consisting in combination of the method of steps and the differential transformation method (DTM). Also some formulas based on DTM are determined for solving certain classes of functional differential equations with proportional delays. The presented technique does not require any symbolic calculations or initial guesstimates in contrast to methods like homotopy analysis method, homotopy perturbation method, variational iteration method or Adomian decomposition method. Some examples are given to demonstrate the validity and applicability of presented technique and a comparison is made with existing results.

Key–Words: Differential transformation method; Functional differential equations; Method of steps.

1 Introduction

Consider the following functional differential equation of n-th order with multiple constant delays

$$\begin{aligned} u^{(n)}(t) = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t),\\ \mathbf{u}_1(t-\tau_1), \mathbf{u}_2(t-\tau_2), \ldots, \mathbf{u}_r(t-\tau_r)), \quad (1) \end{aligned}$$

where

$$\mathbf{u}_i(t-\tau_i) = (u(t-\tau_i), u'(t-\tau_i), \ldots, u^{(m_i)}(t-\tau_i))$$

is m_i-dimensional vector function, $m_i < n$, $i = 1, 2, \ldots, r$, $r \in \mathbb{N}$ and $f\colon [t_0, \infty) \times R^n \times R^\omega$ is a continuous function, $\omega = \sum_{i=1}^{r} m_i$.

Denote

$$t^* = \max\{\tau_1, \tau_2, \ldots, \tau_r\},$$
$$m = \max\{m_1, m_2, \ldots, m_r\}.$$

Initial function $\phi(t)$ needs to be assigned for equation (1) on the interval $[t_0 - t^*, t_0]$. Further, for the sake of simplicity, we assume that $\phi(t) \in C^n([t_0 - t^*, t_0])$.

Also we will consider the following functional differential equations with proportional delays

$$\mathcal{F}\Big(t, \mathbf{u}(t), \mathbf{u}(p_0 t), \mathbf{u}(p_1 t), \ldots, \mathbf{u}(p_k t)\Big) = 0, \quad (2)$$

where

$$\mathbf{u}(t) = \Big(u(t), u'(t), \ldots, u^{(n)}(t)\Big),$$

$t \geq 0$, $\mathcal{F}$ is a given function with appropriate domain of definition, $p_i \in (0, 1)$, $i = 0, 1, \ldots, k$.

Investigation of equations (1),(2) is important since there is plenty of applications of such equations in real life. For example, we mention models for stress-strain states of materials, motion of rigid bodies, models of polymer crystallization, models describing behaviour of the central nervous system in a learning process, species populations struggling for a common food, systems controlled by PI or PID regulators, evolution of population of one species etc. For further models and details, see e.g. [1].

There are several series solution methods such as homotopy analysis method (HAM), homotopy perturbation method (HPM), variational iteration method (VIM), Adomian decomposition method (ADM), Taylor polynomial method, Taylor collocation method and differential transformation method (DTM) which have been considered to approximate solutions of certain classes of equations (1), (2) in a series form. However, in several papers initial problems for equation (1) were not properly defined. The authors used only initial conditions in certain points, not the initial function on the whole interval, thus the way to obtain solutions of illustrative examples was not correct. We propose a correct approach to overcome this vagueness.

2 DTM, basic notions

The concept of differential transformation in the form we use was proposed by Zhou [2] in 1986. It was applied to solve linear and nonlinear initial value problems in electric circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor method for solving of differential equations in the form of a polynomial. It is different from high-order Taylor series method which requires symbolic computation of necessary derivatives of the data functions.

The differential transformation of the k-th derivative of function $u(t)$ is defined as follows:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \qquad (3)$$

where $u(t)$ is the original function and $U(k)$ is the transformed function. Inverse differential transformation of $U(k)$ is defined as follows:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k, \qquad (4)$$

and hence

$$u(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}. \qquad (5)$$

In fact, inverse transformation (5) indicates that the concept of differential transformation is derived from Taylor series expansion. Although DTM is not able to evaluate the derivatives symbolically, relative derivatives can be calculated by an iterative way which is described by the transformed form of the original equation. Since DTM is based on Taylor series, it is clear that conditions for convergence of DTM are the same as for Taylor series.

