Rasch-Andrich Thresholds in Engineering Students’ Attitudes towards Learning Mathematics

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Abstract: Assessment of students’ attitudes is an important aspect of learning mathematics in order to ensure whether there is a connection between students’ liking and understanding of the subject and their attitudes towards the subject. Assessment of attitudes is a major issue, and many experts are still debating about how it should be measured. This paper intends to explain how to assess engineering students’ attitudes towards learning mathematics based on the Rasch-Andrich thresholds probability of five ordered categorical scales. This study was conducted on 342 Engineering students at four engineering faculties in Universiti Teknologi MARA (UiTM), Shah Alam, Malaysia. The study illustrates an application of the Rating Scales (Likert scale) in Rasch measurement model to measure students’ attitudes. The results illustrate that all categories are "modal" and most likely to be observed at some locations on the latent variable. In addition, all positions of category are consistent where functions and curves of category probability and item probability are applied, hence, providing clear assessment of students’ attitudes towards learning mathematics.


1 Introduction
Rasch measurement is model-based approach in a measurement that has become increasingly popular for scale construction in social sciences. Rasch measurement provides a way to convert ordinal observation into linear measures. It is applicable not only to dichotomous data but also to polytomous data. This can be accomplished by adding parameters describing the rating scale functioning to the basic Rasch model for dichotomous data. The two most widely used Rasch models for polytomous data are the rating scale model (RSM) and partial credit model (PCM) [1], [2], [3], and [4].

In attitude, morale and personality construct, it is common to provide one set of response alternatives like strongly disagree, disagree, neutral, agree and strongly agree for use with all items. When the same set of response categories is used with every item in a questionnaire, the instrument is usually called a rating scale. A rating response mechanism for ordered categories is related to threshold formulation where the person and item parameters can be interpreted in terms of thresholds on a latent continuum [2], and [3].

The Rasch-Andrich threshold is defined as the difficulty of observing category $k$ relative to category $k-1$. It can be seen as the points of which categories $k$ and $k-1$ are equally likely to be observed in the continuum. There is a modal perspective on the category boundaries on the latent variable if the categories are visible and ordered. The researchers who illustrate that the Rasch-Andrich thresholds at category probability curves and item probability curves (conditional probability curves) are able to evaluate positions of categories [4], [5], [6], [7], and [8].

Some researchers applied the Rasch model threshold and scale statistics as a tool to find place categories into an empirically optimal ordered to assess the match between theoretical and empirical category orders [9]. Rasch-Andrich threshold used for evaluating functional capacity evaluation (FCEs) while applied Rasch-Andrich threshold in their research to determine positions of response categories [10], [11], and [12].

2 Mathematical Representations for Rating Scale Model
To accommodate multiple response categories for each item, the RSM [13], [14], and [15] adds a threshold parameter ($\tau$) to the model to represent
the relative difficulty of transitioning between rating categories. These threshold parameters are located at the intersections of the probability curve of one rating category with the probability curve of the next rating category. \( \beta_n \) is the location of the person \( n \) and \( \delta_i \) is the location of item \( i \) and \( x \in \{0,1,2,\ldots,m\} \). The RSM can be described mathematically as

\[
\Pr\{0; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -\tau_0 + 0(\beta - \delta) \right]
\]

(1)

\[
\Pr\{1; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1) + 1(\beta - \delta) \right]
\]

(2)

\[
\Pr\{2; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1 + \tau_2) + 2(\beta - \delta) \right]
\]

(3)

\[
\Pr\{3; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1 + \tau_2 + \tau_3) + 3(\beta - \delta) \right]
\]

(4)

\[
\Pr\{4; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4) + 4(\beta - \delta) \right]
\]

(5)

\[
\Pr\{x; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1 + \tau_2 + \tau_3 + \ldots + \tau_x) + x(\beta - \delta) \right]
\]

