Vibration Analysis of Reinforced Multi-Layered Cylindrical Shell under Uniform Internal Pressure Using Sanders Shell Theory

MOHAMMAD REZA ISVANDZIBAEI, HISHAMUDDIN JAMALUDDIN, RAJA ISHAK RAJA HAMZAH
Faculty of Mechanical Engineering, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, MALAYSIA
esvandzebaei@yahoo.com, hishamj@fkm.utm.my, rishak@fkm.utm.my

Abstract: - In this paper a study on the vibration of reinforced multi-layered cylindrical shell under uniform internal pressure is presented. The analysis is carried out with strain–displacement relations from Sanders shell theory. The governing equations of motion were employed by using energy functional and by applying Ritz method. The boundary conditions represented by the end conditions of the multi-layered cylindrical shell are free-sliding and clamped-sliding. The influence of internal pressure on natural frequency characteristics is studied. The present analysis is validated by comparing the results with those available in the literatures.

Key-Words: - Vibration, Cylindrical Shell, Sander Theory, Multi-Layered.

1 Introduction
Cylindrical shells have been used for many years in different engineering applications. They are used as structures in aircrafts, ships, rockets, submarines, missile bodies, pressure vessels, oil tanks and buildings. Studies on the vibrations characteristics of cylindrical shells have been carried out extensively. Among those who have studied the vibrations of cylindrical shells include Arnold and Warburton [1], Chung [2], Bhimaraddi [3], Soldatos and Hajigeoriou [4]. Lam and Loy [5] presented studies on the influence of boundary conditions on the frequencies of a cylindrical shell. The effects of boundary conditions for vibration of cylindrical shell were also presented by Pradhana et al. [6]. The use of stringer and ring stiffeners on the circular cylindrical shell were analysed by Zhao et al. [7], where the effects of rings on the natural frequencies of the cylindrical shells were presented. Multi-layered cylindrical shells are very often more effective and useful than a single layered type of shells, due to the improvements in the mechanical properties of the layers.

The aim of this paper is to present a study on the natural frequency characteristics of reinforced multi-layered cylindrical shells under internal pressure for two boundary conditions. The analysis is carried out using Sanders shell theory. The governing equations of motion are derived using Ritz method with energy functional. The analysis is carried out on the natural frequency characteristics with different boundary conditions by using beam functions as the axial modal functions. The multi-layered cylindrical shell is made up of isotropic three layers where the inner and outer layers are made of stainless steel and the middle layer is aluminium. The boundary conditions of the multi-layered cylindrical shell considered are the combination of free-sliding (F-SL) and clamped-sliding (C-SL). The influence of uniform internal pressure on the natural frequency is studied. The present analysis is validated by comparing the results with those available in the literatures.

2 Sanders Shell Theory
Consider a reinforced multi-layered cylindrical shell under uniform internal pressure which is shown in Fig.1. where $R$ is the radius, $L$ is the length, $P$ is uniform internal pressure and $h$ is the thickness. The reference surface is chosen to be the middle surface of the reinforced multi-layered cylindrical shell where an orthogonal coordinate system $(x, \theta, z)$ is fixed. The deformations of the shell in reference to this coordinate system are denoted by $u$, $v$ and $w$ in the $x$, $\theta$ and $z$ directions respectively.

For a reinforced multi-layered cylindrical shell, plane stress condition can be assumed. The constitutive relation given by Hook's law as,

$$\{\sigma\} = [Q] \{\varepsilon\} \quad (1)$$

where, $\{\sigma\}$ is the stress vector, $\{\varepsilon\}$ is the strain vector and $[Q]$ is the reduced stiffness matrix. The stress vector for plane stress condition is,

$$\{\sigma\}^T = \begin{pmatrix} \sigma_x & \sigma_{x\theta} \\ \sigma_{\theta x} & \sigma_{\theta} \end{pmatrix} \quad (2)$$
where, $\sigma_x$ is the stress in $x$-direction, $\sigma_\theta$ the stress in the $\theta$-direction and $\sigma_{x\theta}$ is the shear stress in the $x\theta$-plane. The strain vector is defined as,

$$\{e\}^T = \begin{bmatrix} e_x \\ e_\theta \\ e_{x\theta} \end{bmatrix}$$

(3)

where, $e_x$ is the strain in $x$-direction, $e_\theta$ the strain in the $\theta$-direction and $e_{x\theta}$ is the shear strain on the $x\theta$-plane. The reduced stiffness $[Q]$ matrix is given as,

$$[Q] = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}$$

(4)

For a cylindrical shell, the reduced stiffness $Q_{ij}$ ($i, j = 1, 2$ and 6) are defined as,

$$Q_{11} = \frac{E}{1-\nu^2} , \quad Q_{12} = \frac{\nu E}{1-\nu^2} , \quad Q_{22} = \frac{E}{1-\nu^2} , \quad Q_{66} = \frac{E}{2(1+\nu)}$$

(5)

From Sanders shell theory [8], the strain components in the strain vector $\{e\}$ are defined as linear functions of the thickness coordinate $z$ as,

$$e_x = e_1 - z k_1 , \quad e_\theta = e_2 - z k_2 , \quad e_{x\theta} = \gamma - 2 z \tau$$

(6)

where $e_1, e_2$, and $\gamma$ are the reference surface strains and $k_1, k_2$ and $\tau$ are the surface curvatures. These surface strains and curvatures are defined as

$$\{e_1, e_2, \gamma\} = \{ \frac{\partial u}{\partial x} - \frac{1}{R} \frac{\partial w}{\partial \theta} , \frac{\partial v}{\partial x} + \frac{1}{R} \frac{\partial w}{\partial \theta} \}$$

(7)

$$\{k_1, k_2, \tau\} = \{ \frac{\partial^2 w}{\partial x^2} - \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta} , \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{4R} \frac{\partial u}{\partial \theta} + \frac{3}{4} \frac{\partial v}{\partial \theta} \}$$

(8)

The force and moment resultants are defined by

$$\{N_x, N_\theta, N_{x\theta}\} = \int_{h/2}^{h/2} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\} \ dz$$

(9)

$$\{M_x, M_\theta, M_{x\theta}\} = \int_{h/2}^{h/2} \{\tau_x, \tau_\theta, \tau_{x\theta}\} \ dz$$

(10)

Substituting into Eq. (1), the substitution of Eq. (6), into Eqs. (9) and (10), the constitutive equation is obtained as,

$$\{N\} = [S] \{e\}$$

(11)

where $\{N\}$ and $\{e\}$ are respectively defined as,

$$\{N\}^T = \{N_x, N_\theta, N_{x\theta}, M_x, M_\theta, M_{x\theta}\}$$

(12)

$$\{e\}^T = \{e_1, e_2, \gamma, k_1, k_2, 2\tau\}$$

(13)

and $[S]$ is defined as,

$$S = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

(14)

where $A$ is the extensional stiffness, $B$ is the coupling stiffness and $D$ is the bending stiffness.
and are given as
\[
A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{nn} \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{nn} \end{pmatrix}
\]

and
\[
A_{ij} = \int_{-h/2}^{h/2} Q_{ij} \, dz, \quad B_{ij} = \int_{-h/2}^{h/2} Q_{ij} \, z \, dz \\
D_{ij} = \int_{-h/2}^{h/2} Q_{ij} \, z^2 \, dz
\]

(16)

For a multi-layered isotropic cylindrical shell, the stiffness are defined as
\[
A_{ij} = \sum_{k=1}^{n} \int_{-h/2}^{h/2} Q_{ij}^k (h_k - h_{k-1}) \\
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \int_{-h/2}^{h/2} Q_{ij}^k (h_k^2 - h_{k-1}^2) \\
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \int_{-h/2}^{h/2} Q_{ij}^k (h_k^3 - h_{k-1}^3)
\]

(17)

The displacement fields for reinforced multi-layered cylindrical shell can be written as,
\[
\begin{align*}
    u &= A \frac{\partial \Omega(x)}{\partial x} \cos(n \theta) \cos(\omega t) \\
    v &= B \Omega(x) \sin(n \theta) \cos(\omega t) \\
    w &= C \Omega(x) \prod_{i=1}^{H} (x - h_i)^{\gamma_m} \cos(n \theta) \cos(\omega t)
\end{align*}
\]

(23)

where \( A, B \) and \( C \) are the constants denoting vibrations in the axial \( u \), circumferential \( v \) and radial \( w \) directions while \( \phi(x) \) is the axial modal function satisfying condition at both ends of the multi-layered cylindrical shell, \( n \) denotes the number of circumferential waves in the mode shape and \( \omega \) is the natural angular frequency of the vibration.

The beam modal function \( \phi(x) \) has been chosen as the axial modal function and expressed in the form of
\[
\phi(x) = \alpha_1 \cosh(\frac{\lambda_m x}{L}) + \alpha_2 \cos(\frac{\lambda_m x}{L}) - \zeta_m
\]

\[
(\alpha_3 \sin(\frac{\lambda_m x}{L}) + \alpha_4 \sin(\frac{\lambda_m x}{L}))
\]

(24)

where, \( \alpha_i \, (i = 1, \ldots, 4) \) are some constants with values of \( 0 \) or \( 1 \) chosen according to the boundary condition. \( \lambda_m \) are the roots of some transcendental equations and \( \zeta_m \) are some parameters dependent on \( \lambda_m \).

The values of \( \alpha_i \, (i = 1, \ldots, 4) \), the transcendental equations and the parameters \( \zeta_m \) for the free-sliding and clamped-sliding boundary condition considered are given in Table 1.

**Table 1. Values of \( \alpha_i, \lambda_m \) and \( \zeta_m \) for two boundary conditions**

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>( \alpha_i )</th>
<th>( \lambda_m )</th>
<th>( \zeta_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free-Sliding (F-SL)</td>
<td>( \Psi_1 = 1, \Psi_2 = 1 )</td>
<td>( (4m - 1)\pi / 4 )</td>
<td>( \cos \phi_a - \cos \phi_m )</td>
</tr>
<tr>
<td></td>
<td>( \Psi_1 = 1, \Psi_2 = 1 )</td>
<td>( (4m - 1)\pi / 4 )</td>
<td>( \sin \phi_a - \sin \phi_m )</td>
</tr>
<tr>
<td>Clamped-Sliding (C-SL)</td>
<td>( \Psi_1 = 1, \Psi_2 = 1 )</td>
<td>( (4m - 1)\pi / 4 )</td>
<td>( \cos \phi_a - \cos \phi_m )</td>
</tr>
<tr>
<td></td>
<td>( \Psi_1 = 1, \Psi_2 = 1 )</td>
<td>( (4m - 1)\pi / 4 )</td>
<td>( \sin \phi_a - \sin \phi_m )</td>
</tr>
</tbody>
</table>
To determine the natural frequencies, Ritz method is used. The energy functional $F$ is defined by the Lagrangian function as,

$$F = U_{\text{max}} - T_{\text{max}} + V_{\text{max}}$$  \hspace{1cm} (25)

Substituting Eq. (23) into Eqs. (18), (19) and (20) and minimizing the energy functional $F$ with respect to the unknown coefficients as follows,

$$\frac{\partial F}{\partial A} = \frac{\partial F}{\partial B} = \frac{\partial F}{\partial C} = 0$$  \hspace{1cm} (26)

From Eq. (26), the three governing eigen value equations can be obtained. These three governing eigen value equations can be expressed in matrix from as:

$$\begin{bmatrix} C_{ij} \end{bmatrix} \{x\} = \{0\}$$  \hspace{1cm} (27)

and

$$\{x\}^T = \{A \quad B \quad C\}$$  \hspace{1cm} (28)

The eigen value equations are solved by imposing the non-trivial solutions condition and equating the determinant of the characteristic matrix $[C_{ij}]$ to zero. Expanding this determinant, a polynomial in even powers of $\omega$ is obtained:

$$\beta_0 \omega^6 + \beta_1 \omega^4 + \beta_2 \omega^2 + \beta_3 = 0$$  \hspace{1cm} (29)

where $\beta_i (i = 0,1,2,3)$ are some constants. Eq. (29) is solved by using Newton-Raphson procedure where three positive and three negative roots are obtained. The three positive roots obtained are the natural angular frequencies of the reinforced multi-layered cylindrical shell in the $x$, $y$ and $z$ directions. The smallest of the three positive roots is the natural angular frequency (Hz) studied in the present study.

### 3 Results and discussion

To validate the analysis, results for simply supported multi-layered cylindrical shells without internal pressure are compared with Arshad et al. [9], see Table 2. The comparisons show that the presented results agree well with those in the literature.

Table 2. Comparison of natural frequency of multi-layered cylindrical shell without internal pressure and reinforce with simply supported-simply supported boundary condition. ($m=1$, $L/R=20$, $h/R=0.002$); Inner and outer layers: Stainless steel $E = 2.1 \times 10^{11}$ (N/m²), $\nu = 0.28$, $\rho = 7.8 \times 10^3$ (kg/m³)

Middle layer: Aluminum $E = 7.0 \times 10^9$ (N/m²), $\nu = 0.35$, $\rho = 2.7 \times 10^3$ (kg/m³)

<table>
<thead>
<tr>
<th>$n$</th>
<th>Arshad et al. [9]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.645</td>
<td>13.632</td>
</tr>
<tr>
<td>2</td>
<td>4.625</td>
<td>4.582</td>
</tr>
<tr>
<td>3</td>
<td>4.331</td>
<td>4.311</td>
</tr>
<tr>
<td>4</td>
<td>7.366</td>
<td>7.340</td>
</tr>
<tr>
<td>5</td>
<td>11.775</td>
<td>10.772</td>
</tr>
</tbody>
</table>

Figures 2 and 3 show the variation of natural frequency characteristics with the circumferential wave numbers $n$ for the reinforced multi-layered cylindrical shell at reinforce position $b = 0.3 L$ with and without internal pressure. The analyses are conducted by assuming internal pressures equal to 1500 kPa. The boundary condition for Fig. 2 is free-sliding (F-SL) and for Fig. 3 is clamped-sliding (C-SL). In both cases with and without internal pressure, the natural frequency increases as the circumferential wave number $n$ is increased.

Fig. 2. Variation of the natural frequency with F-SL boundary condition. ($L/R = 20$, $h/R = 0.002$, $R = 1$, $b = 0.3 L$)

The results show that internal pressure has an effect on the natural frequency characteristics of a reinforced multi-layered cylindrical shell and
causes the natural frequency to increase, and when the value of the internal pressure is large, the natural frequency is higher. The results show that reinforce and pressure have influence on the natural frequencies of multi-layered cylindrical shell. The results obtained also show that the natural frequency characteristics of a reinforced multi-layered cylindrical shell with and without internal pressure are different for different boundary conditions, and the effects of the two boundary conditions are prominent for low circumferential wave numbers. It should be noted that the natural frequency of the reinforced multi-layered cylindrical shell with and without internal pressure for the curves is calculated for $m = 1$. 

Fig. 3. Variation of the natural frequency with C-SL boundary condition. ($L/R = 20$, $h/R = 0.002$, $R = 1$, $b = 0.3L$) 

4 Conclusion

A study on the vibration of reinforced multi-layered cylindrical shell under internal pressure is presented. In multi-layered cylindrical shell, the outer and inner layers are stainless steel, while the middle layer is assumed to be aluminum. The study was carried out using Sanders shell theory. The governing equations were derived using Ritz method. The boundary conditions considered are free-sliding (F-SL) and clamped-sliding (C-SL). The results show that internal pressure has an effect on the natural frequency characteristics and causes the natural frequency to increase, and when the value of the internal pressure is large, the natural frequency is higher. The results obtained also show that the natural frequency characteristics of a reinforced multi-layered cylindrical shell with and without internal pressure are different for different boundary conditions.

References: