A three-part mixed boundary value problem for a heated plate

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Abstract: The work presents an analytical solution for an axisymmetric heat conduction problem of a plate lying on a foundation covered by an annular thermal isolator. The plate is heated by a uniform temperature field on the opposite side over a circular region. The three-part mixed boundary value problem is reduced to a system of triple integral equations by the Hankel integral transforms method. Instead of the Fredholm integral equations approach method, with the help of an integral relation we develop the unknown functions into a series of Bessel functions product. Using the Gegenbauer addition formula for the Bessel function of zero order, we reduce the formulated problem to an infinite system of algebraic equations. Numerical results are also given for the temperature and the flux for different regions of the plate using the Gauss hypergeomertic function proprieties.

Key–Words: Heat conduction, axisymmetric problem, mixed boundary value problem, triple integral equations, Hankel integral transforms, Gegenbauer addition formula, infinite algebraic system.

1 Introduction

The problem of studying the heat conduction in solids is of interest since one encounters mechanical structures subjected to high temperature in different practical areas. During recent years considerable work have been done on calculating temperature in various geometrical configurations such as thick layers and infinite cylinders.
Many physical problems considering electrostatic potential, heat conduction, elastostatic and thermoelastic ones are formulated as mixed boundary values problems Duffy [1]. The Laplace equation is a fundamental tool in studying such problems.
The first works treating simply and doubly mixed boundary value problems for the Laplace equation were published by Dhaliwal [2] and [3]. The dual integral equations were reduced to Fredholm ones then the small parameter method was applied for their numerical solution. Later Mehta [4] gave a closed form solution in terms of hyper-geometrical Gauss function for a layer subjected to a flux over a circular region. The problem for a half-space with various mixed boundary conditions was solved by Lemczyk et al. [5] and [6] by using a Fourier series development for the obtained Abel integro-differential equation. A layered medium was also considered by these authors using the same method. The constriction resistance problem of an isothermal circular spot on a half-space and thick layers was considered by [7] and [8]. A numerical method was proposed for the solution of the corresponding Fredholm integral equation by expanding the kernel into an infinite series.

In this paper a heat conduction problem of a plate lying on a foundation covered by an annular thermal isolator has been considered. The used method is inspired on Shibuya papers dealing with crack and punch elastic problems [9], [10] and [11] . By this approach we reduce the triple integral equations directly to an infinite algebraic system equations.

2 Formulation of the problem

The problem under consideration studies the axisymmetric heat conduction problem of a plate lying on a foundation covered by an annular thermal isolator The crack is situated on the $z = 0$ plane with the inner and outer radii a and b, respectively. It may be treated as a thermal isolator. The heat propagation is due to a uniform temperature field of intensity δ prescribed over a circular region on the bottom medium surface $z = h$ whereas the outer surface is maintained at a zero temperature, as shown in Fig. 1.

The mathematical problem formulation of the equi-

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librium equation in a cylindrical coordinates system is given by

$$\Delta T(r,z)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)T(r,z)=0 \quad (1)$$

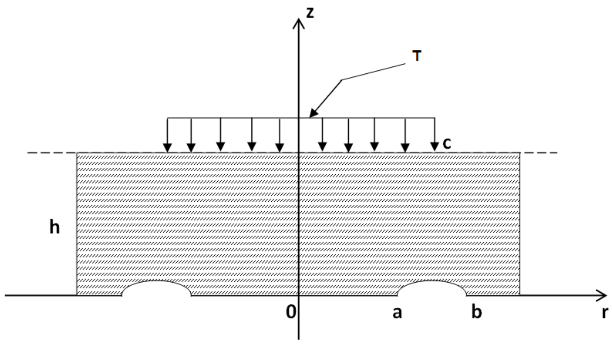

Fig. 1. Geometry of the problem

The boundary conditions of the problem may be written as

$$\begin{cases} T(r,z)|_{z=h}=\begin{cases}\delta, & r<c \quad (a)\\ 0, & r>c \quad (b)\end{cases} \\ \frac{\partial T}{\partial z}(r,z)\big|_{z=0}=0, \qquad a<r<b \qquad (c) \\ T(r,z)|_{z=0}=0, \qquad r<a \ \ and \ \ r>b \quad (d)\end{cases} \quad (2)$$

By using the Hankel transforms method, the solution of Eq. (1) is expressed as

$$\begin{aligned} T(r,z) &= \int_0^\infty \lambda\Big(A(\lambda)\,ch\lambda z \\ &\quad +B(\lambda)\,sh\lambda z\Big)J_0(\lambda r)\,d\lambda \end{aligned} \quad (3)$$

where A and B are functions of λ to be determined, and J_0 is the Bessel function of the first kind of order 0.

Using the non mixed boundary conditions (a) and (b) of Eq. (2), we obtain

$$\begin{aligned} A(\lambda)ch(\lambda h)+B(\lambda)sh(\lambda h) &= \int_0^c \delta r J_0(\lambda r)dr \\ &= \frac{\delta c}{\lambda}J_1(\lambda c) \end{aligned}$$

so

$$A(\lambda)=\frac{\delta c}{\lambda ch(\lambda h)}J_1(\lambda c)-B(\lambda)th(\lambda h)$$

The temperature can be written by

$$\begin{aligned} T(r,z) &= \int_0^\infty \Big[\frac{\delta c}{ch(\lambda h)}J_1(\lambda c)ch(\lambda z) \\ &\quad +\lambda B(\lambda)[sh(\lambda z) \\ &\quad -th(\lambda h)ch(\lambda z)]\Big]J_0(\lambda r)d\lambda \end{aligned} \quad (4)$$

Next, we determine the triple integral equations for calculating the unknown function $C(\lambda)$.

From the condition (c) of Eq. (2), we obtain

$$T(r,z)|_{z=0}=\frac{\delta c}{\lambda ch(\lambda h)}J_1(\lambda c)-B(\lambda)th(\lambda h)$$

Substituting in Eq. (3),we get

$$\int_0^\infty\left[\frac{\delta c}{ch(\lambda h)}J_1(\lambda c)-\lambda B(\lambda)th(\lambda h)\right]J_0(\lambda r)d\lambda=0$$

Putting now

$$\lambda C(\lambda)=\frac{\delta c}{ch(\lambda h)}J_1(\lambda c)-\lambda B(\lambda)th(\lambda h)$$

we get

$$\int_0^\infty \lambda C(\lambda)J_0(\lambda r)d\lambda=0 \qquad r<a \text{ and } r>b \quad (5)$$

