Abstract: An analytical design method is proposed for two-dimensional recursive zero-phase wedge filters. It relies on a frequency mapping applied to a 1D IIR low-pass maximally-flat prototype filter of a specified shape. The design is presented in two versions, one entirely analytical, based on the bilinear transform, and another involving a numerical approximation step. This paper focuses on the design method and provides simulation results of oriented line extraction from a real-life image, to prove the usefulness of these filters in image processing.

Key-Words: 2D IIR filter design, directional filters, frequency transformations, approximation

1 Introduction
The field of two-dimensional filters has known a steady development due to their applications in image processing and various design methods have been proposed [1]. A commonly used design technique is to start with a 1D prototype filter and transform it in order to obtain a 2D filter with desired frequency response. The existing design methods for 2D IIR filters rely on 1D analog prototypes, using spectral transformations from $s$ to $z$ plane via bilinear or Euler method, followed by $z$ to $(z_1,z_2)$ mappings [2]-[4].

There are several types of filters with orientation-selective frequency response, useful in various tasks such as edge detection, motion analysis etc. In [5] a filter bank for directional image decomposition was proposed. A class of linear operators with directional response was introduced in [6]. Wedge filters, named so due to their wedge-like shape in the frequency plane, find useful applications in feature extraction, e.g. in texture classification [7]. Steerable wedge filters were described in [8]. Different design and implementation methods for FIR and IIR fan and wedge filters were introduced in papers like [9]-[14].

In this work two design methods in the frequency domain are proposed for a class of 2D zero-phase wedge filters. The idea of this design approach was first introduced in [15]. Two ideal wedge filters in the frequency plane are displayed in Fig.1. The filter in Fig.1(a) has its frequency response along axis $\omega_2$ which forms an angle $\angle AOB = \theta$ with the frequency axis $\omega_1$. A general wedge filter is considered, with specified values for the aperture angle and orientation angle of its longitudinal axis, as shown in Fig.1(b). A maximally-flat prototype filter will be used for design. We approach here only zero-phase filters, particularly useful in image filtering since they are free of phase distortions.

2 Wedge Filter Design Method

2.1 Prototype Filters and Frequency Transformations
Let us consider the transfer function $H(z)$ of an IIR filter of order $N$ given by:

$$H(z) = \frac{P(z)}{Q(z)} = \sum_{i=1}^{M} p_i \cdot z^i \div \sum_{j=1}^{N} q_j \cdot z^j$$

We can derive a zero-phase prototype filter (with real-valued transfer function), which preserves only the magnitude characteristic of $H(z)$, while its phase is zero throughout the frequency domain $[-\pi,\pi]$. Let us consider the magnitude given by the absolute value of $H(z) = H(e^{j\omega})$ taken from (1):

$$|H(e^{j\omega})| = \left| \sum_{n=0}^{M} p_n \exp(jn\omega) \div \sum_{m=0}^{N} q_m \exp(jm\omega) \right|$$

Using a symbolic calculation software, we derive a rational approximation $H_a(\omega)$ of $|H(e^{j\omega})|$ in powers
of \( \omega^2 \), since \( |H(e^{j\omega})| \) has even parity. An efficient rational approximation is Chebyshev-Padé method. The function \( H_s(\omega) \), assuming the numerator and denominator of the same degree \( N = 4k \) \((k \in \mathbb{N})\), can be factored into functions of the form:

\[
H_s(\omega) = \left( b_0 + b_1 \omega^2 + b_2 \omega^4 \right) / \left( 1 + a_1 \omega^2 + a_2 \omega^4 \right)
\]  

(3)

Usually the value \( N = 4 \) is satisfactory for accuracy and we take \( H_s(\omega) = H_p(\omega) \). The function \( H_s(\omega) \) is a rational approximation of the frequency response magnitude of prototype (1). Let us consider the zero-phase maximally-flat low-pass IIR prototype function in the complex variable \( s \), derived from the general expression (3), where \( \omega^2 = -s^2 \):

\[
H_p(s) = \frac{0.887175 + 0.269975 \cdot s^2 + 0.018905 \cdot s^4}{1 + 0.600346 \cdot s^2 + 5.332057 \cdot s^4}
\]  

