King Type modification of Bernstein-Chlodowsky Operators based on $q$-integers

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Abstract: In this paper we first introduce the King type modification of $q$-Bernstein-Chlodowsky operators, then we examine the rate of convergence of these operators by means of modulus of continuity and with the help of the functions of Lipschitz class. We proved that the error estimation of this modification is better than that of classical $q$-Bernstein-Chlodowsky operators whenever $0 \leq x \leq \frac{b_n}{2[n]+1}$.

Key–Words: Bernstein-Chlodowsky operators, $q$-calculus, King type modification, Korovkin theorem, Rate of convergence

1 Introduction

The classical Bernstein-Chlodowsky polynomials are defined as,

$$C_n(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} b_n \right) \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{n-k}$$

where $0 \leq x \leq b_n$ and $b_n$ is a sequence of positive numbers with $\lim_{n \to \infty} b_n = \infty$ and $\lim_{n \to \infty} \frac{b_n}{n} = 0$.

These operators are introduced by Chlodowsky in 1932 in order to generalize Bernstein polynomials on an unbounded set. We refer to papers [5] and [6] related to this subject. Recall that, for $C_n(e_i; x)$, $i = 1, 2$ one has

1. $C_n(e_0; x) = 1$
2. $C_n(e_1; x) = x$
3. $C_n(e_2; x) = x^2 + \frac{x(b_n-x)}{n}$.

In 2008, Karsli and Gupta [7] defined the $q$-analogue of Chlodowsky operators as

$$C_{n,q}(f; x) = \sum_{k=0}^{n} f \left( \left[ \frac{k}{n} \right] b_n \right) \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{n-k} \times \prod_{s=0}^{n-k-1} \left( 1 - q^s \frac{x}{b_n} \right), \quad 0 \leq x \leq b_n$$

where $0 < q < 1$ and $(b_n)$ is a positive increasing sequence with the property

$$\lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{n} = 0. \quad (2)$$

They studied Korovkin type approximation theorems and rate of convergence of these new operators. They obtained the moments of $C_{n,q}(f; x)$ as

1. $C_{n,q}(e_0; x) = 1$
2. $C_{n,q}(e_1; x) = x$
3. $C_{n,q}(e_2; x) = x^2 + \frac{x(b_n-x)}{n}$.

It is known that obtaining better error estimates is an important concept in approximation theory. For this purpose King [8] presented a new method in which the operators preserve the first and the third test functions. He constructed the operator $V_n$, preserving the functions $e_0$ and $e_2$, and then showed that $V_n$ have a better rate of convergence than the classical Bernstein polynomials whenever $0 \leq x \leq \frac{1}{3}$.

King type modifications of operators have attracted a lot of interest and were studied by many authors. For example Duman and Özsarlan [9, 10] studied this type of modifications for Szasz-Mirakyan and Meyer-König-Zeller (MKZ) operators ; Agratini and Doğru [2] for $q$-Szasz-Mirakyan operators. Duman et.al also obtained better estimates for Szasz-Mirakyan-Kantorovich operators in [11]. Doğru and Örkçü introduced King type modification of both...

In [1], Agratini also gave a technique to construct sequence of operators of discrete type with the same property as in King’s example. He examined the operators -including Bernstein-Chlodowsky operators- with his technique. He showed that the order of approximation of the operator

\[ L_n^r(f) = \sum_{k=0}^{n} f \left( \frac{k}{n} h_n \right) \left( \binom{n}{k} \frac{r_n(x)}{h_n} \right)^k \times \left( 1 - \frac{r_n(x)}{h_n} \right)^{n-k} \]

to the function \( f \) is at least as good as the order of approximation of the classical Bernstein-Chlodowsky operators whenever \( 0 \leq x \leq \frac{h^*}{T} \) where \( h^* = \min_{n \geq 1} h_n \).

Here

\[ r_n(x) = \begin{cases} \frac{1}{2(n-1)} \left( \sqrt{h^2_n + 4n(n-1)x^2} - h_n \right), & x \in [0, b_n] \\ \infty, & x > b_n \end{cases} \]

In the present paper we introduce a King type modification of the q-Bernstein-Chlodowsky operators and show that this modification provides a better error estimation than the classical q-Bernstein-Chlodowsky operators.