On the surface $z=0$ and from the condition (c) of Eq. (2), the flux expressed by

$$\frac{\partial T}{\partial z}(r,z)\bigg|_{z=0} = 0 \Longrightarrow \int_0^\infty \lambda^2 B(\lambda)J_0(\lambda r)d\lambda=0$$

and in function of $C(\lambda)$, we obtain

$$\int_0^\infty \lambda^2 C(\lambda)\frac{1}{th(\lambda h)}J_0(\lambda r)d\lambda = \delta c\int_0^\infty \frac{\lambda}{sh(\lambda h)}J_1(\lambda c)J_0(\lambda r)d\lambda \qquad a<r<b \quad (6)$$

From the Eq. (5) and Eq. (6), we obtain the following triple integral equations for determining the auxiliary function $C(\lambda)$

$$\begin{cases} \int_0^\infty \lambda C(\lambda) J_0(\lambda r) d\lambda = 0, \qquad r < a \;\; and \;\; r > b \\ \int_0^\infty \lambda^2 C(\lambda) \frac{1}{th(\lambda h)} J_0(\lambda r) d\lambda = \\ \delta c \int_0^\infty \frac{\lambda}{sh(\lambda h)} J_1(\lambda c) J_0(\lambda r) d\lambda, \qquad a < r < b \end{cases} \tag{7}$$

In our solution method we use the following integral formula

$$\int_0^\infty \lambda J_0(\lambda r) J_n(\lambda c) J_n(\lambda d)\, d\lambda = \begin{cases} 0, & r < a,\ r > b \\ \frac{\cos n\omega}{\pi c d \sin\omega}, & a < r < b \end{cases} \tag{8}$$

where

$$\omega = arccos\Big(\frac{c^2 + d^2 - r^2}{2cd}\Big)$$

and

$$c = \frac{b+a}{2},\ d = \frac{b-a}{2}$$

Putting $Z_n(\lambda) = J_n(\lambda c) J_n(\lambda d)$,
and taking $G_n(\lambda) = \lambda\left(Z_{n-1}(\lambda) - Z_{n+1}(\lambda)\right)$, we obtain

$$\int_0^\infty J_0(\lambda r) G_n(\lambda)\, d\lambda = \begin{cases} 0, & r < a,\ r > b \\ \frac{2\sin n\omega}{\pi c d}, & a \le r \le b \end{cases} \tag{9}$$

It easy to remark that the first equation of (7) is automatically satisfied by choosing

$$\lambda C(\lambda) = \sum_{n=1}^\infty a_n G_n(\lambda) \tag{10}$$

Substituting Eq. (10) into Eq. (6), we obtain

$$\sum_{n=1}^\infty a_n \int_0^\infty \frac{\lambda}{th(\lambda h)} J_0(\lambda r) G_n(\lambda) d\lambda = \delta c \int_0^\infty \frac{\lambda}{sh(\lambda h)} J_1(\lambda c) J_0(\lambda r) d\lambda \qquad a < r < b \tag{11}$$

For getting the unknowns coefficients a_n, we use the following Gegenbauer formula

$$J_0(\lambda r) = J_0(\lambda c) J_0(\lambda d) + 2\sum_{m=1}^\infty J_m(\lambda c) J_m(\lambda d) \cos m\omega, \quad a < r < b \tag{12}$$

Substituting Eq. (12) into Eq. (11), we obtained

$$\sum_{n=1}^\infty a_n \int_0^\infty \frac{\lambda}{th(\lambda h)} G_n(\lambda) J_m(\lambda c) J_m(\lambda d) d\lambda = \delta c \int_0^\infty \frac{\lambda}{sh(\lambda h)} J_1(\lambda c) J_m(\lambda c) J_m(\lambda d) d\lambda \tag{13}$$

Remarking that the matrix associated to this system is not symmetric. For the simplicity of numerical treatment, taking the difference between the m-th and the (m+2)-th equations, we get the following result that has the symmetric matrix

$$\sum_{n=1}^\infty a_n \int_0^\infty \frac{1}{th(\lambda h)} G_n(\lambda) G_m(\lambda) d\lambda = \delta c \int_0^\infty \frac{1}{sh(\lambda h)} J_1(\lambda c) G_m(\lambda) d\lambda \tag{14}$$

Finally, the unknown coefficients a_n are determined by solving Eq. (14)

In a matrix form, Eq. 14 can be expressed by

$$\sum_{n=1}^\infty a_n A_{mn} = \delta c B_m, \qquad m = 1, 2, 3, ...$$

where

$$A_{mn} = \int_0^\infty \frac{1}{th(\lambda h)} G_n(\lambda) G_m(\lambda) d\lambda$$

and

$$B_m = \int_0^\infty \frac{1}{sh(\lambda h)} J_1(\lambda c) G_m(\lambda) d\lambda$$

3 Temperature and heat conduction problem

The temperature and the flux on the interface $z=0$ are given in a closed form by using some integral relation.

On the surface $z=0$, the temperature can be expressed using Eq. (9) as follows

$$\begin{aligned} T(r,z)|_{z=0} &= \int_0^\infty \lambda C(\lambda) J_0(\lambda r)\, d\lambda \\ &= \sum_{n=1}^{\infty} a_n \int_0^\infty G_n(\lambda) J_0(\lambda r)\, d\lambda \\ &= \begin{cases} 0, & r<a,\ r>b \\ \frac{2}{\pi cd}\sum\limits_{n=1}^{\infty} a_n \sin n\varphi, & a<r<b \end{cases} \end{aligned} \tag{15}$$

On the surface $z=0$, the flux can rewritten as follows

$$\begin{aligned} \frac{\partial T}{\partial z}(r,z)\Big|_{z=0} &= \sum_{n=1}^{\infty} a_n \int_0^\infty \lambda \xi(\lambda) \\ &\quad \times G_n(\lambda) J_0(\lambda r)\, d\lambda \\ &\quad -\delta c \int_0^\infty \eta(\lambda) J_0(\lambda r)\, d\lambda \\ &= \sum_{n=1}^{\infty} a_n \Big[\int_0^\infty \lambda G_n(\lambda) J_0(\lambda r)\, d\lambda \\ &\quad - \int_0^\infty \lambda \big[1-\xi(\lambda)\big] G_n(\lambda) \\ &\quad \times J_0(\lambda r)\, d\lambda \Big] \\ &\quad -\delta c \int_0^\infty \eta(\lambda) J_0(\lambda r)\, d\lambda \\ &= \sum_{n=1}^{\infty} a_n \int_0^\infty \lambda G_n(\lambda) J_0(\lambda r)\, d\lambda \\ &\quad - \sum_{n=1}^{\infty} a_n \int_0^\infty \lambda \big[1-\xi(\lambda)\big] G_n(\lambda) \\ &\quad \times J_0(\lambda r)\, d\lambda \\ &\quad -\delta c \int_0^\infty \eta(\lambda) J_0(\lambda r)\, d\lambda \end{aligned} \tag{16}$$

where

$$\xi(\lambda) = \frac{1}{th(\lambda h)}$$

and

$$\eta(\lambda) = \frac{\lambda}{sh(\lambda h)} J_1(\lambda c)$$

the first term of the right-hand side in Eq. 16, can expressed by (cf. appendix.A)

$$\int_0^\infty \lambda J_0(\lambda r) G_n(\lambda) d\lambda = -\frac{2n}{cd}\left[I_0^n + r\frac{\partial}{\partial r} I_0^n \right] \tag{17}$$

where

$$I_0^n = \int_0^\infty J_0(\lambda r) J_n(\lambda c) J_n(\lambda d)\, d\lambda$$

