Temporal instability of particle-laden curved shallow mixing layers with non-constant friction coefficient

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***Abstract*—** In the present paper we analyze the combined effect of small curvature, non-uniform friction and presence of small heavy particles in a carrier fluid on linear stability of shallow mixing layers. The linear stability problem is solved numerically. Marginal stability curves and marginal stability surfaces are constructed for different values of the parameters of the problem. It is shown that for the case of stably curved mixing layers all three parameters (particle loading parameter, small curvature and non-uniform friction) stabilize the flow.

***Keywords*—**Linear stability, particle-laden flow, shallow water, friction coefficient, curvature

I. Introduction

FLOWS at river junctions or in compound and composite channels represent widespread examples of shallow mixing layers. It is well-known from linear stability analyses of shallow mixing layers that bottom friction in a shallow layer of fluid plays an important role in preventing the development of three-dimensional instabilities [1]-[4]. In addition, bottom friction also stabilizes the flow since growth rates of small perturbations are reduced by the presence of a solid boundary. Experimental investigations show that the growth a mixing layer is also affected by the presence of bottom friction [5]-[8]. In practice shallow mixing layers can be also slightly curved. The effect of small curvature on the stability of free shear layers is investigated in [9] where it is shown that curvature has a stabilizing effect for the case of stably curved mixing layer and destabilizes the flow for unstably curved layer.

The analysis in [1]-[4] is performed for the case where bottom friction is modeled by means of the Chezy formula [10], where the friction coefficient is assumed to be constant in the transverse direction. Recent experimental investigations [11]-[15] showed that there are cases where the friction force changes considerably in the transverse direction. One important application of such a case in practice is flow in compound channels (or rivers) during floods. In this case friction in the floodplain is much higher than the friction in the main channel. It is shown in [11]-[15] that the characteristics of mass and momentum exchange in case of variable friction are different from the corresponding characteristics for the case of constant friction.

In many cases water flows in rivers and channels contain particles [16]. The presence of heavy particles also can affect the dynamics of the flow and, in particular, modify linear stability characteristics of the flow. Spatial and temporal instability of slightly curved particle-laden shallow mixing layers for the case of constant friction is investigated in [17].

In the present paper we investigate the combined effect of small curvature, variable friction in the transverse direction and presence of small heavy particles on the stability characteristics of shallow mixing layers. The corresponding linear stability problem is solved numerically for different values of the parameters of the problem. It is shown that increase of the particle concentration and small curvature, as well as bottom friction has stabilizing effect on the flow.

II. Mathematical Analysis

Shallow water equations under the rigid-lid assumption can be written in the form [17]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f(y)}{2h}u\sqrt{u^2+v^2}=B(u^p-u), \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{R}u^2+\frac{\partial p}{\partial y}+\frac{c_f(y)}{2h}v\sqrt{u^2+v^2}=B(v^p-v), \tag{3}$$

where p is the pressure, u and v are the velocity components of the fluid in the x and y-directions, respectively, u^p and v^p are the velocity components of particles, h is water depth, $c_f(y)$ is the non-constant friction coefficient, B is the particle loading parameter [16], [17], and R is the radius of curvature ($R >> 1$).

The following simplifying assumptions are used to derive (1)-(3): (a) equation (1) represents the rigid-lid assumption (water depth is constant); (b) distribution of particles in the

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fluid is uniform; (c) curvature is small ($1/R << 1$), (d) there is no dynamic interaction between particles and the fluid. Since the continuity equation (1) is the same as in a two-dimensional hydrodynamics we introduce the stream function $\psi(x,y,t)$ by the formulas

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x} \tag{4}$$

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Eliminating the pressure from (1)-(4) we obtain

$$\begin{aligned}&(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \\ &+ \frac{c_f(y)}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} + \frac{c_f(y)}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + \\ &+ 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) + \frac{c_{fy}(y)}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2} \\ &+ B\Delta\psi = 0.\end{aligned} \tag{5}$$

We assume that the friction coefficient $c_f(y)$ can be represented in the form

$$c_f(y) = c_{f_0}\gamma(y), \tag{6}$$

where c_{f_0} is constant and $\gamma(y)$ is a differentiable shape function.