We also estimate the rate of convergence of these operators by using modulus of continuity and functions of Lipschitz class.

**2 Construction of the Operators**

Before proceeding further we recall some basic notations from \( q \)-calculus.

Let \( q > 0 \). For each nonnegative integer \( r \), the q-integer \( [r] \), q-factorial \( [r]! \) and q-binomial \( \binom{n}{r} \), \( n \geq r \geq 0 \) are defined by

\[ [r] := [r]_q := \begin{cases} 1-q^r \frac{1-q^r}{1-q}, & q \neq 1, \\ \frac{r}{1-q}, & q = 1, \end{cases} \]

\[ [r]! := \begin{cases} [r][r-1]...[1] ; & q \geq 1, \\ 1 ; & q = 1, \end{cases} \]

and

\[ \binom{n}{r} := \frac{[n]!}{[n-r]![r]!}, \]

respectively.

We now introduce the King type generalization of q-Bernstein-Chlodowsky operators defined in (1) as follows:

Let \( \{ r_{n,q}(x) \} \) be a sequence of real-valued continuous functions defined on \( [0, b_n] \) with \( 0 \leq r_{n,q}(x) \leq b_n \). Consider

\[ C_{n,q}^r(f; x) = \sum_{k=0}^{n} f \left( \frac{[k]}{[n]} b_n \right) \left( \binom{n}{k} \frac{r_{n,q}(x)}{b_n} \right)^k \times \prod_{s=0}^{n-k-1} \left( 1 - q^s \frac{r_{n,q}(x)}{b_n} \right), \]

where \( f \in C[0, b_n] \) and \( (b_n) \) is a positive increasing sequence with the properties given in (2).

The operators \( C_{n,q}^r(f; x) \) are linear and positive. If we choose \( r_n(x) = x \), then our operators turn out to be classical q-Bernstein-Chlodowsky operators.

**Lemma 1** For each \( x \in [0, b_n] \), \( C_{n,q}^r(f; x) \) satisfy the following identities:

i) \( C_{n,q}^r(e_0; x) = 1 \)

ii) \( C_{n,q}^r(e_1; x) = r_{n,q}(x) \)

iii) \( C_{n,q}^r(e_2; x) = r_{n,q}^2(x) + r_{n,q}(x)(b_n - r_{n,q}(x)) \).

If we take

\[ r_{n,q}(x) = \frac{1}{2q [n-1]} \left( \sqrt{b^2_n + 4q [n-1] x^2} - b_n \right), \]

for \( 0 \leq x \leq b_n \), we get,

\[ C_{n,q}^r(e_0; x) = 1, \quad C_{n,q}^r(e_2; x) = x^2 \quad C_{n,q}^r(e_1 - e_0)(x^2; x) = 2x(x - r_{n,q}(x)). \]

**Remark 2** It is obvious that for \( 0 \leq x \leq b_n \) we have \( r_{n,q}(x) \geq 0 \). On the other hand, we can also write

\[ [n] [n-1] x^2 \leq \left( [n-1] + q [n-1]^2 \right) b_n^2. \]

Therefore we have

\[ b_n^2 + 4q [n-1] x^2 \leq (1 + 2q [n-1]^2) b_n^2 \]

from which we get \( r_{n,q}(x) \leq b_n \).

From the above Lemma and Korovkin’s theorem we can give the following theorem.

**Theorem 3** Let \( \{ q_n \} \) be a sequence of real numbers such that \( 0 < q_n < 1 \) satisfying the conditions given in (2). For \( f \in C[0, \infty) \) and for a fixed positive real number \( A \), \( C_{n,q}^r(f; x) \) converges uniformly to \( f(x) \) on a subinterval \([0, A]\) as \( n \to \infty\).
3 Rates of Convergence

Let \( f \in C[a, b] \). The modulus of continuity of \( f \), denoted by \( \omega(f; \delta) \) is defined by

\[
\omega(f; \delta) = \sup_{|t-x| \leq \delta, t, x \in [a, b]} |f(t) - f(x)|
\]

It is known that for any \( \delta > 0 \) and for each \( t, x \in [a, b] \)

\[
|f(t) - f(x)| \leq \omega(f; \delta) \left(1 + \frac{t-x}{\delta}\right)
\]

holds.