Thus from Eq. 16 we get

$$\begin{aligned} \frac{\partial T}{\partial z}(r,z)\Big|_{z=0} &= -\frac{2}{cd}\sum_{n=1}^{\infty} n a_n \left[I_0^n + r\frac{\partial}{\partial r} I_0^n \right] \\ &\quad - \sum_{n=1}^{\infty} a_n \int_0^\infty \lambda e^{-2\lambda h} \\ &\quad \times G_n(\lambda) J_0(\lambda r)\, d\lambda \\ &\quad -\delta l \int_0^\infty \eta(\lambda) J_0(\lambda r)\, d\lambda \end{aligned} \tag{18}$$

In order to evaluate the integral in Eq. 17, we use the following integral formula (cf. appendix. B)

$$\begin{aligned} &\int_0^\infty J_\xi(\lambda t) J_\mu(\lambda x) J_\nu(\lambda y)\, d\lambda = \\ &\frac{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}{\Gamma(\xi+1)\Gamma(\mu+1)\Gamma\left(\frac{1-\sigma+\nu}{2}\right)} \left(\frac{t}{y}\right)^{\xi} \left(\frac{x}{y}\right)^{\mu} \left(\frac{1}{y}\right) \\ &\times F\left(\frac{1+\sigma-\nu}{2}, \frac{1+\sigma+\nu}{2}; \xi+1; \sin^2\varphi\right) \\ &\times F\left(\frac{1+\sigma-\nu}{2}, \frac{1+\sigma+\nu}{2}; \mu+1; \sin^2\psi\right) \end{aligned} \tag{19}$$

F is a hyperbolic Gauss function, φ and ψ are given by

$$\begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \frac{1}{2}\left[\arcsin\left(\frac{x+t}{y}\right) \pm \arcsin\left(\frac{x-t}{y}\right) \right] \tag{20}$$

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We also use the relation (cf. appendix. C)

$$\begin{aligned}&\frac{\partial}{\partial\varphi}F\left(\alpha,\beta;\gamma;\sin^2\varphi\right)=\\&\frac{\alpha\beta}{\gamma}\sin 2\varphi F\left(\alpha+1,\beta+1;\gamma+1;\sin^2\varphi\right)\end{aligned} \tag{21}$$

For that reason, we distinguish the two following cases

(Case 1) $(0 < r < a)$: replacing $\xi = 0,\ \ \mu = \nu = n$,
$x = r,\ \ y = c$ and $t = d$.
Since $r < b$ we find

$$\begin{bmatrix}\psi\\\varphi\end{bmatrix}=\frac{1}{2}\left[\arcsin\left(\frac{r+d}{c}\right)\pm\arcsin\left(\frac{r-d}{c}\right)\right] \tag{22}$$

Then we get

$$\begin{bmatrix}\frac{\partial\psi}{\partial r}\\ \\ \frac{\partial\varphi}{\partial r}\end{bmatrix}=\frac{1}{2}\left[\frac{1}{\sqrt{(a-r)(b+r)}}\pm\frac{1}{\sqrt{(b-r)(a+r)}}\right] \tag{23}$$

From Eq. 19, we find that

$$\begin{aligned}I_0^n &= \int_0^\infty J_0(\lambda r)\,J_n(\lambda c)\,J_n(\lambda d)\,d\lambda\\&=\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}\frac{1}{c}\left(\frac{d}{c}\right)^n\\&\quad\times F\left(\frac{1}{2},n+\frac{1}{2};1;\sin^2\varphi\right)\\&\quad\times F\left(\frac{1}{2},n+\frac{1}{2};n+1;\sin^2\psi\right)\end{aligned} \tag{24}$$

and

$$\begin{aligned}\frac{\partial I_0^n}{\partial r} &= \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}\frac{1}{c}\left(\frac{d}{c}\right)^n\frac{1+2n}{8}\\&\quad\times\Bigg\{\sin 2\psi F\left(\frac{3}{2},n+\frac{3}{2};2;\sin^2\psi\right)\\&\quad\times F\left(\frac{1}{2},n+\frac{1}{2};n+1;\sin^2\varphi\right)\\&\quad\times\left[\frac{1}{\sqrt{(a-r)(b+r)}}\right.\\&\quad\left.-\frac{1}{\sqrt{(b-r)(a+r)}}\right]\\&\quad+\frac{\sin 2\varphi}{(n+1)}F\left(\frac{3}{2},n+\frac{3}{2};n+2;\sin^2\varphi\right)\\&\quad\times F\left(\frac{1}{2},n+\frac{1}{2};1;\sin^2\psi\right)\\&\quad\times\left[\frac{1}{\sqrt{(a-r)(b+r)}}\right.\\&\quad\left.+\frac{1}{\sqrt{(b-r)(a+r)}}\right]\Bigg\}\end{aligned} \tag{25}$$

(Case 2) $(r > b)$: by setting $\nu = 0$,
$\mu = \xi = n,\ x = c,\ \ y = r$ and $t = d$, we find

$$\begin{bmatrix}\psi\\\varphi\end{bmatrix}=\frac{1}{2}\left[\arcsin\left(\frac{b}{r}\right)\pm\arcsin\left(\frac{a}{r}\right)\right] \tag{26}$$

The derivative functions are given by

$$\begin{bmatrix}\frac{\partial\psi}{\partial r}\\ \\ \frac{\partial\varphi}{\partial r}\end{bmatrix}=\frac{-1}{2r}\left[\frac{b}{\sqrt{r^2-b^2}}\pm\frac{a}{\sqrt{r^2-a^2}}\right] \tag{27}$$

