Abstract: - This paper discusses development of the mathematical model for a planar seven-link bipedal model which is comprised of an upper body and two legs (thigh, shank and feet in every leg). In sagittal plane, the bipedal model possesses seven degrees of freedom (DOF). Procedures of kinematic model and dynamic model constructions are presented in this paper. Kinematic model includes the coordinates of center of mass that used in the mathematical model procedure. The mathematical model for dynamic equations of motion based on the absolute angle is obtained using Lagrange’s equations. Then, a dimension transformation of mathematical model into relative angles was performed. New inertia matrix of the transformed equations was verified to be symmetric. To investigate the performance of the seven-link bipedal robot using the obtained mathematical mode, a simulation was conducted. The results from the simulation show that the dynamic equations of motion of the model are successfully achieved tracking the reference trajectories by giving small error value of every joint in the model. As conclusion, this equation is able to be used to investigate the motion control of seven-link human bipedal model.

Key-Words: - seven-link human biped model, Lagrange’s Equation, computed torque control

1 Introduction
In this modern time, many research groups are actively involved in human bipedal robotics research including humanoid robots and exoskeletons. These types of robots are required to walk naturally to provide a sense of intimacy to human being. To achieve this, an accurate model of human gait has to be obtained to reflect natural human walking trajectories [1]. However, human walking pattern is a complex activity. It is hard for human’s gait pattern to be directly incorporated into the bipedal robot model due to complex bipedal structure and excessive many degrees of freedom (DOF) in human gait [2].

There are 20 or more degrees of freedom (DOF) involved in human walking motion [3]. The inclusion of these high DOF into the mathematical model of human gait pattern could be problematic and may greatly restrict the implication of this model in the following engineering control tasks. Therefore, the human model for the gait analysis has to be as simple as possible. Many research efforts have been devoted into the solution of this complicated control mechanism.

This research study focused on the development of mathematical model for seven-link human bipedal model. Then, the mathematical model is applied in the MATLAB Simulink to observe the performances of the bipedal robot system.

This paper is structured as follows. Section 2 shows the kinetic model and dynamic equations of motion model for seven-link biped robot system using Lagrange’s Equation. Simulation in Section 3 is presented the controller involved to observe the performances of bipedal robot system. This section also shows the simulation parameters setup and results from this study. Finally, the conclusions of the study are made and some recommendations for future work is mentioned in Section 4.

2 Mathematical Modelling of Seven-Link Biped Model
In general, robot manipulators can be defined as a mechanical system that consists of links connected by joints. The links are numbered sequentially from the base (which is link 0) and up to the end-effector which is link n) [4]. The joints coincide to the contact points between two links. An actuator is usually placed at the joint. Therefore, every joint is controlled by an actuator independently and the joint movements give the relative movement of the links.
This section shows the kinematic model and dynamic equations of motion of the seven-link biped model briefly. To obtain the mathematical model of the biped system in this study, Lagrange’s equations of motion have been used. The procedure of the derivation will be shown in this section briefly.

2.1 Kinematic Model

The human biped is modeled as the seven serial links mechanism in sagittal plane as shown in Fig. 1. The seven links consists torso (link 4) and three links in each leg which are thigh (link 3 and 5), shank (link 2 and 6) and feet (link 1 and 7). These links are connected via rotating joints which are two hip joints, two knee joints and two angle joints. The joints are assumed to be frictionless and every of them are driven by an independent DC motor. All the calculations were made by S. Tzafestas et al. [5][3] that already done for five link biped model are referred to build the mathematical model for seven link biped model.

To simplify the analysis, some assumptions have been made which are both left and right sides of the biped model are symmetric, biped locomotion is constrained in the sagittal plane, and the friction of the ground is assumed to be large enough to ensure there is no slipping of the supporting end.

Most of walking dynamics take place on the sagittal plane [6]. This plane is defined as any plane that can be divided body parts into right and left sides [7]. The sagittal plane analysis drives the human biped gait pattern similar to the human gait pattern.