In actual applications the function $u(t)$ is expressed as $u_N(t) + R_M$, where a truncated series is

$$u_N(t) = \sum_{k=0}^{N} U(k)(t-t_0)^k$$

and R_N is the remainder term given by

$$R_N = \frac{1}{(N+1)!}\left.\frac{d^{N+1}u(t)}{dt^{N+1}}\right|_{t=t_1} t^{N+1}$$

for some t_1 such that $0 < t_1 < t$. If the $(N+1)st$ derivative of $u(t)$ is bounded for $t \in (0,1]$, i.e.

$$\left|\frac{d^{N+1}u(t)}{dt^{N+1}}\right| \le K$$

for a certain nonnegative constant K, then the maximum error for $u_N(t)$ in this interval can be estimated from this remainder term as

$$e_{max} = \frac{K}{(N+1)!}.$$

The following well-known formulas for $t_0 = 0$ have been derived from definitions (3), (4):

Theorem 1 *Assume that $F(k)$, $H(k)$ and $U(k)$ are differential transformations of functions $f(t)$, $h(t)$ and $u(t)$, respectively. Then:*

$$\begin{aligned}
\text{If} f(t) &= \frac{d^n u(t)}{dt^n}, \text{then } F(k) = \frac{(k+n)!}{k!}U(k+n).\\
\text{If} f(t) &= u(t)h(t), \text{then } F(k) = \sum_{l=0}^{k} U(l)H(k-l).\\
\text{If} f(t) &= t^n, \text{then } F(k) = \delta(k-n),\ \text{where } \delta\\
&\quad \text{is the Kronecker delta}\\
\text{If} f(t) &= e^{\lambda t},\ \text{then } F(k) = \frac{\lambda^k}{k!}\\
\text{If} f(t) &= \sin t, \text{then}\\
&\quad S(k) = \begin{cases} (-1)^{\frac{k-1}{2}}\frac{1}{k!} & \textit{if}\, k = 2n+1,\\ 0 & \textit{if}\, k = 2n,\end{cases}\\
\text{If} f(t) &= \cos t, \text{then}\\
&\quad C(k) = \begin{cases} (-1)^{\frac{k}{2}}\frac{1}{k!} & \textit{if}\, k = 2n,\\ 0 & \textit{if}\, k = 2n+1.\end{cases}
\end{aligned}$$

More tramsformation formulas and proofs can be found e.g. in [3], [4].

Theorem 2 *Assume that $F(k)$, $G(k)$ are differential transformations of functions $f(t)$, $g(t)$. If $f(t) = g(t-a)$,where $a > 0$ is a real constant, then*

$$F(k) = \sum_{i=k}^{N} (-1)^{i-k}\binom{i}{k} a^{i-k} G(i),\ N \to \infty. \qquad (6)$$

Proof. The proof follows immediately from definition of differential transformation and binomial formula.

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Hence

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} G(k)(t-t_0-a)^k \\ &= \sum_{k=0}^{\infty} G(k) \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (t-t_0)^i a^{k-i} \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (t-t_0)^i a^{k-i} G(k) \\ &= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} (t-t_0)^i (-1)^{k-i} \binom{k}{i} a^{k-i} G(k) \\ &= \sum_{i=0}^{\infty} (t-t_0)^i \sum_{k=i}^{\infty} (-1)^{k-i} \binom{k}{i} a^{k-i} G(k) \\ &= \sum_{k=0}^{\infty} (t-t_0)^k \sum_{i=k}^{\infty} (-1)^{i-k} \binom{i}{k} a^{i-k} G(i). \end{aligned}$$

Using Theorem 2 and the formula i) in Theorem 1 we can easily prove differential transformation formula for function $f(t) = \frac{d^n}{dt^n} g(t-a)$.

Theorem 3 *Assume that $F(k)$, $G(k)$ are differential transformations of functions $f(t)$, $g(t)$, $a > 0$. If*

$$f(t) = \frac{d^n}{dt^n} g(t-a),$$

then

$$\begin{aligned} F(k) &= \frac{(k+n)!}{k!} \sum_{i=k+n}^{N} (-1)^{i-k-n} \binom{i}{k+n} \\ &\times \; a^{i-k-n} G(i), \quad N \to \infty. \end{aligned} \tag{7}$$

Using Theorems 1, 2, 3 differential transformation of any product of functions with delayed arguments and derivatives of that functions can be proved. However, such formulas are complicated and not easy applicable for solving functional differential equations with multiple constant delays (see for example [5], [6], [7]).