(6)

\[
\Pr\{m; \beta, \delta, \tau\} = \frac{1}{\gamma} \exp\left[ -(\tau_0 + \tau_1 + \tau_2 + \tau_3 + \ldots + \tau_m) + m(\beta - \delta) \right]
\]

(7)

\[
\gamma = \sum_{k=0}^{m} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta - \delta) \right] \quad \text{and} \quad \sum_{x=0}^{m} \tau_x = 0
\]

\[
\Pr\{X_{ni} = x, x > 0\} = \frac{1}{\gamma_{ni}} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta_n - \delta_i) \right]
\]

(8)

\[
\Pr\{X_{ni} = 0\} = \frac{1}{\gamma_{ni}}
\]

(9)

For subsequent convenience, let \( \tau_0 = 0 \). Equation (8) can be written as the single expression

\[
\Pr\{X_{ni} = x, x > 0\} = \frac{1}{\gamma_{ni}} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta_n - \delta_i) \right]
\]

(10)

In a subsequent application, the model was written in the form equivalent to

\[
\Pr\{X_{ni} = x\} = \frac{1}{\gamma_{ni}} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta_n - \delta_i) \right]
\]

(11)

which the thresholds \( \tau_x \), \( x = 1,2,\ldots,m; \sum_{x=1}^{m} \tau_x = 0 \)

\[
\gamma = \sum_{k=0}^{m} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta_n - \delta_i) \right]
\]

(11)

\[
\gamma = 1 + \sum_{k=0}^{m} \exp\left[ -(\sum_{x=0}^{k} \tau_x) + k(\beta_n - \delta_i) \right]
\]

(11)

(Normalizing factor or denominator) were therefore subscripted by \( i \) as well as \( \tau_0 = 0 \) remains [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], and [25]. These models have become known as the rating scale model (Eq. 10). The probability of the first and every category is a function of the location of all thresholds. Therefore, all thresholds play role in denominator. Below is a graph of a model with 5 categories that is named category probability curve (CCCs).

3 Conditional Probability of Success at each Latent Threshold

Let \( P_{ni} \) denote the probability of passing the \( x-1 \)th threshold, for example, the threshold between response categories \( x-1 \) and \( x \), is

\[
P_{ni} = \frac{\Pr\{X_{ni} = x\}}{\Pr\{X_{ni} = x-1\} + \Pr\{X_{ni} = x\}}
\]

(12)
[4], [5], [6], [7] and [8] for \( x \in \{1, 2, \ldots, m\} \). So \( P_{\text{nlx}} \) is
the conditional probability of responding in the upper category, given that only two successive
categories are considered. In the dichotomous case, this is the only threshold, namely, the threshold
between categories one and two and its probability is defined by the logistic function of the Rasch
model (RM). As a straightforward generalization to the polytomous case, Andrich (1978) assumed
the same function for all threshold probabilities. For \( x = 1, 2, \ldots, m \), with one item parameter \( \tau_k \)
for each threshold \( x \), for a discrete random variable \( X \) with \( m + 1 \) categories, there are
\( m \) independent probabilities \( P(X = x) \), so in the case of ordered categories, \( m \) independent threshold probability be
defined. From earlier, it can be derived that the response probabilities have the form of \( P_{\text{nlx}} \) (Eq. 12)
is a function that is called item probability function. There are four curves for five responses categories
that it can be seen in following graph

![Graph showing threshold probability curves for 5 categories](image)

**Fig.2 Threshold Probability Curves or Conditional Probability Curves for 5 Categories**

### 4 Rasch-Andrich Thresholds in Rating Scale

The rating scale case is an extension of the dichotomous case; moving from one category to
another implies a dichotomous choice between two levels. Let \( P_{\text{nlx}} \) and \( P_{\text{nlx-1}} \) denote the probabilities of
scoring \( x \) and \( x-1 \) on item \( i \) for a person \( n \), respectively, \( \beta \) denote person \( n \’s \) ability, and
\( \delta_k \) denote the \( k \)-th step difficulty of the item \( i \).