From Fig 1, the parameters of the biped model are shown as follows:

\[
\begin{align*}
&m_i : \text{mass of link} \\
&L_i : \text{length of link} \\
&L_{ic} : \text{distance between the center of mass and the lower joint of link} \\
&I_i : \text{mass moment of Inertia} \\
&\theta_i : \text{angle of link with respect to the horizontal axis (absolute angle)} \\
\end{align*}
\]

\[(x_{lc}, y_{lc}) \text{ are the coordinates of center of mass of link } i \text{ which is shown in (1) based from Fig. 1.}
\]

\[
\begin{align*}
 x_{lc} &= L_{ic} \cos \theta_i \\
 y_{lc} &= L_{ic} \sin \theta_i \\
 x_{2c} &= -L_2 \cos \theta_2 + L_{ic} \cos \theta_i \\
 y_{2c} &= -L_2 \sin \theta_2 + L_{ic} \sin \theta_i \\
 x_{4c} &= -L_4 \cos \theta_4 + L_{ic} \cos \theta_i \\
 y_{4c} &= -L_4 \sin \theta_4 + L_{ic} \sin \theta_i \\
 x_{5c} &= -L_5 \cos \theta_5 + L_{ic} \cos \theta_i \\
 y_{5c} &= -L_5 \sin \theta_5 + L_{ic} \sin \theta_i \\
 x_{6c} &= -L_6 \cos \theta_6 + L_{ic} \cos \theta_i \\
 y_{6c} &= -L_6 \sin \theta_6 + L_{ic} \sin \theta_i \\
 x_{7c} &= -L_7 \cos \theta_7 + L_{ic} \cos \theta_i \\
 y_{7c} &= -L_7 \sin \theta_7 + L_{ic} \sin \theta_i \\
 \end{align*}
\]

(1)

2.2 Dynamic Model

The dynamic equations of a robot manipulator in closed form can be acquired by using Lagrange’s Equations. It is one of the most common approaches used in the computation of robotic dynamic model in closed form. Lagrange’s equations of motion can be used for analysis with increasing number of joints in the robot.

Studies have shown many types of human biped model that can be generated from Lagrange’s Equations [2] which are two-link [8], three-link [9], five-link [5] and seven-link gait model [10]. S. Tzafestas et al. shown the calculations of five-link biped model using relative angles term. However, there is lack information about the construction of seven-link biped model although there are researches [10][11] that have involved for seven-link biped model. This paper presents the calculations for seven-link biped model in briefly.

The biped dynamic modeling is simplified by considering the single-leg support phase only in this study. During the single-leg support phase, the biped has one leg in contact with the surface (support leg) carrying all the weight of biped while the other leg which in the air (swing leg) is in the forward walking direction [12] as shown in Fig. 1. The Lagrange’s equation of motion can be written in the form of
where \( L = K - P \)

\( L \) : Lagrangian of \( n \)-DOF robot manipulator
\( K \): kinetic energy
\( P \): potential energy

These equations can be arranged in the general form as

\[
D(\theta)\ddot{\theta} + H(\theta, \dot{\theta})\dot{\theta} + G(\theta) = T_\theta
\]

(3)

where

\[
\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7]^T
\]

\[
T_\theta = [T_{\theta_1}, T_{\theta_2}, T_{\theta_3}, T_{\theta_4}, T_{\theta_5}, T_{\theta_6}, T_{\theta_7}]^T
\]

\[
H(\theta, \dot{\theta}) = \text{col} \left[ \sum_{j=1 \atop j \neq i}^7 (h_{ij}(\dot{\theta}_j))^2 \right]
\]

\[
G(\theta) = \text{col}[G_i(\theta)]
\]

\[
D(\theta) = [D_{ij}(\theta)] \quad i, j = 1, \ldots, 7
\]

with

\( \theta \) : joint angle vector

\( T_\theta \) : generalized torque corresponds to \( \theta_i \)

\( D(\theta) \) : \( 7 \times 7 \) symmetric, positive-definite inertia matrix

\( H(\theta, \dot{\theta}) \) : \( 7 \times 7 \) vector of Coriolis and Centripetal torques

\( G(\theta) \) : \( 7 \times 1 \) vector of gravitational torques

However, only six of seven DOF can be controlled directly by the driving torques at every joint. The angle \( \theta_1 \) at the contact point with the walking surface which is known as hypothetical joint 0 is controlled indirectly using the gravitational effects [12]. The model in (3) is transformed model using the relative angle for the control purpose.