3 FDEs with constant delays

Consider equation (1) subject to initials conditions

$$u(t_0) = u_0, u'(t_0) = u_1, \ldots, u^{(n-1)}(t_0) = u_{n-1} \tag{8}$$

and subject to initial function $\phi(t)$ on interval $[t_0 - t^*, t_0]$ such that

$$\begin{aligned} &\phi(t_0) = u(t_0), \phi'(t_0) = u'(t_0), \ldots, \phi^{(n-1)}(t_0) \\ &\quad = u^{(n-1)}(t_0). \end{aligned} \tag{9}$$

First we apply the method of steps. We substitute the initial function $\phi(t)$ and its derivatives in all places where unknown functions with deviating arguments and derivatives of that functions appear. Then equation (1) changes to ordinary differential equation

$$\begin{aligned} &u^{(n)}(t) = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t), \\ &\mathbf{\Phi}_1(t-\tau_1), \mathbf{\Phi}_2(t-\tau_2), \ldots, \mathbf{\Phi}_r(t-\tau_r)), \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\mathbf{\Phi}_i(t-\tau_i) = (\phi(t-\tau_i), \phi'(t-\tau_i), \ldots, \\ &\phi^{(m_i)}(t-\tau_i)), \;\; m_i < n, \; i = 1, 2, \ldots, r. \end{aligned}$$

For more details on method of steps, see e.g. [1] or [8].

Now applying DTM we get recurrence equation

$$\begin{aligned} &\frac{(k+n)!}{k!} U(k+n) \\ &= \mathcal{F}\Big(k, U(0), U(1), \ldots, U(k+n-1)\Big), \end{aligned} \tag{11}$$

where, in general, $\mathcal{F}$ is a nonlinear function of its arguments.

Using transformed initial conditions and then inverse transformation rule, we obtain approximate solution of equation (1) in the form of infinite Taylor series

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k$$

on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min\{\tau_1, \tau_2, \ldots, \tau_r\}$, and $u(t) = \phi(t)$ on the interval $[t_0 - t^*, t_0]$. We demonstrate potentiality of this approach on several examples.

Example 1. Consider the following problem for delayed differential equation with variable coefficient that was solved by Arikoglu and Ozkol [5],

$$\begin{aligned} u'''(t) &= -\sin(t) u'\left(t - \frac{\pi}{2}\right) + \cos(t) u'(t) \\ &\quad + \sqrt{2} u\left(t - \frac{\pi}{4}\right) + \sin(t) - 2\cos(t) - 1 \end{aligned} \tag{12}$$

subject to initial conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \tag{13}$$

First, we remark that such formulation of problem is not correct since if we take for instance $t = 0$, then $u(t - \frac{\pi}{2}) = u(-\frac{\pi}{2})$ is not defined at all. Further, since there is no initial function, this is not a Cauchy problem, thus it is not clear what kind of solution are we looking for since uniqueness of solution is not guaranteed.

The authors claim that the exact solution of (12),(13) is $u(t) = \sin(t)$. However, this is only one of many possible solutions, namely it is true for initial function $\phi(t) = \sin(t)$. If, for example, we consider another initial function $\phi(t) = t^3/6 + t$ which also satisfies initial conditions (13), then we obtain completely different solution satisfying (12),(13).

The authors applied Theorems 1, 2, 3 for $N = 5$, 10, 15 and obtained approximate solution of the given initial value problem.