Consider a perturbed stream function $\psi(x,y,t)$ of the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + ... \tag{7}$$

where $\psi_0(y)$ is the stream function of the base flow $U(y)$ and ε is a small parameter. The base flow is assumed in the form

$$U(y) = \frac{1}{2}(1 + \tanh\, y). \tag{8}$$

Using a standard linearization procedure around the base flow $U(y)$ we obtain

$$\begin{aligned}&\psi_{1xxt} + \psi_{1yyt} + \psi_{0y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{0yyy}\psi_{1x} + \\ &+ \frac{c_f(y)}{2h}(\psi_{0y}\psi_{1xx} + 2\psi_{0yy}\psi_{1y} + 2\psi_{0y}\psi_{1yy}) \\ &+ \frac{c_{fy}(y)}{h}\psi_{0y}\psi_{1x} + \frac{2}{R}\psi_{0y}\psi_{1xy} + B(\psi_{1xx} + \psi_{1yy}) = 0.\end{aligned} \tag{9}$$

Using the method of normal modes we assume that ψ_1 is represented in the form

$$\psi_1(x,y,t) = \varphi(y)e^{i\alpha(x-ct)}, \tag{10}$$

where α is the wave number of unsteady perturbation and $c = c_r + ic_i$ is a complex eigenvalue. Substituting (10) into (9) we obtain

$$\begin{aligned}&\varphi_{yy}[\alpha(U-c) - iSU\gamma - iB) - iS(\gamma U_y + \gamma_y U)\varphi_y \\ &+ \varphi[\alpha^3(c-U) - \alpha U_{yy} + i\alpha^2 US\gamma/2 + i\alpha^2 B) = 0,\end{aligned} \tag{11}$$

where $S = \frac{c_{f_0}b}{h}$ is the stability parameter and b is a characteristic length scale of the problem (for example, half-width of the mixing layer).

The boundary conditions are

$$\varphi(\pm\infty) = 0. \tag{12}$$

Linear stability of the base flow (8) is determined by the sign of the imaginary part of the complex eigenvalue $c = c_r + ic_i$ of eigenvalue problem (11), (12). The base flow $U(y)$ is said to be linearly unstable if at least one eigenvalue satisfies the inequality $c_i > 0$. If all $c_i < 0$ then the base flow is linearly stable. Small perturbation is marginally stable if all eigenvalues have negative imaginary parts while one eigenvalue satisfies the condition $c_i = 0$. Problem (11), (12) is solved numerically in the next section for different values of the parameters of the problem.

III. Numerical results

We use the collocation method based on Chebyshev polynomials [4] to solve (11), (12) numerically.

The shape function in (6) is assumed to be of the form

$$\gamma(y) = \frac{\beta+1}{2} + \frac{(\beta-1)}{2}\tanh\, y, \tag{13}$$

where $\beta = \frac{c_{f_1}}{c_{f_0}} \geq 1$ is the ratio of the friction coefficients in the floodplain and main channel.

Fig. 1 plots the marginal stability curves for the case of uniform friction ($\beta = 1$) and straight channel ($R = \infty$). Three curves in Fig. 1 (from top to bottom) correspond to the following three values of the particle loading parameter B: 0, 0.02 and 0.04. As can be seen from the figure, the particle loading parameter has a stabilizing effect on the flow (the critical bed friction number S_{cr} decreases as the parameter B increases.