Recall that in [7], Karsli & Gupta obtained

\[
|C_{n,q}(f; x) - f(x)| \leq 2\omega(f, \delta_n(x))
\]

for \( q \)-Bernstein-Chlodowsky operators for every \( f \in C[0, \infty) \).

Now we give an estimate for the operators \( C_{n,q}^*(f; x) \) given by (6) means of modulus of continuity.

**Theorem 4** Let \((q_n), 0 \leq q_n \leq 1\) be a sequence satisfying the conditions given in (2), for each \( n \geq 2 \). For \( f \in C[0, \infty) \), we have

\[
|C_{n,q}^*(f; x) - f(x)| \leq 2\omega(f, \delta_n(x))
\]

where

\[
\delta_n(x) = \left(2x^2 + x \left[\frac{b_n}{q[n-1]} - \sqrt{\frac{b_n^2}{q[n-1]^2} + \frac{4x^2[n]}{q[n-1]}}\right]\right)^{1/2}
\]

**Proof:** Let \( f \in C[0, \infty) \). From the linearity and positivity of the operators \( C_{n,q}^*(f; q, x) \), we get for each \( n \geq 2 \) and \( 0 \leq x \leq b_n \)

\[
|C_{n,q}^*(f; x) - f(x)| \leq C_{n,q}^*(|f(t) - f(x)|; x)
\]

\[
\leq \omega(f; \delta) \left\{ 1 + C_{n,q}^*(|t-x|; x) \right\}.
\]

Applying Cauchy-Schwarz inequality we have

\[
|C_{n,q}^*(f; x) - f(x)| \leq \omega(f; \delta) \left\{ 1 + \frac{1}{\delta} \sqrt{C_{n,q}^*(|t-x|^2; x)} \right\}.
\]

(11)

On the other hand since

\[
C_{n,q}^* \left((t-x)^2; x\right) = C_{n,q}^* \left(t^2; x\right) - 2x C_{n,q}^* \left(t; x\right) + x^2
\]

we have

\[
C_{n,q}^* \left((t-x)^2; x\right) =
\]

\[
2x^2 + x \left[\frac{b_n}{q[n-1]} - \sqrt{\frac{b_n^2}{q[n-1]^2} + \frac{4x^2[n]}{q[n-1]}}\right].
\]

(12)

Using (12) in (11) and taking \( \delta = \delta_n \) we get the desired result.

**Remark 5** We claim that the error estimation in Theorem 3 is better than that of (9) provided \( f \in C[0, \infty) \) and \( 0 \leq x \leq \frac{b_n}{2[n+1]} \).

Indeed we have to show that

\[
\delta_n(x) \leq \alpha_n(x) = \frac{x(b_n - x)}{[n]}
\]

for all \( 0 \leq x \leq \frac{b_n}{2[n+1]} \).

\[
\delta_n(x) \leq \alpha_n(x) \iff 2x^2 + \frac{x}{q[n-1]} \times \left(\frac{b_n - \sqrt{b_n^2 + 4x^2[n]}[n]}{[n]} - \frac{x(b_n - x)}{[n]}\right) \leq 0
\]

\[
\iff 2x^2 + \frac{x}{q[n-1]} \left(\frac{b_n - \sqrt{b_n^2}}{[n]} - \frac{x(b_n - x)}{[n]}\right) \leq 0
\]

\[
\iff 2x^2 - \frac{x(b_n - x)}{[n]} \leq 0
\]

\[
\iff x \leq \frac{b_n}{2[n+1]].
\]

(14)

**Remark 6** In the view of the inequality

\[
\frac{x(b_n - x)}{[n]} \leq x_0 \frac{b_n}{[n]}
\]

for any fixed point \( x_0 \), Karsli and Gupta [7] obtained some results on the degree of pointwise and uniform convergence of \( q \)-Bernstein-Chlodowsky operators.

Using the same inequality we can give the following theorems for the King type generalization of \( q \)-Bernstein-Chlodowsky operators.