Presented approach is different:
We solve equation (12) on interval $[0, \pi/4]$ with initial function

$$\phi(t) = \sin(t) \tag{14}$$

for $t \in [-\pi/2, 0]$ and initial conditions

$$\begin{aligned} u(0) &= \phi(0) = 0, \\ u'(0) &= \phi'(0) = 1, \\ u''(0) &= \phi''(0) = 0. \end{aligned}$$

Using the method of steps we get

$$u'''(t) = \sin^2(t) + \cos(t)u'(t) - \cos(t) - 1. \tag{15}$$

Applying differential transformation on equation (15) we obtain recurrence relation

$$\begin{aligned} (k+1)(k+2)(k+3)U(k+3) &= \sum_{i=0}^{k} S(i)S(k-i) \\ &+ \sum_{i=0}^{k} C(i)(k-i+1)U(k-i+1) - C(k) - \delta(k), \end{aligned} \tag{16}$$

and from initial conditions $u(0) = 0,\ u'(0) = 1,\ u''(0) = 0$ we get $U(0) = 0,\ U(1) = 1,\ U(2) = 0$. From (16) we have

$$\begin{aligned} U(3) &= \frac{1}{3!}\left(C(0)U(1) + S^2(0) - C(0) - \delta(0)\right) = -\frac{1}{3!}, \\ U(4) &= \frac{1}{4!}[C(0)2U(2) + C(1)U(1) + 2S(0)S(1) \\ &\quad - C(1) - \delta(1)] = 0, \\ U(5) &= \frac{1}{3.4.5}[C(0)3U(3) + C(1)2U(2) + C(2)U(1) \\ &\quad + 2S(0)S(2) + S^2(1) - C(2) - \delta(2)] = \frac{1}{5!}, \\ U(6) &= 0, \\ &\vdots \\ U(k) &= \begin{cases} -\frac{(-1)^k}{k!},\ k = 2n-1,\ n \in N, \\ 0,\ k = 2n,\ n \in N, \end{cases} \\ &\vdots \end{aligned} \tag{17}$$

Therefore, the closed form of the solution can be written as

$$u(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + (-1)^k \frac{t^{2k+1}}{(2k+1)!} + \cdots = \sin t$$

which is unique exact solution of Cauchy problem (12), (14) on $[0, \pi/4]$. It can be easily verified that it is a solution of Cauchy problem (12), (14) on $[0, \infty)$.

Example 2. Consider delayed differential equation of the third order

$$u'''(t) = -u(t) - u(t-0.3) + e^{-t+0.3} \tag{18}$$

subject to the initial function

$$\phi(t) = e^{-t},\ t \leq 0 \tag{19}$$

and conditions

$$\begin{aligned} u(0) &= 1, \\ u'(0) &= -1, \\ u''(0) &= 1. \end{aligned} \tag{20}$$

This problem was solved using the Adomian decomposition method (ADM) by Evans and Raslan [9], later using current DTM approach by Karakoc and Bereketoglu [6] and again using ADM by Blanco-Cocom et al. [10].

Straightforward observation gives the information that, as Blanco-Cocom et al. [10] point out, it is enough to consider only (18) and (19), since conditions (20) are not independent of initial function $\phi(t)$ defined in (19). However, in fact, in all mentioned

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papers authors did not use initial function (19) at all, they solved problem (18), (20) which is not a Cauchy problem. Karakoc and Bereketoglu [6] tried to rectify the situation of not using (19) by excluding this condition from formulation of the studied problem. Unfortunately, this step led to the same curiosity observed in Example 1 when for instance $u(-0.3)$ is not defined. In any of the cases, uniqueness of solution is not guaranteed.

In both papers using ADM the authors obtained approximate solution using iterative scheme containing a triple integral and compared the result to function $u(t) = e^{-t}$ which is a solution of (18), (20) and satisfies (19) as well. Karakoc and Bereketoglu [6] solved equation (18) using current DTM approach without the dependence on the initial function and determined recurrence relation

$$\begin{aligned}&(k+1)(k+2)(k+3)U(k+3)\\&= -U(k) - \sum_{h_1=k}^{N}(-1)^{h_1-k}\binom{h_1}{k}(0.3)^{h_1-k}U(h_1)\\&+ \frac{1}{k!}(-1)^k e^{0.3}. \qquad (21)\end{aligned}$$

The authors solved (21) for $N = 6, 8, 10$ and compared obtained approximate solutions to solution $u(t) = e^{-t}$.

In contrast to complicated formula mentioned above, DTM combined with method of steps gives simple recurrence relation

$$U(k+3) = \frac{-U(k)}{(k+1)(k+2)(k+3)}, \quad k \geq 0. \quad (22)$$

From initial conditions (20) and recurrence relation (22) we have

$$\begin{aligned}&U(0) = 1,\ U(1) = -1,\ U(2) = \frac{1}{2}\ U(3) = \frac{-1}{3!},\\&U(4) = \frac{1}{4!}, \ldots, U(k) = \frac{(-1)^k}{k!}, \ldots\end{aligned}$$

Using inverse differential transformation (4) we obtain a solution of (18), (19) in the form

$$u(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k = e^{-t}. \quad (23)$$

It is the closed form unique solution of Cauchy problem (18), (19) which cannot be reached using either ADM or current approach of using DTM as described in above mentioned papers [9], [10] and [6], only approximation of the solution was possible.

4 FDEs with proportional delays

In this section we investigate equation (2). First, we give several theorems which can be easily proved from definition of the differential transformation method.

Theorem 4 *Assume that $W(k), U(k)$ are the differential transformations of the functions $w(t), u(t)$ and $q \in (0, 1)$, then:*

$$\text{If } w(t) = u(qt), \text{ then } W(k) = q^k U(k).$$

Proof. We calculate $k-th$ derivative of equation $w(t) = u(qt)$. We get

$$\frac{d^k}{dt^k} w(t) = \frac{d^k}{dt^k}[u(qt)] = q^k \frac{d^k}{d\tilde{t}^k} u(\tilde{t}),$$

where $\tilde{t} = qt$, thus

$$\left[\frac{d^k}{dt^k} w(t)\right]_{t=t_0} = q^k \left[\frac{d^k}{d\tilde{t}^k} u(\tilde{t})\right]_{t=t_0} = q^k k! U(k)$$

and using (2) we have

$$W(k) = \frac{1}{k!}\left[\frac{d^k w(t)}{dt^k}\right]_{t=t_0} = \frac{1}{k!} q^k k! U(k) = q^k U(k),$$

where $k \in \mathbb{N} \cup \{0\}$.

Similarly using Theorem 1 and Theorem 4 we can prove the following theorems:

Theorem 5 *Assume that $W(k)$, $U_i(k)$ are the differential transformations of the functions $w(t)$, $u_i(t)$ and $q_i \in (0, 1)$, $i = 1, 2$, then*

$$\text{If } w(t) = u_1(q_1 t) u_2(q_2 t)$$

then

$$W(k) = \sum_{l=0}^{k} q_1^l q_2^{k-l} U_1(l) U_2(k-l).$$

Theorem 6 *Assume that $W(k), U(k)$ are the differential transformations of the functions $w(t), u(t)$ and $q \in (0, 1)$, then:*

$$\text{If } w(t) = \frac{d^m u(qt)}{d(qt)^m}$$

then

$$W(k) = \frac{(k+m)!}{k!} q^k U(k+m).$$

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Example 3 As a practical example, we consider the following pantograph delay equation:

$$u''(t) = \frac{3}{4}u(t)+u\left(\frac{t}{2}\right)-t^2+2, \quad u(0) = u'(0) = 0. \tag{24}$$

Using differential transformation method, the transformed version of equation (24) is

$$\begin{aligned}&(k+1)(k+2)U(k+2)\\ &= \frac{3}{4}U(k) + \left(\frac{1}{2}\right)^k U(k) - \delta(k-2) + 2\delta(k),\end{aligned} \tag{25}$$

$k \geq 0$, and the differential transformation version of the initial conditions $u(0) = u'(0) = 0$ has the form $U(0) = U(1) = 0$. From system (25), we have

$$\begin{aligned}
U(2) &= \frac{1}{2}\left[\frac{3}{4}U(0) + U(0) + 2\right] = 1,\\
U(3) &= \frac{1}{6}\left[\frac{3}{4}U(1) + \frac{1}{2}U(1)\right] = 0,\\
U(4) &= \frac{1}{12}\left[\frac{3}{4}U(2) + \frac{1}{4}U(2) - 1\right] = 0,\\
U(5) &= \frac{1}{20}\left[\frac{3}{4}U(3) + \frac{1}{8}U(3)\right] = 0,\\
&\vdots\\
U(k) &= 0,\\
&\vdots
\end{aligned} \tag{26}$$