The result of the desired integrals are calculated explicitly as follows

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$$\begin{aligned} I_0^n &= \int_0^\infty J_0(\lambda r)\, J_n(\lambda c)\, J_n(\lambda d)\, d\lambda \\ &= \frac{(-1)^n}{\pi}\left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right]^2 \frac{1}{r}\left(\frac{cd}{r^2}\right)^n \\ &\quad \times F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\varphi\right) \\ &\quad \times F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\psi\right) \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{\partial I_0^n}{\partial r} &= \frac{(-1)^{n+1}}{2\pi}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)}\right]^2 \frac{1}{n+1}\frac{1}{r^2}\left(\frac{cd}{r^2}\right)^n \\ &\quad \left\{\frac{4(n+1)}{n+\frac{1}{2}} F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\varphi\right)\right. \\ &\quad \times F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\psi\right) \\ &\quad + \sin 2\varphi F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\varphi\right) \\ &\quad \times F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\psi\right) \\ &\quad \times \left[\frac{b}{\sqrt{r^2-b^2}} - \frac{a}{\sqrt{r^2-a^2}}\right] \\ &\quad + \sin 2\psi F\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; \sin^2\varphi\right) \\ &\quad \times F\left(n+\frac{3}{2}, n+\frac{3}{2}; n+2; \sin^2\psi\right) \\ &\quad \left.\times \left[\frac{b}{\sqrt{r^2-b^2}} + \frac{a}{\sqrt{r^2-a^2}}\right]\right\} \end{aligned} \tag{29}$$

4 Numerical calculations

Next, we give numerical calculations for solving the set of the simultaneous equations Eq. (14). For this purpose, first, we must evaluate the infinite integrals in Eq. (14)

$$A_{mn} = \int_0^\infty \frac{1}{th(\lambda h)} G_n(\lambda) G_m(\lambda) d\lambda$$

Since the term $\frac{1}{th(\lambda h)}$ converge to unity rapidly as the value of λ becomes large, A_{mn} may be expressed approximately by the equation

$$A_{mn} = \int_0^{\lambda_0} \frac{1}{th(\lambda h)} G_n(\lambda) G_m(\lambda) d\lambda + A'_{mn}$$

where $\lambda_0 = 1500$, and A'_{mn} is expressed by the equation

$$A'_{mn} = \int_{\lambda_0}^\infty G_n(\lambda) G_m(\lambda) d\lambda$$

The first integral of A_{mn} is calculated numerically by the Simpson's rule.

Using the integration by parts and introducing the integral sine and cosine functions, we can obtain the approximate formula for A'_{mn} (cf. appendix.D)

$$\begin{aligned} A'_{mn} &= \int_{\lambda_0}^\infty G_m(\lambda)\, G_n(\lambda)\, d\lambda \\ &\simeq \frac{4mn}{\pi^2 (cd)^3}\Big[a^2\Big(\frac{\sin^2\lambda_0 a}{\lambda_0} - a\, si\,(2\lambda_0 a)\Big) \\ &\quad + (-1)^{m+n} b^2\Big(\frac{\cos^2\lambda_0 a}{\lambda_0} + b\, si\,(2\lambda_0 b)\Big) \\ &\quad - [(-1)^m + (-1)^n]\, ab\Big(\frac{\sin\lambda_0 a\cos\lambda_0 b}{\lambda_0} \\ &\quad - c\, ci\,(2\lambda_0 c) + d\, ci\,(2\lambda_0 d)\Big)\Big] \end{aligned} \tag{30}$$

Where

$$si\,(x) = -\int_x^\infty \frac{\sin t}{t} dt$$

and

$$ci\,(x) = -\int_x^\infty \frac{\cos t}{t} dt$$

We have also

$$B_m = \int_0^\infty \frac{1}{sh(\lambda h)} J_1(\lambda c) G_m(\lambda) d\lambda$$

Since the term $\frac{1}{sh(\lambda h)} J_1(\lambda c)$ converge rapidly to zero as the value of λ becomes large, B_m may be expressed by the equation

$$B_m = \int_0^{\lambda_0} \frac{1}{sh(\lambda h)} J_1(\lambda c) G_m(\lambda) d\lambda, \qquad m = 1, 2, 3, \ldots$$

5 Numerical results

We obtain the values of the coefficients a_n in terms of the ratio $(H = h/a)$, by choosing $\lambda_0 = 1500$. The numerical results of a_n are given in tables 1-3, for different values of a/b.

$(a/b = 0.25)$		
$H = 0.5$	$H = 2$	$H = 5$
2.371679310824958	0.343372334094965	0.104827901994758
1.508130681397395	0.190802879858602	0.037072202762871
0.758301068986590	0.076302067359153	0.013634494388821
0.268689772806665	0.028672192046544	0.005748877682817
0.083849475375852	0.012122717797783	0.002670271879081
0.059048077985122	0.005924286716950	0.001311499959137
0.054875177733002	0.003082564493935	0.000665561015023
0.031873688316213	0.001611065311746	0.000343489944993
0.007920726843701	0.000823065528887	0.000175589697838
-0.001300889057132	0.000376476743483	0.000080707935720

Table 1. The coefficients with H

$(a/b = 0.5)$		
$H = 0.5$	$H = 2$	$H = 5$
0.095437486472127	0.023952911732979	0.004013488166229
0.051070554585159	0.005685871177039	0.000642143084823
0.012494608140570	0.001184598622265	0.000128877407387
0.000496371866156	0.000266396125642	0.000030715177399
-0.000361762409437	0.000066959265129	0.000008002236949
0.000110433045478	0.000018140565689	0.000002195226532
0.000121738120850	0.000005089497741	0.000000616880379
0.000023567469967	0.000001503722676	0.000000185757105
-0.000006962832870	0.000000368582772	0.000000040981389
-0.000002886299649	0.000000210920648	0.000000033382432

Table 2. The coefficients a_n with H

$(a/b = 0.75)$		
$H = 0.5$	$H = 2$	$H = 5$
0.010928126394834	0.001811571980819	0.0002364009719648
0.002135525945182	0.000163138710274	0.0000149578911645
0.000202603731088	0.000014226586636	0.0000012880576339
0.000004578572575	0.000001399517562	0.0000001325523842
-0.000000615036062	0.000000136122883	0.0000000126447772
0.000000186074559	0.000000026331492	0.0000000029884951
-0.000000300626002	-0.000000058921800	-0.0000000078584592
0.000000066869413	0.000000026696734	0.0000000040170233
-0.000000922948882	-0.000000161138742	-0.0000000213218373
0.000000265457692	0.000000088563288	0.0000000130718140

Table 3. The coefficients a_n with H

Note that the convergence of coefficients a_n becomes slow with the decreasing the parameter thickness H.

Having calculated the unknowns coefficients a_n, we obtain numerical values for the temperature and the heat flow.

The corresponding plots are shown in the figures 2-8 and given in function of $\rho = r/a$, for various values of $\xi = z/h$. As the graphs have nearly the same behaviour for the different values of a/b, we choose the case $a/b = 0.5$.

