The Change in Impedance of a Coil above a Plate with a Flaw

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Abstract: - In the present paper we consider a coil with alternating current located above a conducting plate with a cylindrical flaw. The axis of the coil coincides with the axis of the flaw. The problem is solved by the method of separation of variables under the assumption that the vector potential is equal to zero at a sufficiently large distance from the axis of the coil. The formula for the induced change in impedance of the coil is obtained. Results of numerical calculations are presented for different values of the parameters of the problem. The method of solution described in the paper can be applied to other axisymmetric flaws.

Key-Words: - Change in impedance, separation of variables, eigenvalues, eddy current testing

1 Introduction

Solutions to eddy current testing problems for the case where a conducting medium is infinite in one or two spatial dimensions are well-known in the literature [1]-[3]. In applications it is often necessary to consider conducting objects of finite size. Since the method of integral transforms for the solution of eddy current problems for infinite media cannot be applied in this case, numerical methods (such as finite element methods) are used [4]. Recently a semi-analytical method (TREE method) is suggested for the solution of eddy current problems [3]. The main idea of the method is that the electromagnetic field is exactly zero at a sufficiently large distance from the source of alternating current. As a result, one obtains a boundary value problem in a finite domain which can be solved by the method of separation of variables. Examples of the use of the TREE method can be found in [6]-[8].

In the present paper we consider the case where a coil with alternating current is located above a conducting plate with a flaw in the form of a cylinder coaxial with the coil. The problem is solved by the TREE method where two steps of the solution process require the use of numerical methods: (a) calculation of complex eigenvalues for the case where a good initial guess for the root is not known and (b) solution of a system of linear algebraic equations. The change in impedance of the coil is computed for different frequencies of the excitation current. The solution of the given problem can be used in practice to model the effect of corrosion in metal plates.

2 Mathematical Formulation

Consider an air-core coil located above a conducting plate with conductivity σ (see Fig. 1).

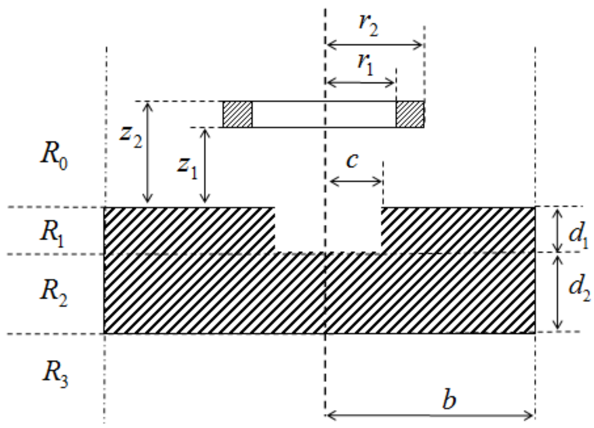

Fig. 1. A coil above a conducting plate.

The parameters of the coil are as follows: r_1 and r_2 are the inner and outer radii, respectively, $z_2 - z_1$ is the height of the coil (z_1 is the distance from the bottom of the coil to the plate), N is the number of turns. The plate has a cylindrical hole of radius c and height d_1. The axis of the coil coincides with the axis of the cylinder. The height of the plate is $d_1 + d_2$.

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The solution of the problem for the coil can be found by the superposition principle if we know the solution for the case where a single-turn coil is located above a plate with a flaw (Fig. 2).

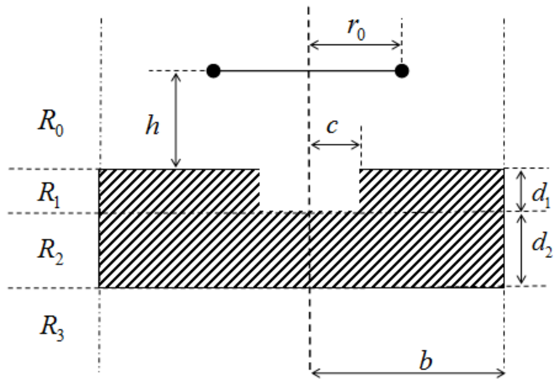

Fig. 2. A single-turn coil above a conducting plate.

Due to axial symmetry the vector potential has only one non-zero component in the azimuthal direction. It is convenient to introduce four regions $R_0 - R_3$ and denote the solutions in each of the regions by A_0, A_1, A_2 and A_3, respectively. The system of equations for the components of the vector potential has the form [8]

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0)\delta(z - h), \quad (1)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma_1\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (2)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (3)$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} + \frac{\partial^2 A_3}{\partial z^2} = 0, \quad (4)$$

where $\delta(x)$ is the Dirac delta-function, $\sigma_1 = 0$ if $0 \le r < c$ and $\sigma_1 = \sigma$ if $c < r < b$, b is the distance from the axis of the coil where the electromagnetic field is assumed to be exactly zero, and ω is the frequency.

The boundary conditions are

$$A_i |_{r=b} = 0, \quad i = 0,12,3, \quad (5)$$

$$A_0 |_{z=0} = A_1^a |_{z=0}, \quad \frac{\partial A_0}{\partial z}\Big|_{z=0} = \frac{\partial A_1^a}{\partial z}\Big|_{z=0}, \quad 0 \le r < c, \quad (6)$$

$$A_0 |_{z=0} = A_1^c |_{z=0}, \quad \frac{\partial A_0}{\partial z}\Big|_{z=0} = \frac{\partial A_1^c}{\partial z}\Big|_{z=0}, \quad c < r < b, \quad (7)$$

$$A_1^c |_{z=-d_1} = A_2 |_{z=-d_1}, \frac{\partial A_1^c}{\partial z}\Big|_{z=-d_1} = \frac{\partial A_2}{\partial z}\Big|_{z=-d_1}, c < r < b, \quad (8)$$

$$A_1^a |_{z=-d_1} = A_2 |_{z=-d_1}, \frac{\partial A_1^a}{\partial z}\Big|_{z=-d_1} = \frac{\partial A_2}{\partial z}\Big|_{z=-d_1}, \ 0 \le r < c, \quad (9)$$

$$A_2 |_{z=-d_3} = A_3 |_{z=-d_3}, \frac{\partial A_2}{\partial z}\Big|_{z=-d_3} = \frac{\partial A_3}{\partial z}\Big|_{z=-d_3}, \quad (10)$$

where $d_3 = d_1 + d_2$ and the superscripts a and c correspond to air and conductive region, respectively.

The interface conditions at $r = c$ have the form

$$A_1^c |_{r=c} = A_1^a |_{r=c}, \quad \frac{\partial A_1^c}{\partial r}\Big|_{r=c} = \frac{\partial A_1^a}{\partial r}\Big|_{r=c}. \quad (11)$$

In addition, vector potential is bounded at infinity in regions R_0 and R_3:

$$A_0 \to 0 \text{ as } z \to +\infty, A_3 \to 0 \text{ as } z \to -\infty. \quad (12)$$

3 Solution for Single-Turn Coil

In order to find the solution to (1) we consider two sub-regions of region R_0, namely, $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$. The solutions in R_{00} and R_{01} are denoted by A_{00} and A_{01}, respectively. Using the principle of superposition we represent the solutions to (1) in R_{00} and R_{01} in the form

$$A_{01}(r,z) = \sum_{i=1}^{\infty} D_{1i} e^{-\lambda_i z} J_1(\lambda_i r), \quad (13)$$

$$A_{00}(r,z) = \sum_{i=1}^{\infty} (D_{2i} e^{-\lambda_i z} + D_{3i} e^{\lambda_i z}) J_1(\lambda_i r), \quad (14)$$

where D_{1i}, D_{2i} and D_{3i} are arbitrary constants, $\lambda_i = \alpha_i / b$ and α_i are the roots of the equation

$$J_1(\alpha) = 0. \quad (15)$$

The vector potential is continuous at $z = h$:

$$A_{00} |_{z=h} = A_{01} |_{z=h}. \quad (16)$$

Integrating (1) with respect to z from $h - \varepsilon$ to $h + \varepsilon$ and considering the limit as $\varepsilon \to +0$ in the resulting equation we obtain

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$$\frac{\partial A_{01}}{\partial z}|_{z=h} - \frac{\partial A_{00}}{\partial z}|_{z=h} = -\mu_0 I\delta(r-r_0). \tag{17}$$

It follows from (13)-(17) that

$$\sum_{i=1}^{\infty} D_{1i}e^{-\lambda_i h}J_1(\lambda_i r) = \sum_{i=1}^{\infty}(D_{2i}e^{-\lambda_i h} + D_{3i}e^{\lambda_i h})J_1(\lambda_i r), \tag{18}$$

$$\sum_{i=1}^{\infty}\lambda_i D_{1i}e^{-\lambda_i h}J_1(\lambda_i r) + \sum_{i=1}^{\infty}(-\lambda_i D_{2i}e^{-\lambda_i h} + \lambda_i D_{3i}e^{\lambda_i h})J_1(\lambda_i r) = \mu_0 I\delta(r-r_0). \tag{19}$$

Multiplying (18) by $rJ_1(\lambda_j r)$, integrating the resulting equation with respect to r from 0 to b and using the orthogonality condition

$$\int_0^b rJ_1(\lambda_j r)J_1(\lambda_i r)dr = \begin{cases} 0, & i \neq j \\ \frac{b^2}{2}J_0^2(\lambda_j b), & i = j \end{cases} \tag{20}$$

the following equation is obtained

$$D_{1j}e^{-\lambda_i h} = D_{2j}e^{-\lambda_j h} + D_{3j}e^{\lambda_j h}. \tag{21}$$

Applying the same procedure to (19) we obtain

$$\left(D_{1j}e^{-\lambda_j h} - D_{2j}e^{-\lambda_j h} + D_{3j}e^{\lambda_j h}\right)\lambda_j \frac{b^2}{2}J_0^2(\lambda_j b) = \mu_0 Ir_0 J_1(\lambda_j r_0). \tag{22}$$

Using (21) and (22) we get

$$D_{3j} = \frac{\mu_0 Ir_0 J_1(\lambda_j r_0)}{\lambda_j b^2 J_0^2(\lambda_j b)}e^{-\lambda_j h}. \tag{23}$$

Substituting (22) and (23) into (13) and (14) we obtain

$$A_{00}(r,z) = \sum_{i=1}^{\infty} D_{2i}e^{-\lambda_i z}J_1(\lambda_i r) + \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i(h-z)}J_1(\lambda_i r), \tag{24}$$

$$A_{01}(r,z) = \sum_{i=1}^{\infty} D_{2i}e^{-\lambda_i z}J_1(\lambda_i r) + \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i(z-h)}J_1(\lambda_i r). \tag{25}$$

Solution to (2) satisfying (11) has the form

$$A_1^a(r,z) = \sum_{i=1}^{\infty} J_1(p_i r)T_1(q_i c)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}), \tag{26}$$

$$A_1^c(r,z) = \sum_{i=1}^{\infty} J_1(p_i c)T_1(q_i r)(\hat{D}_{6i}e^{p_i z} + \hat{D}_{8i}e^{-p_i z}), \tag{27}$$

where

$T_1(q_i r) = J_1(q_i r)Y_1(q_i b) - J_1(q_i b)Y_1(q_i r),1$

$p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$,

and the equation

$$p_i J_1'(p_i c)T_1(q_i c) = q_i T_1'(q_i c)J_1(p_i c) \tag{28}$$

determines complex eigenvalues p_i.

General solution to (3) satisfying (5) is

$$A_2(r,z) = \sum_{i=1}^{\infty}(D_{9i}e^{p_{1i}z} + D_{10i}e^{-p_{1i}z})J_1(\lambda_i r), \tag{29}$$

where

$p_{1i} = \sqrt{\lambda_i^2 + j\omega\sigma\mu_0}$.