Under the partial credit model (Masters, 1982), it is assumed: where after reparameterization, \( \delta \) is the
mean of the step difficulties in item \( i \) and is called the overall difficulty, and \( \tau_k \) is the \( k \)-th deviation
from the mean and is called the \( k \)-th threshold for item \( i \). Suppose the item \( i \) has \( m + 1 \) categories and they are scored as 0, 1, 2, \ldots, \( m \), and then there will be \( m \) step difficulties of \( \delta_k \) for that item. The first step
difficulty describes how difficult it is by moving from category 1 (scoring 0) to category 2 (scoring 1), the second step difficulty describes how difficult it is by moving from category 2 (scoring 1) to
category 3 (scoring 2), and so on, the \( m \)-th step difficulty describes how difficult it is by moving from category \( m \) to category \( m + 1 \). Because the thresholds are identified with each item, it is often
convenient to consider the thresholds of an item as deviations from its overall location. Let
\[ \delta_k = \delta_i + \tau_k \]

where \( \sum_{k=0}^{m} \tau_k = 0 \). Then \( \delta_i \) is the mean of the thresholds \( \delta_k \) and \( \tau_k \) are thresholds which are
deviations from \( \delta_i \). This gives Eq. (10) in the form

\[
\Pr\{X_{ni} = x\} = e^{x(\beta_n - \delta_i - \tau_k)} / \gamma_{ni} = e^{x(\beta_n - \delta_i)} / \gamma_{ni} \quad (13)
\]

This is also a convenient form of the model for estimation of the threshold parameters which can be
reparameterised into principal components of which \( \delta_i \) is the first principal component. Thresholds
are the boundaries between categories. Again they can be conceptualized in various ways. Indeed,
thresholds can be seen as the "Rasch-Andrich threshold" is a parameter of a Rasch rating scale
model. Here is such a model:

\[
\log \left( \frac{P_{\text{nlx}}}{P_{\text{nlx-1}}} \right) = \beta_n - \delta_i - \tau_k \quad (14)
\]

Thresholds \( \{\tau_k\} \) are the locations along the latent variable, relative to the item difficulty, at which
categories \( k - 1 \) and \( k \) are equally likely to be observed. These \( \{\tau_k\} \) are known as "step measures",
"step calibrations", and "Rasch-Andrich thresholds".

### 5 Thresholds Locations in Category Probability and Response Probability

Equation (15) is a functions for category probability as following [8], [14], [15], and [23]. Also,
thresholds as in [14] and [15] assumed the function (16) for all thresholds
\[
\Pr \{ X_m = x \} = \frac{\exp \left[ -\sum_{i=0}^{k} \tau_i + k (\beta_n - \delta) \right]}{\sum_{k=0}^{m} \exp \left[ -\sum_{i=0}^{k} \tau_i + k (\beta_n - \delta) \right]}
\] (15)

probability for \( k \in \{1,2,\ldots,m\} \)

\[
P_{\text{mix}} = \frac{\Pr \{ X_m = x \} \Pr \{ X_m = x-1 \} + \Pr \{ X_m = x \} \Pr \{ X_m = x+1 \}}{1 + \exp \left[ \sum_{i=0}^{k} \tau_i + (\beta_n - \delta) \right]}
\] (16)

\( x \in \{1,2,\ldots,m\} \) [4], [5], [6], [7] and [8],

Fig.3 Category Probability Curves and Threshold Probability Curves for 4 Thresholds

Figure 3 shows curves from the threshold probabilities (\( P_{\text{mix}} \), black curves) and the category probabilities (\( \Pr \{ X_m \} \), blue curves) as a function of the person parameter. It can be seen that the points of inflection of the threshold probabilities and the intersection points of corresponding category probabilities have the same location on the continuum. In addition, these points define a partition of the latent continuum into \( m+1 \) intervals, namely,

\[
I_{i0} = ]-\infty, \tau_1], \ I_{i1} = [\tau_1, \tau_2], \ldots, I_{im} = [\tau_m, +\infty[\]
\]

Within each of these intervals, one of the response categories has the highest choice probability; in some sense the response scale is mapped onto the latent scale; however, this is only the case if all thresholds are ordered, i.e., if their parameters increase with the category numbers. Andrich has pointed out that ordered thresholds are a necessary condition for measuring a trait using items with an ordinal response format [26] and [27].