(4)

This simple prototype will be next used to design a zero-phase wedge filter with specified parameters. A wedge filter having \( O - \omega_2 \) as longitudinal axis results using the 1D to 2D frequency transformation (for \( \omega_2 \neq 0 \)):

\[
\omega \rightarrow f(\omega_1, \omega_2) = a \cdot \omega_1 / \omega_2
\]  

(5)

The coefficient \( a = 1 / \tan(\theta / 2) \) is a measure of the aperture angle \( \theta \). More generally, if the wedge filter axis has an orientation specified by an angle \( \varphi \) about axis \( \omega_2 \), the filter results by rotating the axes of the plane \( (\omega_1, \omega_2) \) by an angle \( \varphi \), as defined by the following linear transformation:

\[
\begin{bmatrix}
\omega_1 \\
\omega_2
\end{bmatrix} = \begin{bmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{bmatrix} \begin{bmatrix}
\widetilde{\omega}_1 \\
\widetilde{\omega}_2
\end{bmatrix}
\]  

(6)

where \( \omega_1, \omega_2 \) are the original and \( \widetilde{\omega}_1, \widetilde{\omega}_2 \) the rotated variables. The 1D to 2D frequency mapping can be written:

\[
\omega \rightarrow f_p(\omega_1, \omega_2) = a \left( \omega_1 - \omega_2 \cdot \tan \varphi \right) / \left( \omega_1 \cdot \tan \varphi + \omega_2 \right)
\]  

(7)

Since \( s_1 = j\omega_1 \) and \( s_2 = j\omega_2 \), the mapping (7) in the complex plane \( (s_1, s_2) \) takes the following form:

\[
\omega \rightarrow f_{sp}(s_1, s_2) = a \left( s_1 - s_2 \cdot \tan \varphi \right) / \left( s_1 \cdot \tan \varphi + s_2 \right)
\]  

(8)

Two design procedures will be next presented, one entirely analytical and another using a numerical approximation step.

### 2.2 Design Method Using the Bilinear Transform

In the general case of an oriented wedge filter, replacing in (3) \( \omega \) by the expression (8), we derive the 2D filter frequency response in \( s_1, s_2 \), namely \( H_5(s_1, s_2) \), a ratio of two-variable polynomials of 4-th degree.

The next step is mapping \( H_5(s_1, s_2) \) onto the complex plane \( (z_1, z_2) \) using bilinear transform. For our purposes the sample interval is \( T = 1 \) so the double bilinear transform for variables \( s_1, s_2 \) in the complex plane \( (s_1, s_2) \) has the form:

\[
s_1 = 2(z_1 - 1) / (z_1 + 1) \quad s_2 = 2(z_2 - 1) / (z_2 + 1)
\]  

(9)

substituting \( s_1, s_2 \) from (9) in the function \( H_5(s_1, s_2) \) found in the previous step, we find a rational function \( H_5(z_1, z_2) \) in \( z_1, z_2 \). The designed 2D filter will have noticeable distortions towards the frequency plane limits compared to the ideal response. This is due to frequency warping effect of the bilinear transform, expressed by the frequency mapping:

\[
\omega = (2/T) \cdot \arctan(\omega, T / 2)
\]  

(10)

where \( \omega \) is the frequency of the discrete-time filter and \( \omega_0 \) is the frequency of the continuous-time filter. In order to correct this error we apply a prewarping, using the inverse of mapping (10). Taking the sampling period \( T = 1 \) in (10) we substitute the mappings:

\[
\omega_1 \rightarrow 2 \omega / (\omega_1 / 2) \quad \omega_2 \rightarrow 2 \omega / (\omega_2 / 2)
\]  

(11)

A more suitable rational approximation of mappings (11) can be obtained using again the Chebyshev-Padé method:

\[
\tan(\omega / 2) \equiv 0.460346 \cdot \omega / (1 - 0.1 \omega^2) = g(\omega)
\]  

(12)

which is very accurate in the frequency range \([-\pi, \pi]\) and has the low-order advantage. Using (7), (11) we get the transformation including frequency pre-warping for \( \omega_1, \omega_2 \):

\[
\omega \rightarrow f_{wp}(\omega_1, \omega_2) = a \left( \tan(\omega_1 / 2) - \tan(\omega_2 / 2) \cdot \tan \varphi \right) / \left( \tan(\omega_1 / 2) \cdot \tan \varphi + \tan(\omega_2 / 2) \right)
\]  

(13)

Substituting the rational approximation (12) written for both frequency variables \( \omega_1 \) and \( \omega_2 \) into (13), we get a rational expression of frequency transformation \( \omega \rightarrow f_{wp}(\omega_1, \omega_2) \) in \( \omega_1 \) and \( \omega_2 \).
Then, taking into account that \( s_1 = j \omega_1 \) and \( s_2 = j \omega_2 \), in the complex plane \((s_1, s_2)\) we obtain the frequency transformation \( \omega \rightarrow f_{\varphi}(s_1, s_2) \):

\[
\omega \rightarrow f_{\varphi}(s_1, s_2) = a \left[ s_1 (1 + 0.1 s_1^2) - t g \varphi \cdot s_1 (1 + 0.1 s_1^2) \right] \left[ t g \varphi \cdot s_1 (1 + 0.1 s_1^2) + s_2 (1 + 0.1 s_1^2) \right]
\]

(14)

This 1D to 2D mapping includes the frequency pre-warping along the two axes. Applying the double bilinear transform (9), after simplification we obtain a mapping \( F: \mathbb{R} \rightarrow \mathbb{C}^2 \):

\[
\omega \rightarrow F(z_1, z_2) = a \left[ f(z_1, z_2) - t g \varphi \cdot g(z_1, z_2) \right] \left[ t g \varphi \cdot f(z_1, z_2) + g(z_1, z_2) \right] A_p(z_1, z_2)
\]

(15)

where \( f(z_1, z_2) \) and \( g(z_1, z_2) \) are symmetric in \( z_1, z_2 \):

\[
f(z_1, z_2) = 1.4 z_1^2 z_2^2 + 1.2 z_1^2 z_2 + 1.4 z_1^2 - 1.4 z_2^2 - 1.2 z_2 - 1.4\]

\[
g(z_1, z_2) = 1.4 z_1^2 z_2^2 + 1.2 z_1^2 z_2^2 + 1.4 z_2^2 - 1.4 z_1^2 - 1.2 z_1 - 1.4
\]

(16)

Next the term template will be used, meaning the coefficient matrices of the numerator and denominator of a transfer function \( H(z_1, z_2) \). Denoting by \( B_{\varphi} \) and \( A_{\varphi} \) the \( 3 \times 3 \) templates for the numerator \( B_{\varphi}(z_1, z_2) \) and denominator \( A_{\varphi}(z_1, z_2) \), we get:

\[
A_{\varphi} = \begin{bmatrix}
-1.4 t g \varphi - 1.4 & -1.2 t g \varphi - 1.4 t g \varphi + 1.4 \\
1.4 t g \varphi - 1.4 & 1.2 t g \varphi + 1.4 t g \varphi + 1.4
\end{bmatrix}
\]

(17)

The matrix \( B_{\varphi} \) is \( A_{\varphi} \) clockwise rotated with \( 90^\circ \) : \( B_{\varphi} = A_{\varphi}^{90^\circ} \). If matrix \( M \) corresponds to \( g(z_1, z_2) \), then \( M^{90^\circ} \) corresponds to \( f(z_1, z_2) \) and \( B_{\varphi} = M^{90^\circ} - t g \varphi \cdot M \), \( A_{\varphi} = t g \varphi \cdot M^{90^\circ} + M \). Finally from (15) the frequency mapping can be expressed as:

\[
\omega \rightarrow F_2(z_1, z_2) = \left( z_1 \times B \times z_2^T \right) / \left( z_1 \times A \times z_2^T \right)
\]

(18)

where \( z_1 \) and \( z_2 \) are the vectors:

\[
z_1 = \begin{bmatrix}
z_1^2 & z_1^1 & z_1^1 & z_1^2
\end{bmatrix}
\]

\[
z_2 = \begin{bmatrix}
z_2^2 & z_2^1 & z_2^1 & z_2^2
\end{bmatrix}
\]

and the \( 5 \times 5 \) matrices \( B = B_{\varphi} \times B_{\varphi} \) and \( A = A_{\varphi} \times A_{\varphi} \) resulted by convolution preserve the property \( B = A^{90^\circ} \). We apply the mapping (18) directly to the 1D prototype (3), obtaining the 2D wedge filter transfer function in \( z_1 \) and \( z_2 \):

\[
H_{\varphi}(z_1, z_2) = \left( Z_1 \times B_{\varphi} \times Z_2^T \right) / \left( Z_1 \times A_{\varphi} \times Z_2^T \right)
\]

(20)

where the vectors \( Z_1 \) and \( Z_2 \) have the form:

\[
Z_1 = \begin{bmatrix}
z^N & z^{N-1} & \ldots & z & 1
\end{bmatrix}
\]

\[
Z_2 = \begin{bmatrix}
z^N & z^{N-1} & \ldots & z & 1
\end{bmatrix}
\]

(21)

with \( N = 9 \); the matrices \( A_{\varphi} \) and \( B_{\varphi} \) result as:

\[
B_{\varphi} = b_0 (A \ast A) + b_1 a^2 (A \ast B) + b_2 a^4 (B \ast B)
\]

\[
A_{\varphi} = A + a_1 a^2 (A \ast B) + a_2 a^4 (B \ast B)
\]

(22)

Even if the templates result relatively large, this is the price paid for ensuring a good linearity of the wedge filter shape in the frequency plane. The frequency pre-warping has increased the filter order, but the filter templates result as a convolution of smaller matrices ( \( 3 \times 3 , 5 \times 5 \) ), being partially separable. At least the numerator of prototype (3) may have real roots and can be factored, involving convolution of smaller matrices. For instance, the numerator in (4) is factored as:

\[
N(\omega) = 0.88717 \cdot (1 - 0.195 \cdot \omega^2) \cdot (1 - 0.10924 \cdot \omega^2)
\]

and is realized as a convolution of the two \( 5 \times 5 \) templates, taking into account (18):

\[
N = 0.88717 \cdot (A - 0.195 \cdot a^2 B) \ast (A - 0.10924 \cdot a^2 B)
\]

(23)

Using the prototype (4), we designed a wedge filter with an aperture angle \( \theta = 0.2 \pi \) and orientation angle \( \varphi = \pi / 5 \). For these specifications we get the values \( a = t g(\theta / 2) = 0.3249 \), \( t g \varphi = 0.7265 \). The frequency response and contour plot of the designed filter are shown in Fig. 2.

### 2.3 Design Method Using Numerical Approximations

The second design method for zero-phase wedge filters starts again from a zero-phase 1D prototype of the general form (3). We use again the 1D-2D frequency mapping (7). Since (3) is a rational function of \( \omega^2 \), the design method requires finding the discrete approximation of the function

\[
F_{\varphi}(\omega_1, \omega_2) = f_{\varphi}^2(\omega_1, \omega_2) = a^2 \left( \omega_1 - \omega_2 \cdot t g \varphi \right) \left( \omega_1 \cdot t g \varphi + \omega_2 \right)^2)
\]

(25)

Fig. 2. Frequency response and contour plot of an oriented flat-top wedge filter with aperture \( \theta = 0.2 \pi \) and orientation \( \varphi = 0.2 \pi \)
The approximation results using the change of variable:
\[ \omega_1 = \arccos x_1, \quad \omega_2 = \arccos x_2 \] (26) and \( F_p(\omega_1, \omega_2) \) is mapped into a function \( G_p(x_1, x_2) \).

The next step is to find a bivariate Taylor series expansion of \( G_p(x_1, x_2) \). Using again the symbolic calculation, we easily determine this series expansion in \( x_1, x_2 \). Then we return to the former variables by substituting back \( x_1 = \cos \omega_1 \) and \( x_2 = \cos \omega_2 \) into \( G_p(x_1, x_2) \). We obtain an approximation of \( F_p(\omega_1, \omega_2) \) in powers of \( \cos \omega_1 \) and \( \cos \omega_2 \). Applying trigonometric identities we find \( F_p(\omega_1, \omega_2) \) as:

\[ F_p(\omega_1, \omega_2) \approx \sum_{m=-N}^{N} \sum_{n=-N}^{N} a_{mn} \cdot \cos(m\omega_1 + n\omega_2) \] (27)

where \( N \) is chosen to ensure a desired precision. Usually we can take the value \( N = 2 \). The coefficients \( a_{mn} \) depend on angle \( \varphi \) and have polynomial expressions in variable \( \tan \varphi \). Let us design a wedge filter with specifications given in Section II, i.e. prototype (4), with parameters: \( a = \tan(\theta/2) = 0.3249 \), \( \tan \varphi = 0.7265 \). Our method gives the approximation:

\[ F_p(\omega_1, \omega_2) \approx a^2 \{0.195736 -0.132213 \cdot \cos(\omega_1) + 0.212134 \cdot \cos(2\omega_1) -0.155057 \cdot (\cos(\omega_1 - \omega_2) + \cos(\omega_1 + \omega_2)) -0.027075 \cdot (\cos(2\omega_1 - \omega_2) + \cos(2\omega_1 + \omega_2)) -0.042024 \cdot (\cos(\omega_1 - 2\omega_2) + \cos(\omega_1 + 2\omega_2)) +0.050075 \cos(2\omega_1) + 0.124584 \cos(2\omega_2) -0.014742 \cdot (\cos(2\omega_1 - 2\omega_2) + \cos(2\omega_1 + 2\omega_2)) \} \] (28)

which corresponds to the 5×5 template:

\[
W = a^2 \cdot \begin{bmatrix}
-0.0073 & -0.0210 & 0.0623 & -0.0210 & -0.0073 \\
-0.0135 & -0.0775 & 0.1060 & -0.0775 & -0.0135 \\
0.0250 & -0.0661 & 0.1957 & -0.0661 & 0.0250 \\
-0.0135 & -0.0775 & 0.1060 & -0.0775 & -0.0135 \\
-0.0073 & -0.0210 & 0.0623 & -0.0210 & -0.0073
\end{bmatrix}
\] (29)

found by identifying coefficients of the 2D Z transform from (28). Once obtained the 1D to 2D frequency mapping of the form \( \omega^2 \rightarrow F_p(\omega_1, \omega_2) \), the next step is to substitute it in \( H_p(\omega) \) from (3). Thus we get the 2D function:

\[ H_1(\omega_1, \omega_2) = B(\omega_1, \omega_2) / A(\omega_1, \omega_2) \] (30)

The templates \( B \) and \( A \) result according to \( H_p(\omega) \) in (3) as:

\[
B = b_0 \cdot E + b_1 \cdot W_b + b_2 \cdot W \cdot W
\]

\[
A = E + a_1 \cdot W_a + a_2 \cdot W \cdot W
\] (31)

The symbol \( \ast \) stands for matrix convolution and \( E \) is a \( 9 \times 9 \) matrix with zero elements and central element equal to 1. The \( 9 \times 9 \) matrix \( W_b \) is obtained by bordering the \( 5 \times 5 \) matrix \( W \) with zeros in order to be summed with matrices \( E \) and \( W \cdot W \).

An advantage of the second design method over the first one is that it avoids the use of bilinear transform, which is known to introduce large distortions unless a frequency pre-warping is applied. The second design approach is somewhat simpler but requires the use of bivariate Taylor series expansion for a given orientation angle \( \varphi \).

### 2.4 Fan Filter Design Example

The proposed method can be used as well in designing fan filters. We consider two types of fan filters specified as in Fig.3(a), (b). The filter with the shape shown in Fig.3(a) is ideally described in the frequency plane as:

\[ H_p(\omega_1, \omega_2) = \begin{cases} 
1, & |\omega_2| \leq |\omega_1| \\
0, & \text{otherwise}
\end{cases} \] (32)

This fan filter is a particular wedge filter with \( \theta = \pi/2 \), \( \varphi = 0 \), so \( a = 1 \), \( \tan \theta = 0 \); the mapping (7) reduces to:

\[ \omega \rightarrow f_p(\omega_1, \omega_2) = \omega_1 / \omega_2 \] (33)

In this particular case the template \( W \) results as:

\[
W = \begin{bmatrix}
0.0072 & -0.0413 & 0.1038 & -0.0413 & 0.0072 \\
0.0134 & -0.1056 & 0.1746 & -0.1056 & 0.0134 \\
0.0281 & -0.1474 & 0.2975 & -0.1474 & 0.0281 \\
0.00134 & -0.1056 & 0.1746 & -0.1056 & 0.0134 \\
0.0072 & -0.0413 & 0.1038 & -0.0413 & 0.0072
\end{bmatrix}
\] (34)

The fan filter based on the prototype (4), shown in Fig.3(c) visibly preserves the maximally-flat response. For the second fan filter type shown in Fig.3(b), \( \theta = \pi/2 \) and \( \varphi = \pi/4 \) so we obtain \( a = 1 \) and \( \tan \varphi = 1 \); the frequency transformation (7) thus simplifies to:

\[ \omega \rightarrow f_p(\omega_1, \omega_2) = (\omega_1 - \omega_2) / (\omega_1 + \omega_2) \] (35)

In this particular case we obtain a particular matrix \( W \) and the final filter templates result again using relations (31).

The stability of the 2D filters designed using the proposed methods will be studied in further work. For 2D filters it is generally difficult to take stability constraints into account during approximation stage.
Various methods were developed to order to separate stability from approximation problem. If the 2D filter becomes unstable, some stabilization procedures are needed.

3 Applications and Simulation Results

A wedge filter can be used in directional image filtering, to select lines with a specified orientation from a given image. The spectrum of a straight line is oriented in the plane \((\omega_1, \omega_2)\) at an angle of \(\pi/2\) with respect to the line direction. Fig.4 shows a real gray-scale image representing a straw texture. The straws have random directions and choosing different filter orientations we can select the straws with roughly the same orientation and filter out the rest. Only the lines with spectrum oriented along filter characteristic remain unchanged, while all the rest are more or less blurred, due to directional filtering. The filter aperture angle was \(\theta = \pi/5\) and three different orientations were used. This simple example was given just to illustrate the potential use of wedge filters in pattern recognition applications. Further research will approach other real-life applications as well.

The advantage of these analytical design methods over other approaches is that the frequency response of the resulted 2D wedge filter depends explicitly on the specified parameters (orientation, aperture, prototype flatness and steepness) and therefore the filter is adjustable. The design need not be remade every time again from the start for different specifications.

4 Conclusion

A design method was proposed for 2D IIR zero-phase wedge filters, oriented along a specified direction. They are based on a 1D LP prototype with given frequency response. A 1D to 2D frequency mapping is applied to the prototype. One method uses the bilinear transform and frequency pre-warping to compensate for distortions. The other avoids the use of bilinear transform and includes instead a numerical approximation. Both methods are fairly accurate and can be used to design wedge filters with various specifications.

References:


