Matrices unitarily similar to symmetric matrices

1 Introduction

Let $A$ be an $n \times n$ complex matrix. The numerical range of $A$ is defined as the set

$$W(A) = \{ \xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1 \}.$$  

The numerical range has been systematically and intensively studied in recent years (cf. [4, 9, 11, 13]). Kippenhahn [13] showed that the numerical range $W(A)$ is completely determined by the following ternary form

$$F_A(x, y, z) = \det(x \Re(A) + y \Im(A) + zI_n),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. More precisely, $W(A)$ is the convex hull of the real affine part of the dual curve of $F_A(x, y, z) = 0$. Two $n \times n$ matrices $A$ and $B$ are unitarily similar if there is a unitary matrix $U$ satisfying $B = U^* AU$. Clearly, unitarily similar matrices $A$ and $B$ have the same associated ternary forms $F_A(x, y, z) = F_B(x, y, z)$, which implies that $W(A) = W(B)$. The converse is in general not true, for instance, we consider two matrices

$$A = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5\sqrt{2} & 0 \\ 0 & 0 & 5\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$  

Then $F_A(x, y, z) = F_B(x, y, z)$. However, $A$ and $B$ are not unitarily similar.

Helton and Spitovsky [10] showed that for every matrix $A$ there exists a complex symmetric matrix $\tilde{B}$ of the same size satisfying $F_{\tilde{B}}(x, y, z) = F_A(x, y, z)$, and hence $W(A) = W(\tilde{B})$. Inspired by the result of Helton and Spitovsky, we ask whether every matrix is unitarily similar to a symmetric matrix. It is known that every matrix is similar to a symmetric matrix (cf. [12, Theorem 4.4.9]), but unitary similarity remains unexplained. Obviously, every normal matrix is unitarily similar to a complex diagonal matrix which is symmetric.

In this paper we prove every matrix in two classes is unitarily similar to a complex symmetric matrix. The two classes are the class of completely non-unitary contractions with deficiency index 1, and the class of Toeplitz matrices. An explicit $3 \times 3$ upper triangular nilpotent matrix which is not unitarily similar to any complex symmetric matrix is provided.

2 Lower dimensions

We start with $2 \times 2$ matrices. The following result is obtained in [14], we give a different proof.

Theorem 1 Let $A$ be a $2 \times 2$ complex matrix. Then $A$ is unitarily similar to a complex symmetric matrix.

Proof. It is known that $A$ is unitarily similar to

$$\tilde{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}$$

for some complex numbers $\alpha, \beta, \gamma$ (cf. [11, Lemma 1.3.1]). By similarity of a unitary matrix $\text{diag}(1, e^{i(\theta_2 - \theta_1)/2})$ and then multiplying a constant $e^{-i(\theta_1 + \theta_2)/2}$, we may assume that $\beta, \gamma$ are non-negative real numbers, where $\beta = |\beta|e^{i\theta_1}$ and $\gamma = |\gamma|e^{i\theta_2}$. Consider a unitary matrix

$$U = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$  

then
\[
U \tilde{A} U^* = \frac{1}{2} \begin{pmatrix}
2\alpha + i(\beta - \gamma) & \beta + \gamma \\
\beta + \gamma & 2\alpha - i(\beta - \gamma)
\end{pmatrix},
\]
and thus \(A\) is unitarily similar to a complex symmetric matrix. \(\square\)

For \(3 \times 3\) matrices, the following typical matrix gives a negative unitary similarity.

**Example 1** Let \(N\) be a \(3 \times 3\) upper triangular nilpotent matrix given by
\[
N = \begin{pmatrix}
0 & 8 & 0 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then \(N\) cannot be unitarily similar to a symmetric matrix.

**Proof.** The imaginary part \(\Im(N) = (N - N^*)/(2i)\) of \(N\) has eigenvalues \(5, 0, -5\). We define \(B = \text{diag}(5, 0, -5)\). The ternary form associated with \(N\) is given by
\[
F_N(x, y, z) = z^3 - 25(x^2 + y^2)z.
\]
Suppose that there is a complex symmetric matrix \(S\) which is unitarily similar to \(N\). We may assume that \(S = A + iB\) for some real symmetric matrix \(A = (a_{ij})\). The condition \(F_S(x, y, z) = F_N(x, y, z)\) implies that
\[
a_{11} = a_{22} = a_{33} = 0,
\]
\[
a_{12} + a_{13} + a_{23} - 25 = 0,
\]
\[
a_{12}a_{13}a_{23} = 0,
\]
\[
a_{22} - a_{23} = 0.
\]
The above simultaneous equations give
\[
a_{13} = \pm 5, \quad a_{12} = a_{23} = 0
\]
(1) or
\[
a_{13} = 0, a_{12} = \epsilon_1 5 \frac{1}{\sqrt{2}}, \quad a_{23} = \epsilon_2 5 \frac{1}{\sqrt{2}},
\]
(2)
\[
\epsilon_1 = \pm 1, \epsilon_2 = \pm 1.
\]
If solution (1) occurs, then the matrix \(S = A + iB\) satisfies \(S^2 = 0\) which cannot be unitarily similar to \(N\) for \(N^2 \neq 0\). In case solution (2) happens, then there is an orthonormal basis \(\{\xi_1, \xi_2, \xi_3\}\) of \(\mathbb{C}^3\) satisfying
\[
S\xi_1 = 0, \quad S\xi_2 = 5\sqrt{2}\xi_1, \quad S\xi_3 = 5\sqrt{2}\xi_2.
\]
We restrict the operator \(N\) to the kernel of \(N^2\), it has a canonical expression
\[
\begin{pmatrix}
0 & 8 \\
0 & 0
\end{pmatrix}.
\]
On the other hand, the operator \(S\) restricted to the kernel of \(S^2\) has a canonical expression
\[
\begin{pmatrix}
0 & 5\sqrt{2} \\
0 & 0
\end{pmatrix}.
\]
These two canonical \(2 \times 2\) matrices are not unitarily similar, which contradicts the assumption of the unitary similarity between \(S\) and \(N\). \(\square\)

Unitary similarity of two matrices \(A\) and \(B\) can be determined in terms of the traces of matrices of certain words in \(A\) and \(B\) (cf. [17, 20]).

**Theorem 2** ([20, Theorem 4.1], see also [17]). Let \(A\) and \(B\) be \(3 \times 3\) complex matrices satisfying the conditions
\[
\text{tr}(A^k) = \text{tr}(B^k), \quad k = 1, 2, 3
\]
and
\[
\text{tr}(A^2A) = \text{tr}(B^2B), \quad \text{tr}(A^2A^2) = \text{tr}(B^2B^2),
\]
\[
\text{tr}(A^2AA^*B) = \text{tr}(B^2BB^*)B.
\]
Then there exists a \(3 \times 3\) unitary matrix \(U\) such that \(B = UAU^*\) or \(B = UA^2U^*\).

As an application of Theorem 2, we provide a \(3 \times 3\) matrix which is unitarily similar to a complex symmetric matrix.

**Example 2** Let \(A\) be a \(3 \times 3\) matrix given by
\[
A = \begin{pmatrix}
\frac{48}{65} & \frac{4i}{13} & \frac{9}{5} \\
\frac{4i}{13} & 0 & \frac{4i}{13} \\
0 & \frac{4i}{13} & \frac{25}{3}
\end{pmatrix}
\]
Then \(A\) is unitarily similar to the complex symmetric matrix \(B = (b_{ij})\) whose entries are
\[
b_{11} = \frac{301737}{61862905} + \frac{i\sqrt{951737}}{1625\sqrt{2}},
\]
\[
b_{12} = \frac{9\sqrt{14694088721}}{1903474},
\]
\[
b_{13} = -\frac{12694959}{118967125},
\]
\[
b_{22} = \frac{284170}{951737},
\]
\[
b_{23} = \frac{9\sqrt{14694088721}}{1903474},
\]
\[
b_{33} = \frac{301737}{61862905} - \frac{i\sqrt{951737}}{1625\sqrt{2}}.
\]

**Proof.** Direct computations show that
\[
\text{tr}(A) = \text{tr}(B) = \frac{44}{65}, \quad \text{tr}(A^2) = \text{tr}(B^2) = \frac{6512}{4225},
\]
\[
\text{tr}(A^3) = \text{tr}(B^3) = \frac{-34624}{274625},
\]
and
\[
\text{tr}(A^* A^* A) = \text{tr}(B^* B^* B) = \frac{586124}{1015625},
\]
\[
\text{tr}(A^* A^* A^2) = \text{tr}(B^* B^* B^2) = \frac{103135881}{66015625},
\]
\[
\text{tr}(A^* A A^* A) = \text{tr}(B^* B B^* B) = \frac{505058466}{244140625}.
\]
Since \( B = B^T \), it follows from Theorem 2, \( A \) is unitarily similar to \( B \).

The matrix \( A \) considered in Example 2 belongs to the class \( S_3 \), completely non-unitary contraction with deficiency index 1, which will be discussed in the next section.

### 3 Main results

Let \( n \) be an integer \( \geq 2 \). Denote by \( S_n \) the class of \( n \times n \) complex matrices \( A \) satisfying the following conditions:

(i) The matrix \( A \) is a contraction, that is, the eigenvalues of \( A^* A \) are less than or equal to 1;

(ii) The matrix \( A \) has no eigenvalue of modulus 1 (this property is called completely non-unitary);

(iii) The dimension of the kernel of \( I_n - A^* A \) is 1 (this quantity is called deficiency index).

A matrix in \( S_n \) is called a completely non-unitary contraction with deficiency index 1. From the geometric viewpoint, the Poncelet porism of the numerical ranges of matrices in \( S_n \) has attracted many authors \([2, 3, 6, 8, 15, 16]\). The following main result asserts that matrices in \( S_n \) are unitarily similar to symmetric matrices.

**Theorem 3** Every matrix in \( S_n \) is unitarily similar to a complex symmetric matrix.

**Proof.** Let \( A \in S_n \). Then by \([7, \text{Corollary 1.3}]\) (see also \([15, \text{Theorem 4}]\)), the matrix \( A \) has a canonical upper triangular form. The matrix \( A \) also dilates to \( A_i \), \( i = 1, 2, \ldots, n \), \( A_i \) are expressed as

\[
A_i = \begin{pmatrix}
\lambda_i & \xi_i & \cdots & 0 \\
0 & \lambda_i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_i
\end{pmatrix},
\]

where \( \lambda_i \) is an eigenvalue of \( A \) and \( \xi_i \) is the corresponding eigenvector.

We assume the distinct eigenvalues of \( A \) are given by \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n \), and their respective corresponding eigenvectors are \( f_1, f_2, \cdots, f_{n-1}, f_n \). Let \( P \) be the \( n \)-dimensional orthogonal projection satisfying \( A = (PWP)_{\mathbb{C}^n} \). By replacing \( f_j \) by \( \exp(i\theta_j) f_j \) for some angles \( \theta_1, \ldots, \theta_{n+1} \), the space \( \mathbb{C}^n = P(\mathbb{C}^{n+1}) \) is expressed as

\[
\mathbb{C}^n = \{ z_1 f_1 + z_2 f_2 + \cdots + z_n f_n : (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}, \quad b_1 z_1 + b_2 z_2 + \cdots + b_{n+1} z_{n+1} = 0 \}
\]

for some non-negative real numbers \( b_1, b_2, \ldots, b_{n+1} \). Since the modulus of any eigenvalue of \( A \) is strictly less than 1, the numbers \( b_j \) are positive. Then the space \( \mathbb{C}^n = P(\mathbb{C}^{n+1}) \) consists of the linear spans of

\[
b_1 f_2 - b_2 f_1, \quad b_1 f_3 - b_3 f_1, \ldots, \quad b_1 f_{n+1} - b_{n+1} f_1.
\]

Let \( \{\xi_1, \xi_2, \ldots, \xi_n\} \) be an orthonormal basis of \( \mathbb{C}^n = P(\mathbb{C}^{n+1}) \) obtained by the Gram-Schmidt orthonormalization of \( n \) independent vectors in \( (3) \). The vectors \( \xi_j \) are expressed as

\[
\xi_j = \xi_{j,1} f_1 + \xi_{j,2} f_2 + \cdots + \xi_{j,n+1} f_{n+1}
\]

for some real numbers \( \xi_{j,k} \) with \( \xi_{j,j+1} > 0 \) and \( \xi_{j,j+2} = \cdots = 0, \quad j = 1, 2, \ldots, n \). With respect to the orthonormal basis \( \{\xi_1, \ldots, \xi_n\} \), the operator \( A \) on the \( n \)-dimensional Hilbert space \( \mathbb{C}^n \) satisfies the property

\[
\langle A \xi_{\ell}, \xi_{k} \rangle = \sum_{j=1}^{n+1} c_j \xi_{\ell,j} \xi_{k,j} = \sum_{j=1}^{n+1} c_j \xi_{k,j} \xi_{\ell,j} = \langle A \xi_{k}, \xi_{\ell} \rangle.
\]

Thus the operator \( A \) has a symmetric matrix representation with respect to this orthonormal basis \( \{\xi_1, \ldots, \xi_n\} \).

An \( n \times n \) matrix \( A = (a_{ij}) \) is called a Toeplitz matrix if \( a_{ij} = a_{k\ell} \) for every pairs \( (i, j), (k, \ell) \) satisfying \( i - j = k - \ell \). In this case, \( a_{ij} \) is denoted by \( a_{i-j} \) for some \( a_0, a_1, a_2, \ldots \). A typical Toeplitz matrix is the \( n \times n \) Jordan block \( J_n(0) \) corresponding to the eigenvalue 0. It is mentioned in \([12, \text{page 208}]\) that \( J_n(0) \) is unitarily similar to a complex symmetric matrix. In the following, we obtain symmetric unitary similarity of general Toeplitz matrices (An equivalent result can be found in \([19]\)).

**Theorem 4** Every Toeplitz matrix is unitarily similar to a complex symmetric matrix.

**Proof.** Let \( A = (a_{ij}) \) be an \( n \times n \) Toeplitz matrix with \( a_{ij} = a_{i-j} \). We regard \( A \) as a linear operator on the \( n \)-dimensional column vector space \( V = \{(y_1, y_2, \ldots, y_n)^T : y_j \in \mathbb{C}, 1 \leq j \leq n\} \):

\[
(y_1, y_2, \ldots, y_n)^T \rightarrow A(y_1, y_2, \ldots, y_n)^T = (Y_1, Y_2, \ldots, Y_n)^T
\]
We construct a circulant matrix $S$ acting on the $(2n-1)$-dimensional column vector space $W = \{(x_{-n+1}, x_{-n+2}, \ldots, x_0, \ldots, x_{n-2}, x_{n-1})^T : x_j \in \mathbb{C}, -n + 1 \leq j \leq n - 1\}$:

$$(x_{-n+1}, \ldots, x_{n-1})^T \rightarrow S(x_{-n+1}, \ldots, x_{n-1})^T = (X_{n+1}, \ldots, X_{n-1})^T$$

with

$$X_{n+1} = a_0 x_{-n+1} + a_{-1} x_{-n+2} + \cdots + a_{-n+1} x_{-n+2} + \cdots + a_{n-1} x_1 + a_n x_0 + a_0 x_{n+1},$$

$$X_{n+2} = a_1 x_{-n+1} + a_0 x_{-n+2} + \cdots + a_{n-1} x_1 + a_n x_0 + a_0 x_{n+1} + a_1 x_{n+2},$$

and so on.

Then $S$ is a normal matrix. In the case $n = 2m \geq 4$ is an even number, we consider the following embedding of $V$ into $W$:

$$(y_1, \ldots, y_m, y_{m+1}, \ldots, y_{2m})^T \rightarrow (y_{m+1}, \ldots, y_{2m}, 0_{n-1}, y_1, \ldots, y_m)^T \in W,$$

where $0_{n-1}$ is the $n-1$ copies of 0. In the case $n = 2m + 1 \geq 3$ is an odd number, we consider the following embedding $V$ into $W$:

$$(y_1, \ldots, y_{2m+1})^T \rightarrow (0_m, y_1, \ldots, y_{2m+1}, 0_{m})^T \in W.$$
1 \leq k, \ell \leq n$, and this proves that $A$ is unitarily similar to a complex symmetric matrix. \hfill \square

For a finite cyclic $G$, its dual group $\hat{G}$ is isomorphic to $G$. The harmonic analysis on $G$ is viewed as a special case of the analysis on a discrete abelian group or a compact group (cf. [18]). We explicitly construct, based on Theorem 4, a complex symmetric matrix which is unitarily similar to a given $n \times n$ Toeplitz matrix $A = (a_{ij})$ with $a_{ij} = a_{i-j}$.

Firstly, for $n = 2m$, we set
\[
\phi_p = \frac{1}{\sqrt{2}}(\chi_p + \chi_{-p}), \quad \psi_p = \frac{1}{i\sqrt{2}}(\chi_p - \chi_{-p}),
\]
$m \leq p \leq 2m - 1$. Then for the normal dilation $S$ of $A$ in Theorem 4, the following equations hold:
\[
\langle S\phi_p, \phi_q \rangle = \frac{1}{2}(a_{2m-1-p-q} + a_{-4m+1+p+q} + a_{-p+q} + a_{p-q}),
\]
\[
\langle S\psi_p, \psi_q \rangle = \frac{1}{2}(a_{2m-1-p-q} - a_{4m-1+p+q} - a_{-4m+1+p+q}),
\]
\[
\langle S\phi_p, \psi_q \rangle = \langle S\psi_p, \phi_q \rangle = \frac{-i}{2}(a_{2m-1-p-q} - a_{4m-1+p+q} + a_{p-q} - a_{-p+q}),
\]
$m \leq p, q \leq 2m - 1$.

Secondly, for $n = 2m + 1$, we set
\[
\phi_0 = \chi_0, \quad \phi_p = \frac{1}{\sqrt{2}}(\chi_p + \chi_{-p}), \quad \psi_p = \frac{1}{i\sqrt{2}}(\chi_p - \chi_{-p}),
\]
$1 \leq p \leq m$. Then the following equations hold:
\[
\langle S\phi_0, \phi_0 \rangle = a_0,
\]
\[
\langle S\phi_0, \phi_p \rangle = \langle N\phi_p, \phi_0 \rangle = \frac{1}{\sqrt{2}}(a_p + a_{-p}),
\]
\[
\langle S\phi_0, \psi_p \rangle = \langle N\psi_p, \phi_0 \rangle = \frac{i}{\sqrt{2}}(a_p - a_{-p}),
\]
\[
\langle S\phi_p, \phi_q \rangle = \frac{1}{2}(a_{p+q} + a_{-p-q} + a_{-p+q} + a_{p-q}),
\]
\[
\langle S\psi_p, \psi_q \rangle = \frac{1}{2}(a_{p+q} + a_{-p-q} - a_{p+q} - a_{-p-q}),
\]
\[
\langle S\phi_p, \psi_q \rangle = \langle S\psi_p, \phi_q \rangle = \frac{i}{2}(a_{p+q} + a_{-p+q} - a_{p-q} - a_{-p-q}).
\]

**Example 3** Consider a $4 \times 4$ Toeplitz matrix $A = (a_{ij})$ with $a_{ij} = a_{i-j}$, the complex symmetric matrix $B = (b_{ij})$ constructed above which is unitarily similar to $A$ is given by
\[
\begin{align*}
b_{11} &= (a_3 + a_{-4} + 2a_0)/2, \\
b_{12} &= (a_2 + a_{-2} + a_1 + a_{-1})/2, \\
b_{13} &= (-i(a_3 - a_{-3}))/2, \\
b_{14} &= (-i(a_2 - a_{-2} + a_{-1} - a_1))/2, \\
b_{22} &= (a_1 + a_{-1} + 2a_0)/2, \\
b_{23} &= (-i(a_2 - a_{-2} + a_1 - a_{-1}))/2, \\
b_{24} &= (-i(a_1 - a_{-1}))/2, \\
b_{33} &= (2a_0 - a_3 - a_{-3})/2, \\
b_{34} &= (a_1 + a_{-1} - a_2 - a_{-2})/2, \\
b_{44} &= (2a_0 - a_1 - a_{-1})/2,
\end{align*}
\]

The $c$-numerical range of a special upper triangular nilpotent Toeplitz matrix $A$ with the first row $(0, a_1, a_2, \ldots, a_{n-1})$ satisfying $a_1 = a_2 = \cdots = a_{n-1}$ is investigated in [1]. We have proved in Theorem 4 that such matrix $A$ is unitarily similar to a symmetric matrix. In a generic case, the Hermitian matrix $\mathfrak{H}(A)$ has no repeated eigenvalues. We conjecture, under this assumption, the matrix $A$ is unitarily similar to the symmetric matrix $B = H + iK$, where $H$ and $K$ are real symmetric matrix obtained from the determinantal representation $F_A(x, y, z) = \det(\sigma H + yK + zI_n)$ via Fiedler’s formula in [5, Theorem 1].

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**References:**