6 Interval Advance for Rasch-Andrich Threshold

6.1 Thresholds are advance monotonically with the categories

In rating scale, threshold \( \tau_i \), does not have the subscript of \( i \), suggesting all the items share the same set of thresholds. Under the Andersen formulation the parameters do not have such a simple graphical interpretation. The parameter for each category is the sum of masters’ item parameters up to that category.

When using the rating scale model we generally parameterize the difficulty of achieving a score of \( j \) on item \( i \) and represent it with \( \delta_{ik} \). That is, \( \delta_{ik} \) is the proficiency level required to expect an equal chance of responding in category \( k \) or in category \( k-1 \) on item \( i \). Alternatively, we might think of the average of the \( \delta_{ik} \) as an overall item difficulty, and the step difficulties as each step's deviance from the average. In looking at item difficulties in this way we are saying that each \( \delta_{ik} \) can be formulated as \( \delta_i + \tau_k \), where \( \tau_k \) is the deviance from the average item difficulty for item \( i \) at step \( j \). Note that in this case the last tau parameter is equal to the negative sum of the others so that the sum of all the tau parameters equals zero. For example, consider an item with five categories, scored 0, 1, 2, 3, 4. The mean of the \( m \) step parameters is called \( \delta_i \), which is the overall difficulty. Under the master’s formulation of the model the item parameters, \( \delta_{ik} \) \( k=1,2,\ldots,m \) are the points at which \( P(X_m = k-1) = P(X_m = k) \) that is, they are the intersection points of the successive pairs of category probability curves. Under the Andrich formulation, the item difficulty parameter, \( \delta_i \), is the point at which \( P(X_m = 0) = P(X_m = m) \) that is, the intersection point of the highest and lowest categories, and \( \tau_k \), \( k=1,2,\ldots,m \) are the distances between \( \delta_i \) and each of the intersection points of the successive pairs of category probability curves, respectively. In addition, it can be shown that the Andrich item difficulty parameter \( \delta_i \) is the average of the set of master’s item parameters.

Therefore, there is

\[
\delta_i = \frac{\delta_{i1} + \delta_{i2} + \ldots + \delta_{im}}{m}
\] (17)

and \( \tau_1 = \delta_{i1} - \delta_i \)
\[
\tau_2 = \delta_{i2} - \delta_i \\
\tau_3 = \delta_{i3} - \delta_i \\
\tau_m = \delta_{im} - \delta_i
\]

According to Andrich there are \( \delta_{i1} < \delta_{i2} < \ldots < \delta_{im} \) in
ordered category. There are
\[ \delta_{1} - \delta_{i} < \delta_{i+1} - \delta_{i} < \ldots < \delta_{m} - \delta_{i} \Rightarrow \tau_{1} < \tau_{2} < \ldots < \tau_{m} \]
\[ \sum_{k=1}^{m} \tau_{k} = \sum_{k=1}^{m} \delta_{k} - m\delta_{i} + \frac{1}{m} \sum_{k=1}^{m} \tau_{k} = \sum_{k=1}^{m} \delta_{k} - m\delta_{i} \]
\[ \Rightarrow \sum_{k=1}^{m} \tau_{k} = \frac{1}{m} \sum_{k=1}^{m} \delta_{k} - \frac{m}{m} \delta_{i} = \delta_{i} - \delta_{i} = 0 \Rightarrow \sum_{k=1}^{m} \tau_{k} = 0 \]
(18) [14], and [23]. Therefore thresholds are monotonically increasing in ordered categories for rating scale. This assertion corresponds to probability characteristic curves, like those in which each category in turn is the most probable, i.e., modal. These probability curves look like a range of hills. The extreme categories always approach a probability of 1.0. Asymptotically, because the model specifies that respondents with infinitely high (or low) measures must be observed in the highest (or lowest) categories [28].