The equations of motion are calculated in the terms of relative angles the link as shown in Fig. 2 which is

\[
D(q)\ddot{q} + H(q, \dot{q})\dot{q} + G(q) = T_q
\]

(4)

Based from Fig. 2, \( q_1, q_2, q_3, q_4, q_5, q_6 \) and \( q_7 \) are the relative angle deflections of the corresponding joints and can be calculated as shown in (5).

\[
q_0 = \theta_1
\]

\[
q_1 = \frac{\pi}{2} - \theta_1 - \theta_2
\]

\[
q_2 = \theta_3 - \theta_2
\]

\[
q_3 = \theta_4 - \theta_3
\]

\[
q_4 = \pi - \theta_4 - \theta_5
\]

\[
q_5 = \theta_6 - \theta_5
\]

\[
q_6 = \frac{\pi}{2} - \theta_6 - \theta_7
\]

and when

\( \tau_0 \) : driving torque at toes of the supporting leg

\( \tau_1 \) : driving torque at the ankle of the supporting leg

\( \tau_2 \) : driving torque at knee of the supporting leg

\( \tau_3 \) : driving torque at hip of the supporting leg

\( \tau_4 \) : driving torque at hip of the free leg

\( \tau_5 \) : driving torque at knee of the free leg

\( \tau_6 \) : driving torque at ankle of the free leg

By using the relationship using (4) and (5),

\[
T_{\theta_i} = \sum_{j=1}^6 \tau_j \left. \frac{\partial q_j}{\partial \theta_i} \right|_{\theta_0}
\]

(6)

where \( E \) is \( 7 \times 6 \) matrix and \( \tau \) is the \( 6 \times 1 \) matrix.
The generalized torques $T_{qi}$ corresponds to the relative angle displacements which are:

$$T_{q0} = 0 \quad \text{and} \quad T_{qi} = \tau_i \quad (8)$$

where $\tau_i$ is the actual driving torques at the joints of the model.

The angle displacement of every link can be written in terms of $q_i$ which are:

$$\theta_1 = q_0$$
$$\theta_2 = \frac{g}{2} - q_0 - q_1$$
$$\theta_3 = \frac{g}{2} - q_0 - q_1 + q_2$$
$$\theta_4 = \frac{g}{2} - q_0 - q_1 + q_2 + q_3$$
$$\theta_5 = \frac{g}{2} + q_0 + q_1 - q_2 - q_3 - q_4$$
$$\theta_6 = \frac{g}{2} + q_0 + q_1 - q_2 - q_3 - q_4 + q_5$$
$$\theta_7 = -q_0 - q_1 + q_2 + q_3 + q_4 - q_5 - q_6 \quad (9)$$

From the relationship [5] which is

$$T_{qi} = \sum_{j=1}^{7} T_{qj} \frac{d\theta_j}{d\theta_i} \quad i = 1, \ldots, 7, \quad (10)$$

The generalized torques $T_{qi}$ are obtained as:

$$T_{q0} = -T_{\theta1} + T_{\theta2} + T_{\theta3} + T_{\theta4} - T_{\theta5} - T_{\theta6} + T_{\theta7}$$
$$T_{q1} = -T_{\theta2} - T_{\theta3} - T_{\theta4} + T_{\theta5} + T_{\theta6} - T_{\theta7}$$
$$T_{q2} = -T_{\theta3} + T_{\theta4} - T_{\theta5} - T_{\theta6} + T_{\theta7}$$
$$T_{q3} = T_{\theta4} - T_{\theta5} - T_{\theta6} + T_{\theta7}$$
$$T_{q4} = -T_{\theta5} - T_{\theta6} + T_{\theta7}$$
$$T_{q5} = T_{\theta6} - T_{\theta7}$$
$$T_{q6} = -T_{\theta7} \quad (11)$$

Using the same relationship as (11), the equations of motion are transformed into the following forms which are:

$$A_{ij}\ddot{\theta}_i + \dot{A}_{ij}\dot{\theta}_j + A_{ij}\dot{\theta}_4 + A_{ij}\dot{\theta}_5 + A_{ij}\dot{\theta}_6 + A_{ij}\dot{\theta}_7 + H_{qi} + G_{qi} = T_{qi} = 0 \quad (12)$$

where

\[
\begin{align*}
A_{ij} &= -D_{ij} + D_{ij} + D_{ij} - D_{ij} - D_{ij}, \quad j = 1, 2, 3, 4, 5, 6, 7 \\
H_{qi} &= -H_i + H_i + H_i - H_i - H_i - H_i + H_i, \\
G_{qi} &= -G_i + G_i + G_i - G_i - G_i + G_i.
\end{align*}
\]

Using the same relations again, the equation of motion is further modified and finally transformed into the equation using the relative angle which is

$$D_\eta(q)\ddot{q} + H_\eta(q, \dot{q})\dot{q} + G_\eta(q) = T_q \quad (13)$$

where

\[
\begin{align*}
D_\eta(j, 1) &= -A_{j1} + A_{j2} + A_{j3} + A_{j4} - A_{j5} - A_{j6} + A_{j7} \\
D_\eta(j, 2) &= -A_{j2} - A_{j3} - A_{j4} + A_{j5} + A_{j6} - A_{j7} \\
D_\eta(j, 3) &= A_{j3} + A_{j4} - A_{j5} - A_{j6} + A_{j7} \\
D_\eta(j, 4) &= A_{j4} + A_{j5} + A_{j6} - A_{j7} \\
D_\eta(j, 5) &= A_{j5} - A_{j6} + A_{j7} \\
D_\eta(j, 6) &= A_{j6} - A_{j7} \\
D_\eta(j, 7) &= -A_{j7} \quad \text{with} \; j = 1, 2, 3, 4, 5, 6, 7
\end{align*}
\]
\[ H_q(q, \dot{q}) = \begin{bmatrix} H_{q_1}, H_{q_2}, H_{q_3}, H_{q_4}, H_{q_5}, H_{q_6} \end{bmatrix}^T \]
\[ G_q(q) = \begin{bmatrix} G_{q_1}, G_{q_2}, G_{q_3}, G_{q_4}, G_{q_5}, G_{q_6} \end{bmatrix}^T \]
\[ T_q = \begin{bmatrix} T_{q_1}, T_{q_2}, T_{q_3}, T_{q_4}, T_{q_5}, T_{q_6} \end{bmatrix}^T \]

\( D_q(q) \) is a 7×7 symmetric, positive definite inertia matrix, \( H_q(q, \dot{q}) \) is the 7×1 vector of centripetal and Coriolis torques, \( G_q(q) \) is the 7×1 vector of gravitational torques and \( T_q \) is the vector of control torques applied at each joint. This mathematical model of human biped will be carried out using relative angles in terms of absolute angles \( \theta_i \) (i = 1, 2, 3, 4, 5, 6, 7). To verify the new inertia matrix \( D_q(q) \) from the obtained transformed human biped model, this matrix is symmetric.

### 3 Simulation

In this section, (13) can be verified by using computed torque control done in MATLAB Simulink. Next, the simulation parameters setup and the results from controller are shown in this section.

#### 3.1 Computed Torque Control

Computed torque control based on Proportional-Derivative (PD) control is used to track the planned trajectory. Based from Fig. 3, the generalised torque equation is:

\[ T_q = D_q(q)u + H_q(q, \dot{q}) + G_q(q) \] (14)

The trajectory error, \( e \) is:

\[ e = q - q_r \] (15)

where

- \( q \): actual joint trajectory
- \( q_r \): reference joint trajectory

Fig. 3: PD-based computed torque control scheme [4]

Then, (15) is differentiate with respect to time up to second order which are

\[ \ddot{e} = \dot{q} - \dot{q}_r \] (16)
\[ \dddot{e} = \ddot{q} - \ddot{q}_r \] (17)

The control law based from PD control law is:

\[ u = \ddot{q}_r - K_D \dddot{e} + K_P e \] (18)

where

- \( K_D = \text{diag}[k_D] \)
- \( K_P = \text{diag}[k_P] \)

The closed-loop equation for the error tracking is

\[ \dddot{q} + K_D \dddot{q} + K_P q = 0 \] (19)