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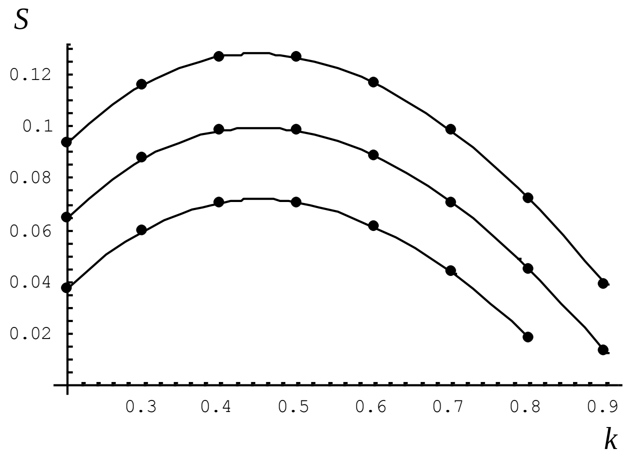

Fig. 1. Marginal stability curves for the case $R=\infty, \beta=1$ and three values of $B: B=0, 0.02$ and $B=0.04$ (from top to bottom).

The effect of non-uniform friction on the marginal stability curves for the case $R=\infty, \beta=1.5$ is shown in Fig. 2. The three curves in Fig. 2 correspond to the same values of B as in Fig. 1. Comparing Figs. 1 and 2 we can see that non-uniform friction stabilizes the flow: the maxima of the marginal stability curves in Fig. 2 occur at lower values of S than in Fig. 1 (uniform friction).

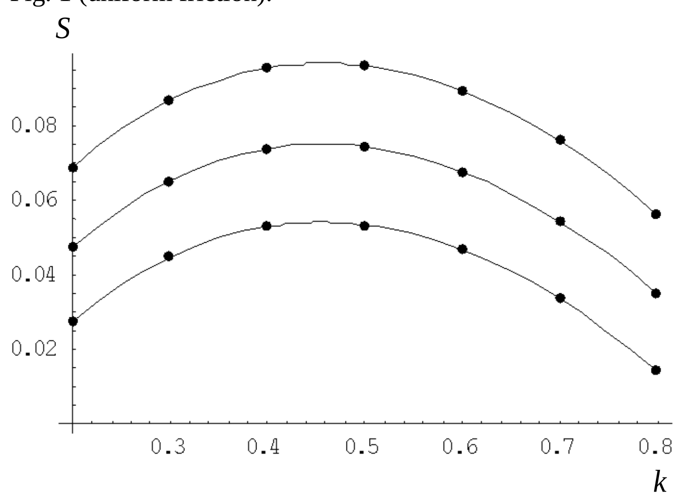

Fig. 2. Marginal stability curves for the case $R=\infty, \beta=1.5$ and three values of $B: B=0, 0.02$ and $B=0.04$ (from top to bottom).

In order to analyze the combined effect of the three parameters R, β and B on the stability boundary we calculated the critical values of the bed friction number S, namely, $S_{cr}=\max_k S(k)$ for several values of the parameters. The results (marginal stability surfaces) are shown in Figs. 3 – 5. Fig. 3 plots the critical values of S (the vertical ass) for different values of β and $B: 1 \le \beta \le 3$ and $0 \le B \le 0.05$ for $R=\infty$.

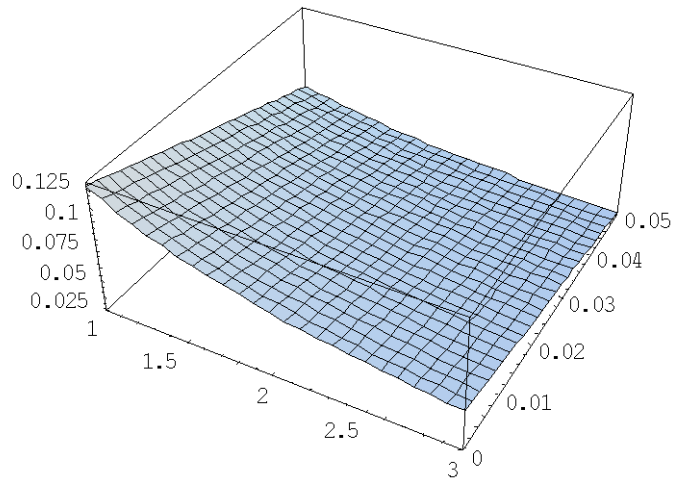

Fig. 3. Marginal stability surfaces for the case $R=\infty$ and different values of β and B.