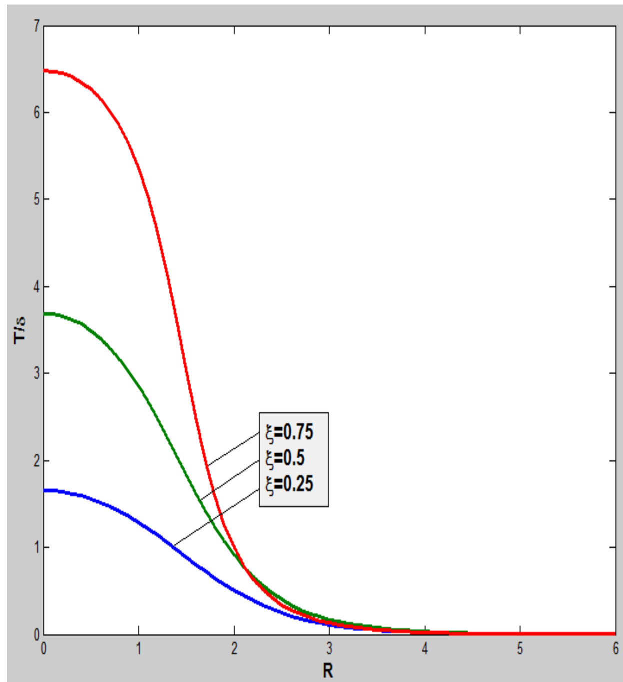

Fig. 2. Distribution of the temperature

Fig. 2 shows the variations of temperature with R for various values of $\xi = z/h$. The temperature decrease in $r < a$ $(R < 1)$ and very rapidly in $r \geq a$ $(R \geq 1)$, then tends to zero as the value of R becomes large. In the same figure, as the increasing of ξ, T decreases.

In Fig. 3, we show the variations of temperature with R for various small values of ξ (ξ approaches to 0). The temperature decreases in $r < a$ $(R < 1)$ but increases and tends T_{max} in $a \leq r \leq b$ $(1 \leq R \leq 2)$, and tends to zero as the value of R becomes large $(R > 2)$. In the same figure, we plot the variation of temperature between a and b, at $\xi = 0$ using Eq. (15) . As the increasing of ξ, T increases and tends T_{max}, but comeback to zero when $R = 2$ making a hyperbolic curve.

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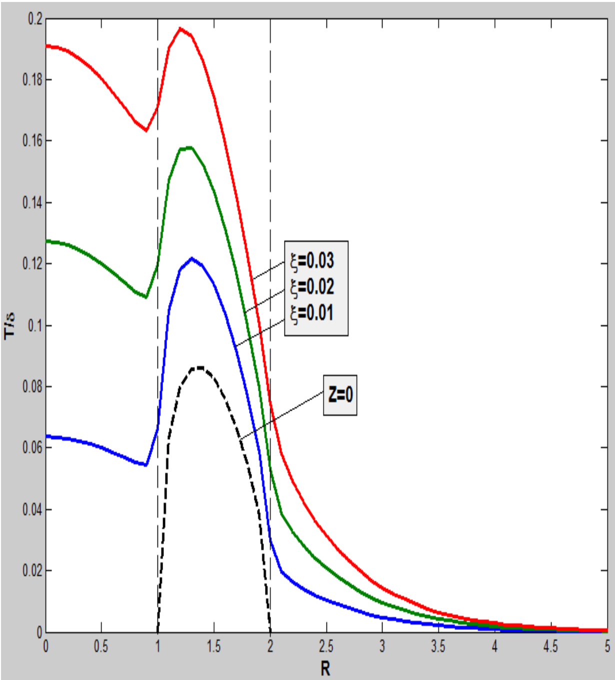

Fig. 3. Distribution of the temperature

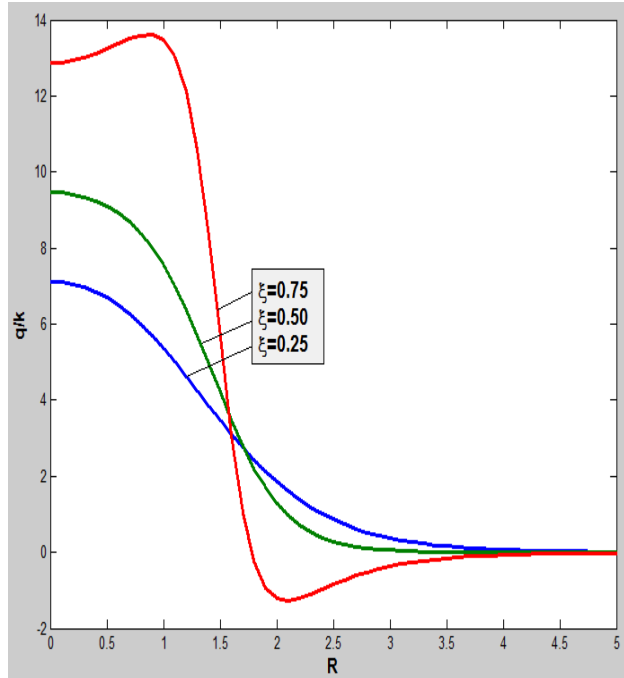

Fig. 5. Distribution of the flux

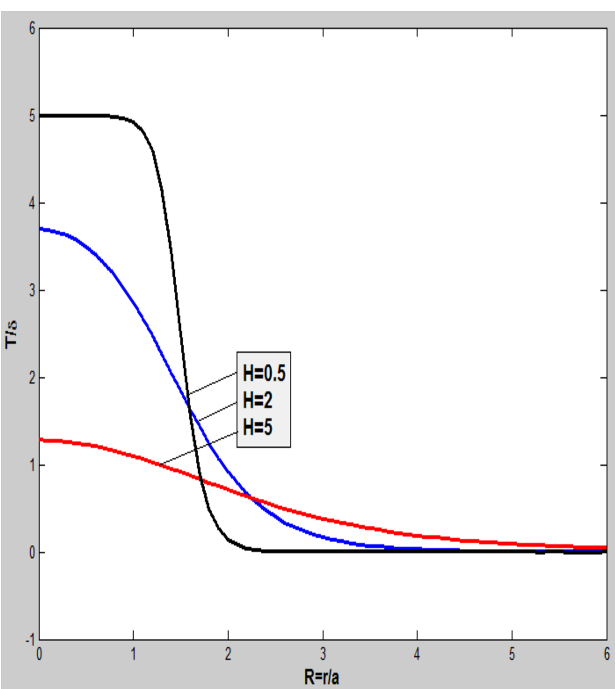

Fig. 4. Distribution of the temperature

Fig. 4 shows the variations of temperature with R for various values of H. the temperature decrease in $r < a$ $(R < 1)$ and very rapidly in $r \geq a$ $(R \geq 1)$, then tends to zero as the value of R becomes large. In the same figure, as the increasing of H, T/δ decreases.