6.2 Boundary of among Rasch-Andrich Thresholds’ Intervals
Thresholds show the points where the probability of a response of either 0 or 1, and finally \( m \) or \( m+1 \) respectively, are equally likely. This conceptualizes the rating scale as a set of dichotomous items. Under Rasch model conditions, a test of \( m \) independent dichotomous items is always mathematically equivalent to a rating scale of \( m+1 \) categories. But a rating scale of \( m+1 \) category is only equivalent to test of \( m \) independent dichotomous items under specific conditions.

6.2.1 Minimum of boundary
The Rasch-Andrich thresholds will be ordered if the thresholds will ascend with the category numbers. We need four thresholds of five categories of rating scale. Advances of at least 1.0 logit between thresholds or step calibrations are required. Thresholds that advance too little suggest that the two categories are not separable. When two thresholds are too close together, combining the two categories together might indicate more meanings to the selected categories with attention undertaken on the measured construct [29] and [30].

6.2.2 Maximum of boundary
Step difficulties advance should be less than 5.0 logits. When a category represents a very wide range of performance, so that is too far apart in the middle of the category is an indication of a “dead zone” where the measurement loses its precision. This suggests there is a need for an additional rating category between these categories in order to increase precise calibrations [3], [29] and [30]. Thus, in ordered category there are relations between the Rasch-Andrich thresholds
\[ 1 \leq |\tau_{m+1} - \tau_{m}| \leq 5 \] (19)

7 Methodology
In this study, a survey method was used to gather primary data from 342 undergraduate students at four engineering faculties namely, mechanical, chemical, and electrical and civil engineering in UiTM, Shah Alam. The students were intercepted at the end of their class lesson. The data was obtained using the attitude instrument based on the modified Aiken Scales towards learning Calculus. Students provided their responses to 20 items relating to their attitudes toward learning mathematics. The items were constructed according to the five-point Likert scale category ranging from strongly disagree, disagree, neutral, agree and strongly agree.

8 Results
The RSM-based analysis was applied by using the computer program WINSTEPS 3.81.0 [31]. The purpose is to look for the relationship between Rasch-Andrich thresholds location in category probability curves (CCCs) and threshold probability curves (TCCs). The results of the Rasch-Andrich thresholds status are shown in Table 1.

<table>
<thead>
<tr>
<th>CATEGORY LABEL</th>
<th>RASCH-ANDRICH THRESHOLD BOUNDARY</th>
<th>CATEGORY NUMBER</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD</td>
<td>NONE</td>
<td>-1.82</td>
</tr>
<tr>
<td>D</td>
<td>-1.82</td>
<td>-0.99</td>
</tr>
<tr>
<td>N</td>
<td>-0.83</td>
<td>0.02</td>
</tr>
<tr>
<td>A</td>
<td>0.19</td>
<td>2.27</td>
</tr>
<tr>
<td>SA</td>
<td>2.46</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 show that The Rasch-Andrich Thresholds can be approximated by \( \{-1.82, -0.83, 0.19, 2.46\} \) as shown by \( \tau \) which increase with the category numbers. There are four ordered Rasch-Andrich threshold that increase with category numbers, i.e.,
-1.82 < -0.83 < 0.19 < 2.46 as in \( \tau_i < \tau_s < \tau_i < \tau_s \). In this example, they are located on the latent variable at logits relative to the overall item difficulty. The overall item difficulty \( \delta_i \) which can be seen in Eq. 17.