**Theorem 7** Let \((q_n), \) be a sequence of real numbers with \( 0 < q_n < 1 \), satisfying conditions in (2). If \( f \in C[0, b_n] \), then for all \( x \in [0, \frac{b_n}{2[n+1]}] \)

\[
|C_{n,q}^*(f; x_0) - f(x_0)| \leq 2\omega \left(f, \sqrt{x_0 \frac{b_n}{[n]}}\right)
\]

where \( x_0 \) is a fixed point.
Proof: From Theorem 3 we have
\[ |C_{n,q}^* f (x) - f (x)| \leq 2 \omega (f, \delta_n (x)) \]
where \( \delta_n (x) \) is given by (10). Since \( x \in \left[ 0, \frac{b_n}{2[n]+1} \right] \), from Remark 4 and Remark 5 we can write
\[ \delta_n (x) \leq \alpha_n (x) = \frac{x(b_n - x)}{n} \leq x_0 \frac{b_n}{n}. \]
In the view of the above inequality and properties of modulus of continuity we get the desired result.

We can also give the following theorem, similarly.

Theorem 8 Let \((q_n)\), be a sequence of real numbers with \( 0 < q_n < 1 \), satisfying conditions in (2). If \( f \in C[0, \infty) \), then for all \( x \in \left[ 0, \frac{b_n}{2[n]+1} \right] \)
\[ \| (C_{n,q}^* f) - f \|_{C[0,b_n]} \leq 2 \omega \left( f, \frac{ \sqrt{A} b_n}{n} \right). \]
where \( A \) is constant given in Theorem 2.

Proof: The proof of theorem is analogues to the proof of theorem 6, so we omit it.

The following theorem gives us the rate of convergence of King type generalization of Bernstein-Chlodowsky operators by means of the elements of Lipschitz Class \( Lip_{\alpha} (\alpha) \).

Recall that a function \( f \in C[0, b_n] \) belongs to \( Lip_{\alpha} (\alpha) \) if
\[ | f (t) - f (x) | \leq M | t - x |^{\alpha}, \ (t, x \in [0, b_n]) \]
holds.

Theorem 9 If \( f \in Lip_{\alpha} [0, b_n] \) and \( x \in [0, A] \), \( A > 0 \) a constant, we have
\[ \| (C_{n,q}^* f) - f \|_{C[0,b_n]} \leq M \left\{ A \frac{b_n}{n} \right\}^{\frac{\alpha}{2}}. \]
for all \( x \in \left[ 0, \frac{b_n}{2[n]+1} \right] \).

Proof:
\[ \| (C_{n,q}^* f) (x) - f (x) \| \leq C_{n,q} (| f (t) - f (x) | ; x) \]
\[ \leq MC_{n,q}^* (| t - x |^{\alpha} ; x) \]
Applying Hölder’s inequality with \( p = \frac{2}{\alpha} \) and \( q = \frac{2}{\alpha - \alpha} \) we get,
\[ \| (C_{n,q}^* f) (x) - f (x) \| \leq M \left\{ C_{n,q}^* \left( (t - x)^{2} ; x \right) \right\}^{\frac{\alpha}{2}}. \]
(16)
Since \( x \leq \frac{b_n}{2[n]+1} \), proceeding similarly as in the previous theorem, we get the desired result.

Lemma 10 For \( f \in C[0, \infty) \), \( x \in [0, b_n] \) and \( n \in \mathbb{N} \) we have
\[ \| C_{n,q}^* (f ; q ; x) \| \leq \| f \|. \]

Proof: From the definition of the operator \( C_{n,q}^* (f ; q ; x) \) and from Lemma 1, we have
\[ C_{n,q}^* (f ; q ; x) \leq C_{n,q}^* (1 ; q ; x) \| f \| = \| f \|. \]