Fig. 5 shows the variations of flux with R for various values of ξ. the flux decrease in $r < a$ $(R < 1)$ and very rapidly in $r \geq a$ $(R \geq 1)$, and tends to zero as the value of R becomes large. In the same figure, as the increasing of ξ, q decreases.

In Fig. 6, we show the variations of flux with R for various small values of ξ (ξ approaches to 0). The flux decreases in $r < a$ $(R < 1)$ and rapidly between a and b, and tends to zero as the value of R becomes large $(R > 2)$. In the same figure, we plot the variation of flux in $r < a$ and $r > b$, at $\xi = 0$ using Eq. 16 (the ones with dashed lines are obtained from the analytical formulas). When $r < a$, q/k decreases but increases rapidly when r close to a, and tends to infinity when $R = 1$ and $R = 2$, whereas q/k decreases and tends to zero when R becomes large $(R > 2)$.

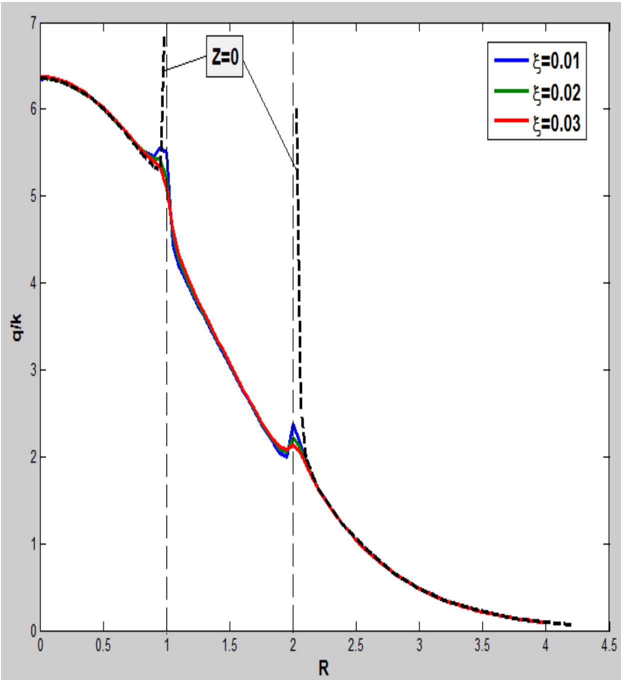

Fig. 6. Distribution of the flux

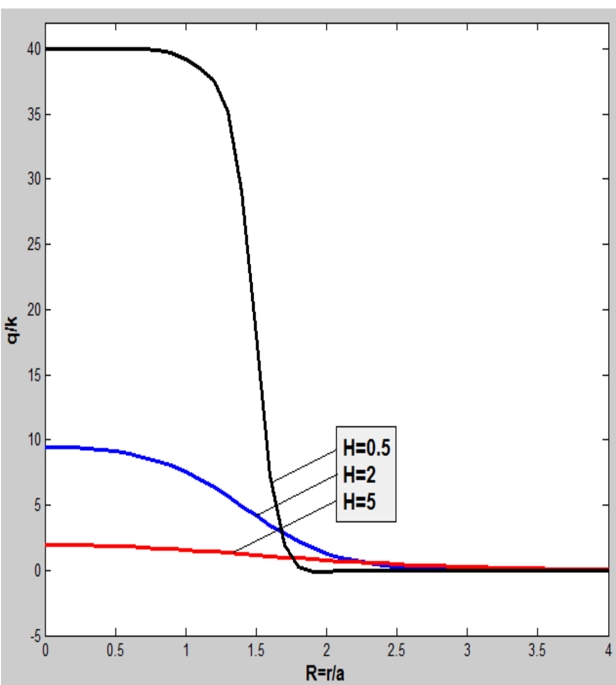

Fig. 7. Distribution of the flux

Fig. 7 shows the variations of flux with R for various values of H. the flux decrease in $r < a$ $(R < 1)$ and very rapidly in $r \geq a$ $(R \geq 1)$, then tends to zero as the value of R becomes large. In the same figure, as the increasing of H, q/k decreases.

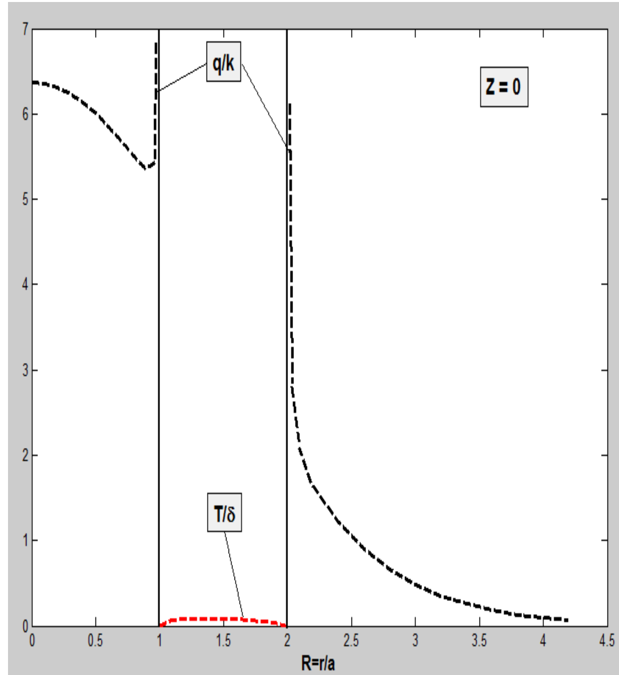

Fig. 8. Distribution of the temperature and the flux

Fig. 8 shows the variations of temperature and flux at $\xi = 0$ with R.

We can note the good agreement between the values obtained numerically and analytically.

6 Conclusion

Instead of the traditional method of reducing the triple integral equations to a system of Fredholm integral equation, we get directly an infinite system of algebraic equations for determining the unknown function. The obtained results are compatible with the physical meaning of the problem. Whereas, the approximate temperature and flux values converge to the exact analytical ones on the interface of the medium.