Similar graphs are shown in Fig. 4 where the marginal stability surfaces are shown for the case $1/R=0.03$ (slightly curved mixing layer).

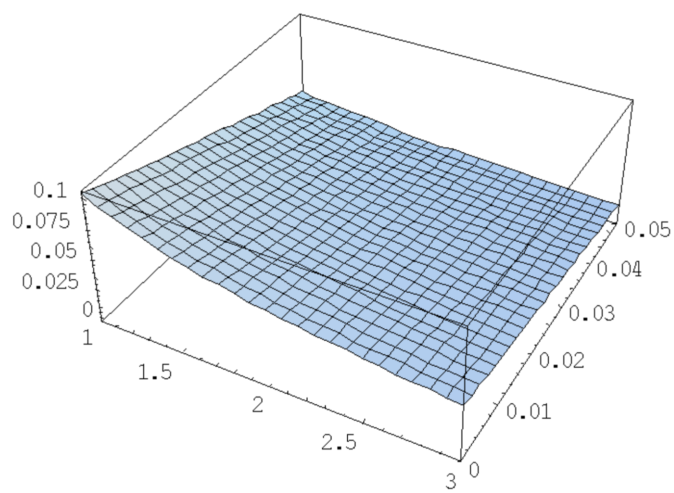

Fig. 4. Marginal stability surfaces for the case $1/R=0.03$ and different values of β and B.

The stabilizing effect of small curvature can be seen from the analysis of Figs. 3 and 4: critical bed friction numbers decrease as $1/R$ increases.

Larger value of the parameter $1/R$ is shown in Fig. 5 (the marginal stability surfaces are constructed for the same range of β and B values as in Figs. 3 and 4). Stabilization of the base flow is even more pronounced for the case $1/R=0.06$.

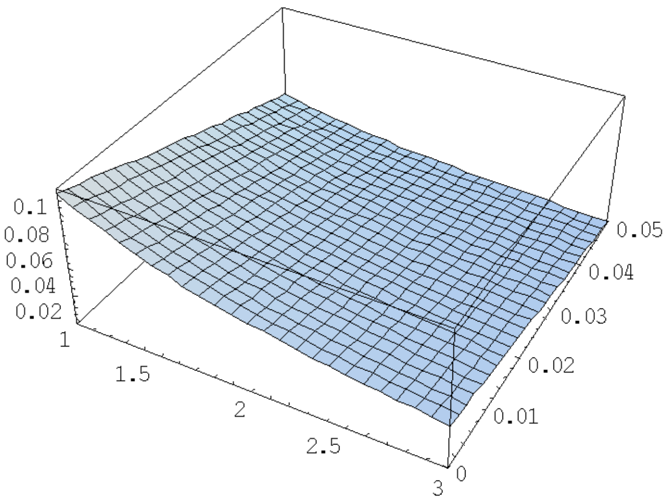

Fig. 5. Marginal stability surfaces for the case $1/R = 0.06$ and and different values of β and B.

Comparing Figs. 3 – 5 we see that all three parameters R, β and B have a stabilizing effect on the flow: the critical bed friction number decreases as all parameters increase. It would be interesting to see what happens in the case of unstably curved mixing layer since previous studies have shown that curvature has a destabilizing effect on unstably curved mixing layers [9]. Thus, one can expect to see a competition between destabilizing effect of curvature and stabilizing influence of non-uniform friction and presence of particles in the flow. The authors are currently working on this topic.

IV. Conclusion and Direction for Future Work

Linear stability of shallow mixing layers in the presence of a non-uniform friction in the transverse direction is investigated. It is assumed that carrier fluid contains small heavy particles. The flow is also slightly curved in a longitudinal direction. The combined effect of three parameters: small curvature, particle loading parameter and non-uniform friction is investigated. It is shown that all three parameters have a stabilizing influence on the flow.

