A comparison of Lee-Carter and Cairns-Blake-Dowd stochastic mortality model

JÁN GOGOLA
Department of Mathematics and Quantitative Methods
Faculty of Economics and Administration, University of Pardubice
Studentská 84, 532 10 Pardubice
CZECH REPUBLIC
jan.gogola@upce.cz http://www.upce.cz

Abstract: - Insurance companies are affected by many different kinds of risks. In the case of life insurance there are two main risks: the investment risk and the demographic risk. The latter can be split into insurance risk due to the random deviation of the number of deaths from its expected value, and longevity risk deriving from the improvement in mortality rates. Numbers of stochastic models have been developed to analyse the mortality improvement. This paper focuses on Lee-Carter and Cairns-Blake-Dowd models. We use data on male’s deaths and exposures for the Czech Republic from the Human Mortality Database. We write the code associated with models in R.

In this paper we propose using the CBD model as a longevity risk indicator. The indicator contains only two set of numbers, \( \kappa_t^{(1)} \) and \( \kappa_t^{(2)} \), each of which is readily interpretable and they together tell how mortality rates at different ages change with time. It has the new-data invariant property.

Key-Words: - Mortality, Lee-Carter, Cairns-Blake Dowd model, forecasting, R language, force of mortality

JEL Classification: C22, J11

1 Introduction
The mortality of the population in developed countries has improved rapidly over the last thirty years and this has important financial implications for the insurance industry, since several important classes of liability are sensitive to the direction of future mortality trends. This uncertainty about the future development of mortality gives rise to longevity risk. Longevity risk plays a central role in the insurance company management since only careful assumptions about future evolution of mortality phenomenon allow the company to correctly face its future obligations. Longevity risk represents a sub-modul of the underwriting risk module in the Solvency II framework. The most popular and widely used model for projecting longevity is the well-known Lee-Carter model. This paper follows on article Jindrová, Slavíček [6]. They deal with the development and the prediction of life expectancy in selected European countries (Czech Republic, Slovakia, Finland and Spain) by applying Lee-Carter model.

Most stochastic mortality models are constructed in a similar manner. Specifically, when they are fitted to historical data, one or more time-varying parameters (\( \kappa_t \)) are identified. By extrapolating these parameters to the future, we can obtain a forecast of future death probabilities and consequently other demographic quantities such as life expectancies. They are important for quantifying longevity in pension risks and for constructing benchmarks for longevity-linked capital markets.

2 Data and notation
We use data on male deaths and exposure to risk between 1960 and 2011 from the Human Mortality Database (www.mortality.org) [4]. We consider the restricted age range from 60 to 90, the range of interest to providers of pensions.

Let calendar year \( t \) runs from exact time \( t \) to exact time \( t+1 \) and let \( d_{x,t} \) be the number of deaths aged \( x \) last birthday in the calendar year \( t \). We suppose that the data on deaths are arranged in a
matrix $D = (d_{x,t})$. In a similar way, the data on exposure are arranged in a matrix $E = (e_{x,t})$ where $e_{x,t}$ is a measure of the average population size aged $x$ last birthday in calendar year $t$, the so-called central exposed to risk. We suppose that $(d_{x,t})$ and $(e_{x,t})$ are each $n_a \times n_y$ matrices, so that we have $n_a$ ages and $n_y$ years.

We denote the force of mortality (or hazard rate) at exact time $t$ for lives with exact age $x$ by $\mu_{x,t}$. The force of mortality can be thought as an instantaneous death rate the probability that a life subject to a force of mortality $\mu_{x,t}$ dies in the interval of time $(t, t + dt)$ is approximately $\mu_{x,t} \cdot dt$ where $dt$ is small.

The force of mortality $\mu_{x,t}$ for human populations varies slowly in both $x$ and $t$ and a standard assumption is that $\mu_{x,t}$ is constant over each year of age, i.e., from exact age $x$ to exact age $x+1$, and over each calendar year, i.e., from exact time $t$ to exact time $t+1$. Thus,

$$\mu_{x+u,t+v} = \mu_{x,t} \text{ for } 0 \leq u < 1, 0 \leq v < 1,$$

and so $\mu_{x,t}$ approximate the mid-year force of mortality $\mu_{x+0.5,t+0.5}$.

We suppose that $d_{x,t}$ is a realization of a Poisson variable $D_{x,t}$:

$$D_{x,t} \sim P(e_{x,t}, \mu_{x,t}),$$

The expected values are the product of exposures $e_{x,t}$ and the force of mortality $\mu_{x,t}$.