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Appendix

A. Evaluation of the integral

$$\int_0^\infty \lambda J_0\left(\lambda r\right) G_n(\lambda) d\lambda$$

Using the following proprieties of the Bessel function J_n

$$J_n'\left(x\right) = \frac{1}{2}\left[J_{n-1}\left(x\right) - J_{n+1}\left(x\right)\right] \tag{31}$$

$$J_n\left(x\right) = \frac{x}{2n}\left[J_{n-1}\left(x\right) + J_{n+1}\left(x\right)\right] \tag{32}$$

We find that

$$\begin{aligned} \frac{d}{d\lambda} Z_n\left(\lambda\right) &= c\frac{d}{d(\lambda c)} J_n\left(\lambda c\right) J_n\left(\lambda d\right) \\ &\quad + d\frac{d}{d(\lambda d)} J_n\left(\lambda c\right) J_n\left(\lambda d\right) \\ &= \frac{cd}{2n} G_n\left(\lambda\right) \end{aligned}$$

Thus we get

$$G_n\left(\lambda\right) = \frac{2n}{cd}\frac{d}{d\lambda} Z_n\left(\lambda\right) \tag{33}$$

By integrating by parts, and using the following formulas

$$\lambda\frac{\partial}{\partial\lambda} J_0\left(\lambda r\right) = \lambda r\frac{\partial}{\partial\lambda r} J_0\left(\lambda r\right) = r\frac{\partial}{\partial r} J_0\left(\lambda r\right) \tag{34}$$

$$\lambda J_0\left(\lambda r\right) Z_n\left(\lambda\right)\Big|_0^\infty = 0, \quad J_0\left(0\right) = 1 \tag{35}$$

we obtain

$$\begin{aligned} \int_0^\infty \lambda J_0\left(\lambda r\right) G_n(\lambda) d\lambda &= -\frac{2n}{cd}\Big[\int_0^\infty J_0\left(\lambda r\right) \\ &\quad \times J_n\left(\lambda c\right) J_n\left(\lambda d\right) d\lambda \\ &\quad + r\frac{\partial}{\partial r}\int_0^\infty J_0\left(\lambda r\right) \\ &\quad \times J_n\left(\lambda c\right) J_n\left(\lambda d\right) d\lambda\Big] \\ &= -\frac{2n}{cd}\Big[I_0^n + r\frac{\partial}{\partial r} I_0^n\Big] \end{aligned} \tag{36}$$

where

$$I_0^n = \int_0^\infty J_0\left(\lambda r\right) J_n\left(\lambda c\right) J_n\left(\lambda d\right) d\lambda$$

B. verification of the relation (19)

Next, we use the following integral formula [12]

$$\int_0^\infty J_\xi(\lambda t)\, J_\mu(\lambda x)\, J_\nu(\lambda y)\, d\lambda = \frac{t^\xi x^\mu \Gamma\left(\frac{\xi+\mu+\nu+1}{2}\right)}{y^{\xi+\mu+1}\Gamma(\xi+1)\Gamma(\mu+1)\Gamma\left(\frac{1-\xi-\mu+\nu}{2}\right)} \times F_4\left(\frac{\xi+\mu-\nu+1}{2}, \frac{\xi+\mu+\nu+1}{2}; \xi+1, \mu+1; \frac{t^2}{y^2}, \frac{x^2}{y^2}\right) \quad (37)$$

for

$$[Re(\xi+\mu+\nu) > -1,\ t, x > 0,\ \ y > t+x]$$

where F_4 is the Apell function given by

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (38)$$

where

$$\left[\left|\sqrt{x}\right| + \left|\sqrt{y}\right| < 1\right]$$

It verifies the relation

$$F_4\big(\alpha, \gamma+\gamma'-\alpha-1, \gamma, \gamma'; x(1-y), y(1-x)\big) = F(\alpha, \gamma+\gamma'-\alpha-1; \gamma; x)\, F(\alpha, \gamma+\gamma'-\alpha-1; \gamma'; y)$$

where F is the Gauss hyper-geometric function

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^\infty \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!} \quad (39)$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

In order to express F_4 in terms of F, we put **a)**

$$\begin{cases} \alpha = \frac{\xi+\mu-\nu+1}{2} \\ \gamma = \xi+1 \\ \gamma' = \mu+1 \end{cases}$$

then

$$\begin{aligned} \gamma+\gamma'-\alpha-1 &= \xi+\mu+1-\frac{\xi+\mu-\nu+1}{2} \\ &= \frac{\xi+\mu+\nu+1}{2} \end{aligned}$$

b) choosing

$$\begin{cases} \frac{t^2}{y^2} = \alpha_0(1-\beta_0) \\ \frac{x^2}{y^2} = \beta_0(1-\alpha_0) \end{cases}$$

where α_0 and β_0 are to be calculated.

Putting $t = y\sin\varphi\cos\psi$,
$x = y\cos\varphi\sin\psi;\quad 0 < \varphi,\ \psi < \frac{\pi}{2}$, then

$$\begin{cases} \frac{t^2}{y^2} = \sin^2\varphi\cos^2\psi = \sin^2\varphi\left(1-\sin^2\psi\right) \\ \frac{x^2}{y^2} = \sin^2\psi\left(1-\sin^2\varphi\right) \end{cases}$$

Next, we put

$$\frac{x+t}{y} = \sin(\psi+\varphi), \quad \frac{x-t}{y} = \sin(\psi-\varphi)$$

i.e

$$\psi+\varphi = \arcsin\left(\frac{x+t}{y}\right), \quad \psi-\varphi = \arcsin\left(\frac{x-t}{y}\right)$$

Then, we obtain

$$\begin{bmatrix} \psi \\ \\ \varphi \end{bmatrix} = \frac{1}{2}\left[\arcsin\left(\frac{x+t}{y}\right) \pm \arcsin\left(\frac{x-t}{y}\right)\right]$$

Finally replacing σ by $\xi+\mu$, the relation (37) can be written as

$$\int_0^\infty J_\xi(\lambda t)\, J_\mu(\lambda x)\, J_\nu(\lambda y)\, d\lambda = \frac{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}{\Gamma(\xi+1)\Gamma(\mu+1)\Gamma\left(\frac{1-\sigma+\nu}{2}\right)}\left(\frac{t}{y}\right)^\xi\left(\frac{x}{y}\right)^\mu\left(\frac{1}{y}\right) \times F\left(\frac{1+\sigma-\nu}{2}, \frac{1+\sigma+\nu}{2}; \xi+1; \sin^2\varphi\right) \times F\left(\frac{1+\sigma-\nu}{2}, \frac{1+\sigma+\nu}{2}; \mu+1; \sin^2\psi\right) \quad (40)$$

C. How to obtain the relation (21)

Using the following formulas

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$$

$$\frac{\partial}{\partial\varphi}\sin^{2n}\varphi=2n\cos\varphi\sin^{2n-1}\varphi=n\sin 2\varphi\sin^{2n-2}\varphi$$

it is easy to find that

$$\begin{aligned}\frac{\partial}{\partial\varphi}F\left(\alpha,\beta;\gamma;\sin^2\varphi\right) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sin 2\varphi \\ &\sum_{n=1}^{\infty}\left(\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \times\frac{\left(\sin^2\varphi\right)^{n-1}}{(n-1)!}\right) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sin 2\varphi \\ &\sum_{n=0}^{\infty}\left(\frac{\Gamma(\alpha+1+n)}{\Gamma(\gamma+1+n)} \times\Gamma(\beta+1+n) \times\frac{\left(\sin^2\varphi\right)^{n}}{n!}\right) \\ &= \frac{\alpha\beta}{\gamma}\sin 2\varphi \\ &\times F\Big(\alpha+1,\beta+1;\gamma+1; \sin^2\varphi\Big) \qquad (41)\end{aligned}$$