Solution to (4), (12) has the form

$$A_3(r,z) = \sum_{i=1}^{\infty} D_{11i}e^{\lambda_i z}J_1(\lambda_i r). \tag{30}$$

The six sets of constants in (24)-(27), (29) and (30),namely, $D_{2i}, \hat{D}_{6i}, \hat{D}_{8i}$, D_{9i}, D_{10i} and D_{11i} can be determined from the boundary conditions (6) –(10). Eliminating D_{2i}, D_{9i}, D_{10i} and D_{11i}, we obtain the following system of algebraic equations for the coefficients $\hat{D}_{6i}$ and $\hat{D}_{8i}$:

$$\sum_{i=1}^{n}[(\lambda_j + p_i)\hat{D}_{6i} + (\lambda_j - p_i)e\hat{D}_{8i}]a_{ji} = \mu_0 Ir_0 J_1(\lambda_j r_0)e^{-\lambda_j h} \tag{31}$$

$$\sum_{i=1}^{n}[(g_{ij}\hat{D}_{6i} + f_{ij}\hat{D}_{8i}]a_{ji} = 0, \tag{32}$$

where

$$g_{ij} = (\lambda_j - p_{1j})(p_{1j} + p_i)e^{(p_{1j}-p_i)d - p_{1j_1}d_3} + (\lambda_j + p_{1j})(p_{1j} - p_i)e^{-(p_{1j}+p_i)d + p_{1j_1}d_3},$$

$$f_{ij} = (\lambda_j - p_{1j})(p_{1j} - p_i)e^{(p_{1j}-p_i)d - p_{1j_1}d_3} + (\lambda_j + p_{1j})(p_{1j} + p_i)e^{-(p_{1j}-p_i)d + p_{1j_1}d_3},$$

$$a_{ji} = T_1(q_i c)\tilde{a}_{ji} + J_1(p_i c)\tilde{\tilde{a}}_{ji},$$

$$\tilde{a}_{ji} = \int_0^c rJ_1(\lambda_j r)J_1(p_i r)dr$$
$$= \frac{c}{\lambda_j^2 - p_i^2}\left(\lambda_j J_2(\lambda_j c)J_1(p_i c) - p_i J_1(\lambda_j c)J_2(p_i c)\right)$$
$$\tilde{\tilde{a}}_{ji} = \int_c^b rJ_1(\lambda_j r)T_1(q_i r)dr$$
$$= Y_1(q_i b)\int_c^b rJ_1(\lambda_j r)J_1(q_i r)dr - J_1(q_i b)\int_c^b rJ_1(\lambda_j r)Y_1(q_i r)dr$$
$$= \frac{1}{\lambda_j^2 - q_i^2}\left\{\begin{array}{l} bq_i J_1(\lambda_j b)[J_1(q_i b)Y_2(q_i b) - J_2(q_i b)Y_1(q_i b)] + \\ + c\lambda_j J_2(\lambda_j c)[J_1(q_i b)Y_1(q_i c) - J_1(q_i c)Y_1(q_i b)] + \\ + cq_i J_1(\lambda_j c)[J_2(q_i c)Y_1(q_i b) - J_1(q_i b)Y_2(q_i c)] \end{array}\right\}$$

Note that the upper limit of the index of summation in (31) and (32) is finite (it represents the number of terms in the series). System (31), (32) has to be solved numerically. Solving (31), (32) we obtain the coefficients $\hat{D}_{6i}$ and $\hat{D}_{8i}$.

The induced vector potential has the form

$$A_0^{ind}(r_0, h) = \sum_{j=1}^{n} D_{2j} e^{-\lambda_j h} J_1(\lambda_j r_0), \qquad (33)$$

where

$$D_{2j} = \frac{2}{b^2 J_0^2(\lambda_j b)} \sum_{i=1}^{n} a_{ji}(\hat{D}_{6i} + \hat{D}_{8i}) - \frac{\mu_0 I r_0 J_1(\lambda_j r_0) e^{-\lambda_j h}}{\lambda_j b^2 J_0^2(\lambda_j b)}. \qquad (34)$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I} 2\pi r_0 A_0^{ind}(r_0, h). \qquad (35)$$

The induced change in impedance is computed for the following values of the parameters of the problem: $\mu_0 = 4 \cdot 10^{-7}\pi$, $\sigma = 3.0$ Ms/m, $c = 2.2$ mm, $b = 55$ mm, $d_1 = 0.5$ mm, $d_2 = 10$ mm, $r_0 = 4.5$ mm, $h = 0.2$ mm. The results are shown in Fig. 3.