Assumption (2) leads us to the estimates of $\mu_{x,t}$ as

$$\hat{\mu}_{x,t} = \frac{d_{x,t}}{e_{x,t}},$$

or in a matrix form $\hat{\mu} = \frac{D}{E}$, that means element-wise division in $\mathbf{R}$.

We also consider the mortality rate $q_{x,t}$. This is the probability that an individual aged exactly $x$ at exact time $t$ will die between $t$ and $t + 1$.

We have the following relation between the force of mortality and the mortality rate:

$$q_{x,t} = 1 - \exp \left( \int_0^1 \mu_{x+s,t+s} \, ds \right) \approx 1 - e^{-\mu_{x,t}}. \quad (4)$$

### 3 The mortality models

We use the following conventions for our models:

- The $\alpha_x, \beta_x^{(1)}$ coefficients will reflect age-related effects
- The $\kappa_{i}^{(1)}, \kappa_{i}^{(2)}$ coefficients will reflect time-related effects

Our models are fitting to historical data.

#### a) Lee-Carter model

The Lee-Carter model was introduced by Ronald D. Lee and Lawrence Carter in 1992 with the article [7]. The model grew out of their work in the late 1980s and early 1990s attempting to use inverse projection to infer rates in historical demography. The model has been used by the United States Social Security Administration, the US Census Bureau and the United Nations. It has become the most widely used mortality forecasting technique in the world today.

Lee and Carter proposed the following model for the force of mortality:

$$\log \mu_{x,t} = \alpha_x + \beta_{x}^{(1)} \cdot \kappa_{i}^{(1)} + \kappa_{i}^{(2)} \mu_{x,t}, \quad (5)$$

with constraints

$$\sum_{i=1}^{n_{\kappa}} \beta_{i}^{(1)} = 1, \quad (6)$$

$$\sum_{i=1}^{n_{\kappa}} \kappa_{i}^{(1)} = 0. \quad (7)$$

The second constraint implies that, for each $x$, the estimate for $\alpha_x$ will be equal (at least approximately) to the mean over $t$ of the $\log \mu_{x,t}$.

By the equation (5) the log of the force mortality is expressed as the sum of an age-specific component $\alpha_x$ that is independent of time and another component that is the product of a time-varying parameter $\kappa_{i}^{(1)}$ reflecting the general level of mortality and an age-specific component $\beta_{x}^{(1)}$ that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

Interpretation of the parameters in Lee-Carter model is quite simple: $\exp(\alpha_x)$ is the general shape of the mortality schedule and the actual forces of mortality change according to overall mortality index $\kappa_{i}^{(1)}$. 
modulated by an age response $\beta_x^{(1)}$ (the shape of the $\beta_x^{(1)}$ profile tells which rates decline rapidly and which slowly over time in response of change in $\kappa_t^{(1)}$).

b) Cairns-Blake-Dowd model (CBD model)
The original CBD model was published in Cairns et al. [2].
The model fits mortality rates $q_{x,t}$:
\[
\logit q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(1)}(x - \bar{x}),
\]
where $\logit x = \log \left( \frac{x}{1-x} \right)$, $x \in (0, 1)$, and $\bar{x}$ is the mean age in the sample range (in our case $\bar{x} = 75$).
This model has no constraints.
We calculate the likelihood for all models based on the $\mu_{x,t}$. For a given model we use $\phi$ to represent the full set of a parameters and the notation for $\mu_{x,t}$ is extended to $\mu_{x,t}(\phi)$, to indicate its dependence on these parameters.
For both models the log-likelihood is:
\[
\ell(\phi; D, E) = \sum_{x,t} (d_{x,t} \cdot \log [e_{x,t} \cdot \mu_{x,t}(\phi)] - e_{x,t} \cdot \mu_{x,t}(\phi) - \log(d_{x,t}!))
\]
and estimation is by maximum likelihood.

The Lee-Carter model deals with the force of mortality $\mu_{x,t}$, whereas the CBD model with the mortality rate $q_{x,t}$. To ensure a valid comparison between the different models, our analysis of the models for $q_{x,t}$ involve an additional step. For a given set of parameters we calculate the $q_{x,t}$ then we transform these into force of mortality using the identity $\mu_{x,t} = -\log(1 - q_{x,t})$ which comes from (4). (log x means natural logarithm of x throughout the article)
We can calculate the likelihood for all models consistently based on the $\mu_{x,t}$.
For a model with $q_{x,t}$ we use notation $q_{x,t} = q_{x,t}(\phi)$ and we define
\[
\mu_{x,t}(\phi) = -\log(1 - q_{x,t}(\phi)),
\]
For practice the fitting of a model is usually only the first step and the main purpose is the forecasting of mortality. For forecasting-time series we use Box-Jenkins approach [5]. We apply the R package Forecast - methods and tools for displaying and analysing univariate time series forecasts including automatic ARIMA modelling. The estimated parameters ($\kappa_t^{(1)}$, $\kappa_t^{(2)}$) create a bivariate vector time-series and it is modelled by a multivariate approach.
The estimated age parameters, $\alpha_x$, $\beta_x^{(1)}$, are assumed invariant over time. This last assumption is certainly an approximation but the method has been very thoroughly tested in Booth at al. [1] and found to work.

4 Results
In Figures 1.-3. we have plotted the maximum likelihood estimates for the parameters for each model, using the Czech republic male’s data, aged 60-90.
Figure 1. Shows that the parameter $\kappa_t^{(1)}$ values from the Lee-Carter model changes if new data are considered.
• When one more year of data is included, the maximum likelihood estimates of all model parameters, that is, $\alpha_t$, $\beta_t$ and $\kappa_t$ for all $x$ and $t$ will be updated.
• Parameter constraints are involved in the estimation process. In particular, the constraint $\sum \kappa_t$ re-scales the series of $\kappa_t$ as new data are included.
For CBD model the inclusion of new data will not affect previous parameters values. We can call this property as “new data invariant”.
Reasons for this special property, that adding new data will have no effect on the parameters that are already estimated, is due to no constraint in this model.
Figure 3. shows the data-invariant property of MLE estimates of mortality parameters from the CBD model using Czech Republic male data.
$\kappa_t^{(1)}$ in CBD model presents the level of the logit-transformed mortality curve. A reduction in $\kappa_t^{(1)}$, that is a parallel downward shift of the logit-transformed mortality curve, represents an overall mortality improvement.
\( \kappa_t^{(2)} \) presents the steepness of the logit-transformed mortality curve. An increase in \( \kappa_t^{(2)} \), that is an increase in the steepness of the logit-transformed mortality curve, means that mortality (in logit scale) at younger ages improves more rapidly than at older ages.

5 Conclusion

National governments and the WHO announce life expectancies of different populations every year. To financial institutions, life expectancy is not an adequate measure of risk, because it does not give any idea about how mortality rates at different ages vary over time. On the other hand, indicators of longevity risk cannot be too complicated. An indicator that is composed by a huge array of numbers is difficult to interpret and will lose the purpose as a “summary” of a mortality pattern. We propose using the CBD model mortality parameters (\( \kappa_t^{(1)} \), \( \kappa_t^{(2)} \)) as a longevity risk indicator. It is a “simple” summary of a mortality pattern. The indicator contains only two set of numbers, \( \kappa_t^{(1)} \) and \( \kappa_t^{(2)} \), each of which is readily interpretable and they together tell how mortality rates at different ages change with time. It has the new-data invariant property. This property is important; because, as a proper indicator, we cannot allow new data to alter the index values of previous years.

We have presented stochastic models to analyse the mortality and shown how they may be fitted. Afterwards we can turn to the industry requirement to forecast future mortality. But forecasting of mortality should be approached with both caution and humility. Any prediction is unlikely to be correct. There is a need for awareness of model risk when assessing longevity-related liabilities, especially for annuities and pensions. The fact that parameters can be estimated does not imply that they can sensibly be forecast. Such forecasting should enable actuaries to examine the financial consequences with different models and hence to come to an informed assessment of the impact of longevity risk on the portfolios in their care.

Acknowledgements I am grateful to Iain Currie and Andrew Cairns, from Heriot-Watt University in Edinburgh, for discussion on this paper and for valuable suggestions.

Model fitting was done in R, which was also used for all graphs.

References:


Fig. 1. Parameter $\kappa_t^{(1)}$ estimates for Lee-Carter model. Source: Own Processing

Fig. 2. Parameter $\kappa_t^{(1)}$ estimates for Cairns-Blake-Dowd model. Source: Own Processing

Fig. 3. Parameter $\kappa_t^{(2)}$ estimates for Cairns-Blake-Dowd model. Source: Own Processing