Using the inverse transformation rule (4) , we obtain the following solution

$$u(t) = t^2$$

which is the exact solution of equation (24).
The same equation was solved by Ghomanjani and Farahi [11] using the Bezier control points method. They obtained the same solution in very complicated form

$$\begin{aligned}u(t) &= t^2(1-t)^6 + 6t^3(1-t)^5 + 15t^4(1-t)^4\\ &+ 20t^5(1-t)^3 + 15t^6(1-t)^2 + 6t^7(1-t) + t^8.\end{aligned}$$

Example 4 Consider the following neutral differential equation with proportional delays of the third order:

$$\begin{aligned}u'''(t) &= u(t) + u'\left(\frac{t}{2}\right) + u''\left(\frac{t}{3}\right) + \frac{1}{2}u'''\left(\frac{t}{4}\right)\\ &- t^4 - \frac{t^3}{2} - \frac{4}{3}t^2 + 21t\end{aligned} \tag{27}$$

subject to initial conditions

$$u(0) = u'(0) = u''(0) = 0. \tag{28}$$

The differential transformation version of equation (27) has the form

$$\begin{aligned}U(k+3) &= \frac{1}{\left(1-\frac{1}{2^{2k+1}}\right)(k+3)(k+2)(k+1)}\\
&\times \Bigg[U(k) + \frac{1}{2^k}(k+1)U(k+1)\\
&+ \frac{1}{3^k}(k+1)(k+2)U(k+2) - \delta(k-4)\\
&- \frac{1}{2}\delta(k-3) - \frac{4}{3}\delta(k-2) + 21\delta(k-1)\Bigg].\end{aligned} \tag{29}$$

Differential transformation version of initial conditions (28) is

$$U(0) = 0,\ U(1) = 0,\ U(2) = 0.$$

Solving recurrence equation (29) we get

$$\begin{aligned}
U(3) &= \frac{1}{3}\Big[U(0) + U(1) + 2U(2)\Big] = 0,\\
U(4) &= \frac{1}{21}\left[U(1) + U(2) + 2U(3) + 21\right] = 1,\\
U(5) &= \frac{8}{465}\left[U(2) + \frac{3}{4}U(3) + \frac{4}{3}U(4) - \frac{4}{3}\right] = 0,\\
U(6) &= \frac{16}{1905}\left[U(3) + \frac{1}{2}U(4) + \frac{20}{27}U(5) - \frac{1}{2}\right] = 0,\\
&\vdots\\
U(k) &= 0,\\
&\vdots
\end{aligned}$$

From here we get

$$u(t) = t^4$$

which is the exact solution of equation (27).

Chen and Wang [12] solved equation (27) using variational iteration method and obtained sequence of approximate solutions in the form

$$\begin{aligned}u_1(t) &= \frac{7}{8}t^4 - \frac{1}{45}t^5 - \frac{1}{240}t^6 - \frac{1}{210}t^7,\\ u_2(t) &= \frac{63}{64}t^4 - \frac{1}{288}t^5 - \frac{1031}{1492992}t^6 - \ldots.\end{aligned}$$

Ghomani and Farahi [11] solved equation (27) using Bezier control points method and obtained the exact solution in complicated form

$$\begin{aligned}u(t) &= t^4(1-t)^4 + 4t^5(1-t)^3 + 6t^6(1-t)^2\\ &+ 4t^7(1-t) + t^8.\end{aligned}$$

5 Conclusion

In the present paper, we have shown that the differential transformation method and the differential transformation method in combination with the method of steps can be successfully used for solving functional differential equations with constant delays and proportional delays, respectively. The results obtained using the proposed method are in good agreement with those obtained by other methods. It can be concluded that DTM is powerful and efficient in finding analytical as well as numerical solutions for wide class of functional differential equations.

The main advantage of presented approach is that it can be applied directly to functional differential equations without requiring linearization, discretization or perturbation. Another important advantage is that this technique is capable of greatly reducing the size of computational work and also reduces solving of an initial value problem to solving of a system of recurrence algebraic equations.

It is necessary to point out that all the other mentioned methods are more complicated in comparison with the differential transformation method. Moreover, the other methods usually give only approximate solutions whereas using differential transformation method it is possible to obtain solutions in closed form.

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