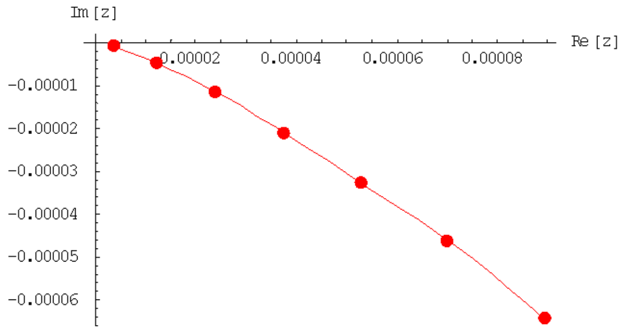

Fig. 3. The change in impedance of a single-turn coil for seven different frequencies.

The real and imaginary parts of the change in impedance are shown for seven frequencies (1 kHz, 2 kHz, …,7 kHz from top to bottom).

4 Solution for a Coil of Finite Dimensions

The induced vector potential for a coil with finite dimensions shown in Fig. 1 can be computed using the principle of superposition:

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2}\int_{z_1}^{z_2} A_0^{ind}(r, z, r_0, h)dr_0 dh. \qquad (36)$$

The induced change in impedance of the coil is given by

$$Z^{ind} = \frac{2\pi j\omega}{I}\frac{N}{(r_2 - r_1)(z_2 - z_1)}\int_{r_1}^{r_2}\int_{z_1}^{z_2} rA_{0coil}^{ind}(r, z)drdz. \qquad (37)$$

Substituting (33) into (36) and (37) we obtain

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2(z_2 - z_1)^2}\sum_{i=1}^{n}\frac{\left(e^{-\lambda_j z_1} - e^{-\lambda_j z_2}\right)}{\lambda_i^3}\int_{\lambda_j r_1}^{\lambda_j r_2}\xi J_1(\xi)d\xi$$
$$\sum_{k=1}^{n} Y_{ik}\frac{\left(e^{-\lambda_k z_1} - e^{-\lambda_{ik} z_2}\right)}{\lambda_k^3}\int_{\lambda_k r_1}^{\lambda_k r_2}\xi J_1(\xi)d\xi \qquad (38)$$

where the elements of the matrix Y are not shown for brevity.

Formula (38) is used to compute the change in impedance of the coil for seven frequencies from 1 kHz to 7 kHz. The results are shown in Fig. 4.

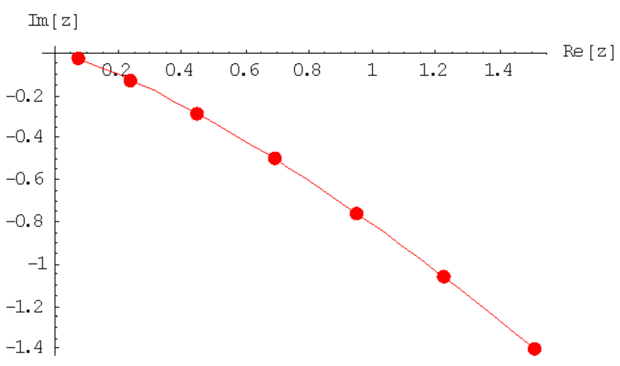

Fig. 4. The change in impedance of a coil for seven different frequencies (from top to bottom).

The parameters of the coil are as follows: $z_1 = 0.3\,\text{mm}, z_2 = 2.6\,\text{mm}, r_1 = 3.5\,\text{mm}$, $r_2 = 5.5\,\text{mm}$, $N = 200$. The other parameters are as in Fig. 3. As can be seen from Fig. 4, the modulus of the change in impedance increases as the frequency increases.

4 Conclusion

The method of truncated eigenfunction expansions is used in the present paper to compute the change in impedance of a coil due to a cylindrical flaw in a conducting plate. The problem is solved by the method of separation of variables. The obtained solution is semi-analytical since the method of separation of variables is combined in the paper with numerical methods in order to compute complex eigenvalues and solve systems of linear algebraic equations. The method can be generalized for other problems with axial symmetry.

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