In order to attain the critical damped closed loop performance, the value of \( K_D \) and \( K_P \) are:

\[ K_D = \text{diag}[2\lambda] \]
\[ K_P = \text{diag}[\lambda^2] \]

Due to leg contact with ground, the component \( u_0 \) cannot be computed with (18) because it is chosen that \( T_q(1) = 0 \) which the biped model has non-controllable joint \( q_0 \). Therefore, the equation to estimate the value of \( u_0 \) and \( u_{i+1} \) are:

\[ u_0 = -\frac{1}{D_q(3,3)} \cdot \left[ \sum_{j=1}^{6} [D_q(1,j+1) \cdot D_{q,j}(1,1) + H_q(1) + G_q(1)] \right] \] (20)
\[ u_{i+1} = \ddot{q}_r i - K_{Di} \dddot{e}_i - K_{Pi} \dot{e}_i \] (21)

where \( i = 0, 1, 2, 3, 4, 5 \)

#### 3.2 Simulation Parameters Setup

The parameters of the biped robot model achieved from [13][14] is shown in Table 1 below:

<table>
<thead>
<tr>
<th>Link</th>
<th>Link Number</th>
<th>Mass, m (kg)</th>
<th>Length, L (m)</th>
<th>Locat of Centre of Mass, Lc (m)</th>
<th>Moment of Inertia, I (kgm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torso</td>
<td>4</td>
<td>0.678M=44.07</td>
<td>0.75</td>
<td>0.375</td>
<td>0.99077 6</td>
</tr>
<tr>
<td>Thigh</td>
<td>3.5</td>
<td>0.1M=6.5</td>
<td>0.52</td>
<td>0.260</td>
<td>0.13895 2</td>
</tr>
<tr>
<td>Shank</td>
<td>2.6</td>
<td>0.046M=3.0225</td>
<td>0.37</td>
<td>0.185</td>
<td>0.06543 5</td>
</tr>
<tr>
<td>Foot</td>
<td>1</td>
<td>0.0145M=0.942</td>
<td>0.27</td>
<td>0.180</td>
<td>0.00877 4</td>
</tr>
</tbody>
</table>

Table 1: Parameters of human biped model [13][14]
By using Heuristic Method, the obtained $\lambda$ value is $28\text{rad/s}$. Therefore the controller gains for PD controller are:

\[
k_p = 784/\text{s}^2
\]
\[
k_D = 56/\text{s}
\]

Fig. 4 shows the walking trajectories of every joint for biped robot model [15].

![Fig. 4: Walking trajectories of every joint](image)

3.3 Simulation Results and Discussion
The computed torque control is driven the tracking performance of the biped robot system. Fig. 5 until Fig. 10 show the tracking errors of every joint in the biped system. These figures show that the system with computed torque control is stable and controllable.

![Fig. 5: Tracking error of joint 1](image)

![Fig. 6: Tracking error of joint 2](image)

![Fig. 7: Tracking error of joint 3](image)
Table 2: Averaged tracking error of every joint obtained using computed torque control

<table>
<thead>
<tr>
<th>Joint</th>
<th>Averaged Tracking Error (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0261</td>
</tr>
<tr>
<td>2</td>
<td>0.0377</td>
</tr>
<tr>
<td>3</td>
<td>0.0190</td>
</tr>
<tr>
<td>4</td>
<td>0.0295</td>
</tr>
<tr>
<td>5</td>
<td>0.0347</td>
</tr>
<tr>
<td>6</td>
<td>0.0219</td>
</tr>
</tbody>
</table>

4 Conclusion

This paper presents the methodology for derivation of equations motion to describe the seven-link biped robot model walking on a flat horizontal surface. The equations are developed from kinematic model and dynamic equations of motion model are shown in this paper. The equations of motion for the single support phase were constructed by using the biped model with one support leg. These equations were also developed using Lagrange’s Equations using relative angles. The symmetrical matrix of new inertia matrix \( D_q(q) \) shows that the obtained transformed human biped model is verified. Simulation is done to explore the motion control performances of the seven-link biped robot using the obtained mathematical model using MATLAB Simulink is successful in tracking the reference trajectories by giving small error value of every joint in the human bipedal model. The controller will be extended and modified with different types of intelligent techniques to provide continuous, automatic and online computation of required inertia matrix while the system is in motion.

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