Linear stability can be also analyzed for the case where mixing layer is unstably curved. It is known from the previous studies that in this case small curvature destabilizes the flow. On the other hand, stabilizing influence of particle loading parameter is confirmed for the case of fluid flow with uniform friction. The authors are currently analyzing the combined effect of all parameters on linear stability of unstably curved mixing layer. In addition, it is planned to develop weakly nonlinear theory for the case where the bed friction number is slightly smaller than the critical value.

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References

[1] V.H. Chu, J.H. Wu, and R.E. Khayat, “Stability of transverse shear flows in shallow open channels”, *J. Hydr. Engng.*, vol. 117, pp. 1370-1388, 1991.

[2] D. Chen, and G.H. Jirka, “Linear stability analysis of turbulent mixing layers and jets in shallow water”, *J. Hydr. Res.*, vol. 36, pp. 815-830, 1998.

[3] M.S. Ghidaoui, and A.A. Kolyshkin, “Linear stability analysis of motions in compound open channels”, *J. Hydr. Engng.*, vol. 125, pp. 871-880, 1999.

[4] A.A. Kolyshkin, and M.S. Ghidaoui, “Gravitational and shear instabilities in compound and composite channels, *J. Hydr. Engng.*, vol. 128, pp. 1076-1086, 2002.

[5] V.H. Chu, and S. Babarutsi, “Confinement and bed-friction effects in shallow turbulent mixing layers”, *J. Hydr. Engng*., vol. 114, pp. 1257-1274, 1988.

[6] W.S.J Uijttewaal, and J. Tukker, “Development of quasi two-dimensional structures in shallow free-surface mixing layer,” *Exp. in Fluids*, vol. 24, pp. 192–200, 1998.

[7] W.S.J Uijttewaal, and R. Booij, “Effect of shallowness on the development of free-surface mixing layers,” *Phys. Fluids*, vol. 12, pp. 392–402, 2000.

[8] B.C. Prooijen, and W.S.J Uijttewaal, “A linear approach for the evolution of coherent structures in shallow mixing layers”, *Phys. Fluids*, vol. 14, pp. 4105-4114, 2002.

[9] W. Liou, “Linear instability of free curved shear layers”, *Phys. Fluids*, vol. 6, pp. 541-549, 1993.

[10] V.L. Streeter, and E.B. Wylie, *Fluid mechanics*, Mc-Graw Hill, 1998.

[11] H. Nepf, “Hydrodynamics of vegetated channels,” *J. Hydr. Res.*, vol. 50, pp. 262–279, 2012.

[12] H. Nepf, “Flow and transport in regions with aquatic vegetation,” *Ann. Rev. of Fluid Mech.*, vol. 44, pp. 123–142, 2012.

[13] X. Zhang, and H. Nepf, “Exchange flow between open water and floating vegetation,” *Environ. Fluid Mech.*, vol. 11, pp. 531–546, 2011.

[14] Jamali, M., X. Zhang, and H. Nepf, “Exchange flow between canopy and open water,” *J. Fluid Mech.*, vol. 611, pp. 237–254, 2008.

[15] B. White, and H. Nepf, “Shear instability and coherent structures in shallow flow adjacent to a porous layer,” *J. Fluid Mech.*, vol. 593, pp. 1–32, 2007.

[16] Y.Yang, J.N. Chung, T.R. Troutt, and C.T. Crowe, “The influence of particles on the stability of two-phase mixing layers” *Phys. Fluids*, vol. A2, pp. 1839-1845, 1990.

[17] I. Eglite, and A. Kolyshkin,“Spatial and temporal instability of slightly curved particle-laden shallow mixing layers”, *Proceedings of the V international conference on computational methods for coupled problems in science and engineering,* Ibiza, Spain, June 17-19, 2013, 8 pages (publication on USB).