D. The approximation formula of A'_{mn} expressed in Eq. (30)

the integrand function is given by

$$\begin{aligned}&G_m(\lambda)G_n(\lambda)= \\ &\lambda^2\Big[J_{m-1}(\lambda c)J_{m-1}(\lambda d)-J_{m+1}(\lambda c)J_{m+1}(\lambda d)\Big] \\ &\times\Big[J_{n-1}(\lambda c)J_{n-1}(\lambda d)-J_{n+1}(\lambda c)J_{n+1}(\lambda d)\Big]\end{aligned}$$

For the large values of x, the approximate function of J_n is [12]

$$\begin{aligned}J_n(x)=\sqrt{\frac{2}{\pi x}}\Bigg[&\cos\Big(x-(2n+1)\frac{\pi}{4}\Big)-\Bigg(\frac{4n^2-1}{8x} \\ &\times\sin\Big(x-(2n+1)\frac{\pi}{4}\Big)\Bigg)+O\left(\frac{1}{x^2}\right)\Bigg]\end{aligned}$$

Then, we get

$$\begin{aligned}J_n(x)J_n(y) &= \frac{2}{\pi\sqrt{xy}}\Bigg\{\Bigg[\cos\Big(x-(2n+1)\frac{\pi}{4}\Big) \\ &-\frac{4n^2-1}{8x}\sin\Big(x-(2n+1)\frac{\pi}{4}\Big)\Bigg] \\ &\Bigg[\cos\Big(y-(2n+1)\frac{\pi}{4}\Big)-\frac{4n^2-1}{8y} \\ &\times\sin\Big(y-(2n+1)\frac{\pi}{4}\Big)\Bigg] \\ &+O\left(\frac{1}{(xy)^2}\right)\Bigg\} \\ &= \frac{1}{\pi\sqrt{xy}}\Bigg\{\cos(x-y) \\ &+\cos\Big(x+y-(2n+1)\frac{\pi}{2}\Big) \\ &-\frac{4n^2-1}{8}\Bigg[\frac{1}{x}\Big[\sin(x-y) \\ &+\sin\Big(x+y-(2n+1)\frac{\pi}{2}\Big)\Big]\Bigg] \\ &+\frac{1}{y}\Big[\sin(y-x)+\sin\Big(x+y \\ &-(2n+1)\frac{\pi}{2}\Big)\Big]+O\left(\frac{1}{(xy)^2}\right)\Bigg\}\end{aligned}$$

We can find easily that

$$\begin{aligned} & J_{m-1}(x) J_{m-1}(y) - J_{m+1}(x) J_{m+1}(y) \\ = & \frac{1}{\pi\sqrt{xy}}\Bigg\{ \left[\frac{4(m+1)^2-1}{8} - \frac{4(m-1)^2-1}{8} \right] \\ & \times \frac{1}{xy}\Bigg[(y-x)\sin(x-y) \\ & + (-1)^m (x+y)\cos(x+y) \Bigg] + O\left(\frac{1}{(xy)^2}\right) \Bigg\} \\ = & \frac{2m}{\pi (xy)^{\frac{3}{2}}}\Big[-(x-y)\sin(x-y) \\ & + (-1)^m (x+y)\cos(x+y) \Big] + O\left(\frac{1}{(xy)^2}\right) \end{aligned}$$

The approximate value of the function is

$$\begin{aligned} G_m(\lambda) G_n(\lambda) = & \frac{4mn}{\pi^2 (cd)^3} \frac{1}{\lambda^2}\Big\{ a^2 \sin^2(\lambda a) \\ & + (-1)^{m+n} b^2 \cos^2(\lambda b) \\ & - \Big[(-1)^m + (-1)^n \Big] ab \\ & \times \sin\lambda a \cos\lambda b \Big\} \\ & + O\left(\frac{1}{\lambda^3}\right) \end{aligned}$$

Then the integral A'_{mn} can be expressed by

$$\begin{aligned} A'_{mn} = & \int_{\lambda_0}^{\infty} G_m(\lambda) G_n(\lambda)\, d\lambda \\ \simeq & \frac{4mn}{\pi^2 (cd)^3} \int_{\lambda_0}^{\infty} \frac{1}{\lambda^2}\Big\{ a^2 \sin^2(\lambda a) \\ & + (-1)^{m+n} b^2 \cos^2(\lambda b) \\ & - [(-1)^m + (-1)^n]\, ab \sin\lambda a \cos\lambda b \Big\} d\lambda \end{aligned}$$

Finally, we obtain

$$\begin{aligned} A'_{mn} = & \int_{\lambda_0}^{\infty} G_m(\lambda) G_n(\lambda)\, d\lambda \\ \simeq & \frac{4mn}{\pi^2 (cd)^3}\Bigg[a^2 \left(\frac{\sin^2\lambda_0 a}{\lambda_0} - a\, si(2\lambda_0 a) \right) \\ & + (-1)^{m+n} b^2 \left(\frac{\cos^2\lambda_0 b}{\lambda_0} + b\, si(2\lambda_0 b) \right) \\ & - [(-1)^m + (-1)^n]\, ab \Bigg(\frac{\sin\lambda_0 a \cos\lambda_0 b}{\lambda_0} \\ & - c\, ci(2\lambda_0 c) + d\, ci(2\lambda_0 d) \Bigg) \Bigg] \end{aligned}$$

where we used the values of the following integrals:

$$\int_{\lambda_0}^{\infty} 2\sin^2\lambda a\, d\lambda = \frac{2\sin 2\lambda_0 a}{\lambda_0} - 2a\, si(2\lambda_0 a)$$

we have also

$$\int_{\lambda_0}^{\infty} 2\cos^2\lambda b\, d\lambda = \frac{2\cos 2\lambda_0 b}{\lambda_0} + 2b\, si(2\lambda_0 b)$$

and

$$\begin{aligned} \int_{\lambda_0}^{\infty} 2\sin\lambda a \cos\lambda b\, d\lambda = & \frac{1}{\lambda_0}\sin\lambda_0 a \cos\lambda_0 b \\ & - c\, ci(2\lambda_0 c) + d\, ci(2\lambda_0 d) \end{aligned}$$

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