From Eq. 18 there is

\[
\sum_{k=1}^{m} \tau_k = \tau_1 + \tau_2 + \tau_3 + \tau_4 = -1.82 - 0.83 + 0.19 + 2.46 = 0
\]

TCCs in Figure 5 show the conditional probability of success at each latent threshold while CCCs in Figure 4 display the probability of each response category across the whole continuum. Figure 6 is a combination of both TCCs and CCCs curves. From Figure 6, it can be seen that the location of Rasch-Andrich thresholds is the same in TCCs and CCCs curves.

Figures 4, 5 and 6 show the threshold probabilities in CCCs( \( \{ C_1, C_2, C_3, C_4 \} \) as well as the TCCs (\( \{ T_1, T_2, T_3, T_4 \} \) as Rasch-Andrich thresholds. \( \{ C_1, C_2, C_3, C_4 \} \) are the points of intersection between the conditional probability and 0.5 probability line. The points at which adjacent categories are equally probable (indicated by arrows) are the Rasch-Andrich thresholds or step calibrations. They are the response-structure parameters of the RSM model. The Rasch-Andrich thresholds (arrowed) between categories A and SA, and categories SA, are in ascending order of the categories of the latent variable. This is termed "ordered". So, overall, the Rasch-Andrich thresholds are ordered for this example.

In addition, these points define a partition of the latent continuum into \( m + 1 \) intervals, namely,

\[
I_0 = [-\infty, -1.82], I_1 = [-1.82, -0.83],
I_2 = [-1.82, -0.83], I_3 = [0.19, 2.46], I_4 = [2.46, +\infty]
\]

That \( I_m \) shows boundaries in Table 1.

On the other hand, from Table 1 (boundary), the intervals correspond to the tops of the hill is shown in Figure 4. These are SD: \(-\infty \) to \(-1.82 \); D: \(-1.82 \) to \(-0.83 \); N: \(-0.83 \) to \(0.19 \); A: \(0.19 \) to \(2.46 \); SA: \(2.46 \) to \(+\infty \). This can be summarized as

\[
\{-1.82, -0.83, 0.19, 2.46\}
\]

When the Rasch-Andrich thresholds are ordered, then the modal values coincide with those threshold values, and all categories appear on the latent variable.
RSM category probability curves that are shown in Figure 4 has five visible curves. Starting from the left, SD is most likely to be observed as low-measure respondents. Then as respondent measures increase, the probability of observing D increases. With increasing measure, category N becomes most probable, then A, and finally SA.

\[ \text{Boundary intervals} = \{-1.82, -0.99, 0.02, 2.27\} \]

is a set of distances between the thresholds which are located between categories 1 to 5. All the members of boundary intervals set are between 1 and 5 (\(-0.99\approx 1\)). In this research, as mentioned in section 6.2, all thresholds intervals follow the acceptable distances. Therefore, categories are nicely position.

9 Conclusions
This research is undertaken to estimate students’ attitudes towards learning mathematics using the Rasch-Andrich threshold. Based on the modified Aiken Attitude Scales towards learning Calculus, 342 undergraduate students from four engineering faculties at UiTM Shah Alam have provided their responses. The locations of their attitudes were shown in the Rasch-Andrich threshold where the conditional probability curves intersect the 0.5 probability line. This study shows the conditional probabilities of observing adjacent categories. These are a series of Rasch dichotomous ogives. The intercepts with 0.5 probabilities line are the Rasch-Andrich thresholds. They are at the intersection of adjacent probability curves where the probability of being observed in the higher category starts to exceed that of being observed in the adjacent lower category. This considers the two categories at a time in each threshold. The results of this study have provided an important insight in the function of the conditional probability in assessing students’ attitudes based on the conditional probability of ordered rating scale categories. The researchers able to determine by Rasch-Andrich thresholds have identified which category is modal and nicely position. This study also shows that all categories curves are visible. So, the response categories are nicely position. In conclusion, students with increasing categories number, i.e., category 1(SD) to category 2 and then category 3(D, N) is most probable and visible then category 4 (A). Finally the last category (SA) is the most probable and visible in expressing their attitudes’ towards learning calculus.

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