Definition 11 For \( f \in C[a, b] \) and \( \delta > 0 \), the Peetre-K functional are defined by
\[ K_2 (f; \delta) = \inf_{g \in C^2 [a,b]} \left\{ \| f - g \|_{C[a,b]} + \delta \| g'' \|_{C[a,b]} \right\}. \]
(17)
Here \( C^2 [a, b] \) is the space of functions \( f \) such that \( f, f', f'' \in C[a,b] \). The norm on \( C^2 [a, b] \) is defined as
\[ \| g \|_{C^2 [a,b]} = \| g \|_{C[a,b]} + \| g' \|_{C[a,b]} + \| g'' \|_{C[a,b]} \]. \]
(18)
There exists a positive constant \( C > 0 \) such that
\[ K_2 (f; \delta) \leq w_2 (f; \sqrt{\delta}) \]
(19)
where
\[ w_2 (f; \sqrt{\delta}) = \sup_{0 \leq \delta \leq \delta} \sup_{x+h \in [0,b_n]} | f (x + 2h) - 2f (x + h) + f (x) | \]
is the second order modulus of smoothness of \( f \), and
\[ w (f; \delta) = \sup_{0 \leq \delta \leq \delta} \sup_{x+h \in [0,b_n]} | f (x + h) - f (x) | \]
is the usual modulus of continuity of \( f \).

Theorem 12 Let \((q_n)\) be a sequence of real numbers with \( 0 < q_n < 1 \), satisfying conditions in (2). If \( f \in C[0, \infty) \), then
\[ \| (C_{n,q}^* f) (x) - f (x) \| \leq C w_2 (f; \sqrt{\phi_{n,q} (x)}) + w (f; r_{n,q} (x) - x) \]
where \( \phi_{n,q} (x) = C_{n,q}^* (\delta_n^2 ; x) + (r_{n,q} (x) - x)^2 \) and \( r_{n,q} (x) \) and \( \delta_n (x) \) are given in (7) and (10), respectively.
Taking the infimum on the right hand side over all \( g \in C^2[0,b_n] \), By Taylor’s Theorem we have
\[
g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du.
\]

Applying \( \tilde{C}_{n,q}^* \) to the both side of the above equality, we get
\[
\tilde{C}_{n,q}^* (g(t) - g(x); x) = \left| g'(x)\tilde{C}_{n,q}^*(t - x; x) \right|
\]
\[
+ \tilde{C}_{n,q}^* \left( \int_x^t (t - u)g''(u)du; x \right)
\]

Since \( \tilde{C}_{n,q}^* (t - x; x) = 0 \), we can write
\[
\left| \tilde{C}_{n,q}^* (g(t) - g(x); x) \right| 
\leq C_{n,q}^* \left( \int_x^t |t - u| |g''(u)| du; x \right)
+ \left( \int_x^t |r_{n,q}(x) - u| |g''(u)| du; x \right)
\]
\[
\leq \|g''\| \left[ C_{n,q}^* ((t - x)^2; x) + (r_{n,q}(x) - x)^2 \right]
\]
\[
= \left[ \delta_n^2 + (r_{n,q}(x) - x)^2 \right] \|g''\|_{C[0,b_n]}
\]

Then from (20) and from Lemma 9 we get
\[
\left| \tilde{C}_{n,q}^* (f; x) \right| 
\leq \left| C_{n,q}^* (f; x) \right| + 2 \|f\|
\leq 3 \|f\|.
\]

Thus for \( x \in [0,b_n] \) and \( n \geq 2 \) we have,
\[
\left| C_{n,q}^* (f; x) - f(x) \right| 
\leq \left| \tilde{C}_{n,q}^* (g - f; x) \right| + \left| \tilde{C}_{n,q}^* (g(x) - g(x)) \right|
+ |g(x) - f(x)| + \left| f(r_{n,q}(x)) - f(x) \right|
\]
\[
\leq 4 \|f - g\| + \left( \delta_n^2 + (r_{n,q}(x) - x)^2 \right) \|g''\|
\]
\[
+ \left| f(r_{n,q}(x)) - f(x) \right|.
\]

Taking the infimum on the right hand side over all \( g \in C^2[0,b_n] \) and using (19) we get the required result.

4 Conclusion

In this study we present a modification of Chlodowsky type q-Bernstein operators so that the modified operators preserve the test function \( e^2(x) = x^2 \). We get an estimate for the rate of convergence of these operators and then compared it with estimates of classical q-Bernstein- Chlodowsky operators. We obtained that on some appropriate intervals our modified operator has a better error estimation than that of classical one.

References:


